Transport in media, interaction models

Consider a particle moving in the z-direction in a medium in which the medium can change along its path

The survival probability $p_s(z)$ can be expressed in general differential forms as

$$dp_s(z) = -p_s(z)\,\mu(z)\,dz \qquad (3.2)$$

($\mu(z)$: interaction probability / unit length)

Assuming that $p_s(0) = 1$, we can integrate

$$p_s(z) = \exp\left(-\int_0^z dz'\,\mu(z')\right) \qquad (3.3)$$

$p_s(0) = 1$ i.e., if $\mu(z) > 0$, $p_s(\infty) = 0$

The probability that a particle has interacted between $0 \le z' \le z$ is

$$c(z) = 1 - p_s(z) = 1 - \exp\left(-\int_0^z dz'\,\mu(z')\right) \qquad (3.4)$$

The probability of interaction is $\frac{d}{dz}c(z) = p(z)$

$$p(z) = \mu(z)\,p_s(z) \qquad (3.5)$$

8.1.1 Uniform, ∞, homogeneous media

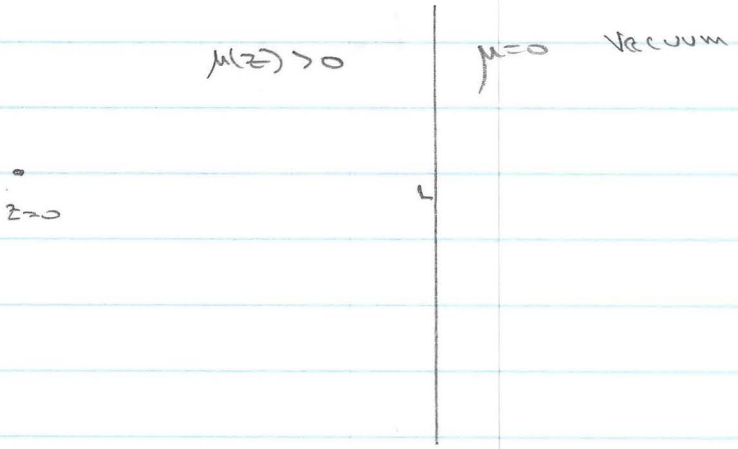
In this case $\mu = \mu(z)$, a constant

$$p_s(z) = e^{-\mu z} \qquad (8.6)$$

$$c(z) = 1 - e^{-\mu z} \qquad (8.7)$$

$$p(z) = \mu p_s(z) \qquad (8.8)$$

8.2 Finite media



In the above case

since $\mu(z)=0$ for $z>L$

$$c(\infty) = 1 - p_s(\infty) = 1 - \exp\left(-\int_0^\infty dz' \mu(z')\right) < 1 \qquad (8.9)$$

which implies that not all probability has been accounted for

Rewrite $C(z)$ as

$$C(z) = 1 - P_s(z) = 1 - \exp\left(-\int_0^\infty dz' \mu(z')\right) + \exp\left(-\int_0^L dz' \mu(z)\,\Theta(z-L)\right) \quad (8.10)$$

$$p(z) = \mu(z)\exp\left(-\int_0^z dz' \mu(z')\right) + \underbrace{P_s(L)\,\delta(z-L)}_{\text{Survival at } z=L}$$

8.3 Generalization to many regions with different scattering characteristics



$$\mu(z) = \mu_1(z)\Theta(z)\Theta(b_1-z) + \mu_2(z)\Theta(z-b_1)\Theta(b_2-z) + \quad (8.12)$$

$$\vdots$$

$$p(z) = \Theta(z)\Theta(b_1-z)\,\mu_1(z)\exp\left(-\int_0^z dz' \mu_1(z')\right) + \Theta(z-b_1)\Theta(b_2-z)\,\mu_2(z)\underbrace{\exp\left(-\int_0^{b_1} dz' \mu_1(z')\right)}_{P_s(b_1)}\exp\left(-\int_{b_1}^z dz' \mu_2(z')\right)$$

$$\vdots \quad (8.13)$$

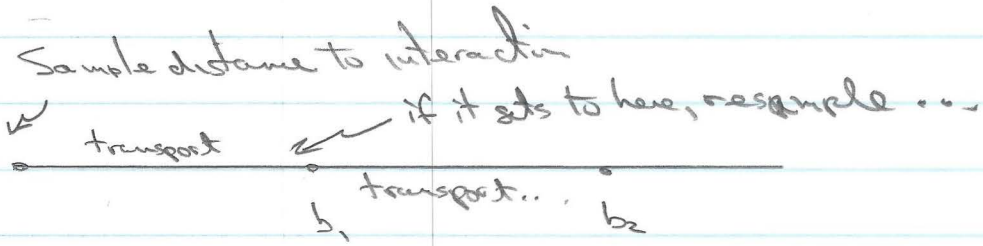
or

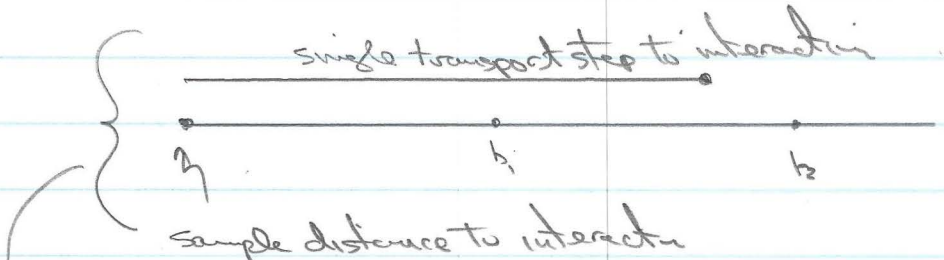
$$p(z) = \Theta(z)\Theta(b_1 - z)\,\mu_1(z)\exp\left(-\int_0^z dz'\,\mu_1(z')\right)$$

$$+ p_s(b_1)\,\Theta(z - b_1)\Theta(b_2 - z)\,\mu_2(z)\exp\left(-\int_{b_1}^z dz'\,\mu_2(z')\right)$$

$$\vdots \qquad (3.14)$$

Implications (e.g. 2 medium scattering)





sample distance to interaction

more efficient scheme for analog transport

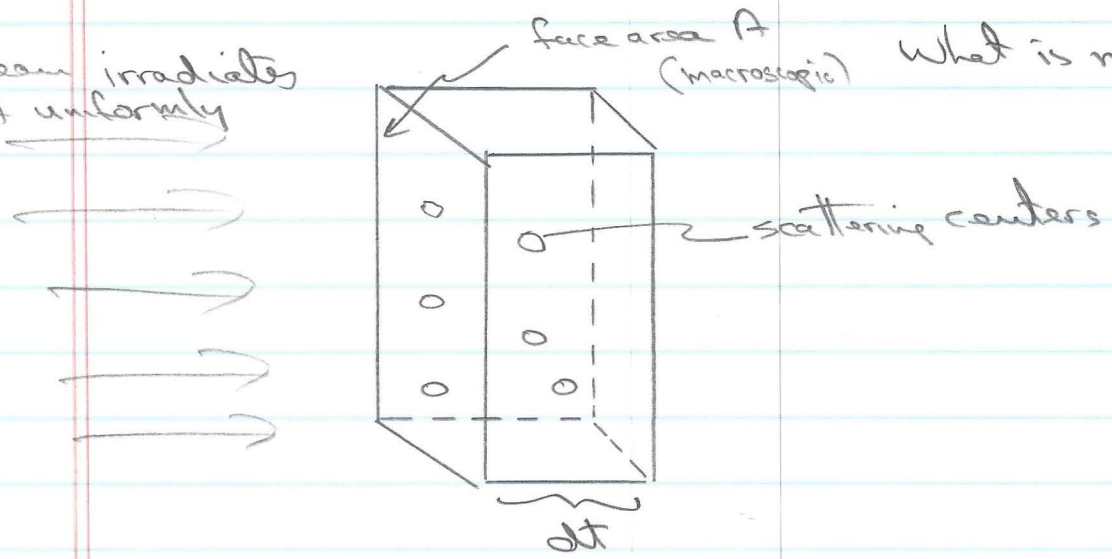
(not for electrons)

(8.4) Obtaining μ from microscopic cross sections

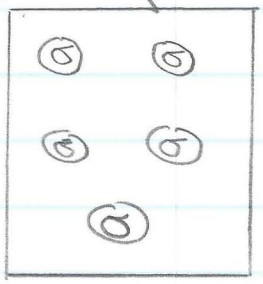
$$\sigma = \frac{dp}{n\,dz}$$

Unit		
L^2	σ =	single particle cross section causing an "event"
none	dp =	probability that an "event" occurs
$1/L^3$	n =	density of scatterers
L	dz =	thickness of target

beam irradiates A uniformly



Face on, the beam "sees" (this is called the "beam's eye view") individual scattering centers, each with area σ



Total # of scatterers

$$dN = \frac{N_A \rho A\,dz}{M}$$

N_A = Avogadro's number,
$6.022140857(74) \times 10^{23}$ /mole

ρ = mass density [e.g. kg/L^3]

M = molar mass [kg/mole] of the material being irradiated

So,

$$n = \frac{N_A \rho}{M}$$ is the number density

$$dp = \left(\frac{N_A \rho \sigma}{M}\right) dz$$

$\frac{N_A \rho \sigma}{M}$ is called the "macroscopic" cross section (units $1/L$)

Common symbols μ, or Σ

↑ don't confuse with $\vec{\mu}$, a direction cosine in the book

8.5 Compounds & Mixtures

Recall, for a single medium

$$dp(z) = \mu(\varepsilon)\, dz$$

↖ emphasizes that μ can be (usually is) ε

Generalizing to compounds (non molecular) & mixtures

$$\mu(\varepsilon) = \sum_{i=1}^{N_e} n_i \sigma_i$$

N_e = # of elements

where $n_i = \dfrac{N_A \rho_i}{M_i}$

← in the book M_i & A are used interchangably

Molecular compound

A molecular compound is an "entity" of two or more atoms. e.g. H_2O, NH_3

For high energy applications, meaning that the quantum wavelength $\lambda\!\!\!^{-} = \dfrac{\lambda_h}{p}$

($\lambda\!\!\!^{-} = \lambda/2\pi$, $\lambda_h = \lambda_h/2\pi$)

↖ Planck's constant

is much smaller than the molecule, elemental cross sections are employed.

e.g. pair production, bremsstahlung } in their σ-dominance regim

When the wavelength becomes of the order of the size of the molecule,

eg photoelectric effect, Rayleigh (coherent) scattering } in their dominance regim

The Compton (or incoherent) interaction's dominance regim spans both regimes

Discuss graphite!

Mixtures

$$\mu(E) = \sum_{i=1}^{N} \frac{\rho_i^2}{M_i} N_A \sigma_i$$

(ρ_i: actual density)

$$= \sum_{i=1}^{N} \rho_i \left(\frac{\mu_i}{\rho_i}\right) = \rho \sum_{i=1}^{N} w_i \left(\frac{\mu_i}{\rho_i}\right)$$

(ρ: of the mixture; ρ_i: normal density)

$$\frac{\mu(E)}{\rho} = \sum_{i=1}^{N} w_i \left(\frac{\mu_i}{\rho_i}\right) \qquad (3.37)$$

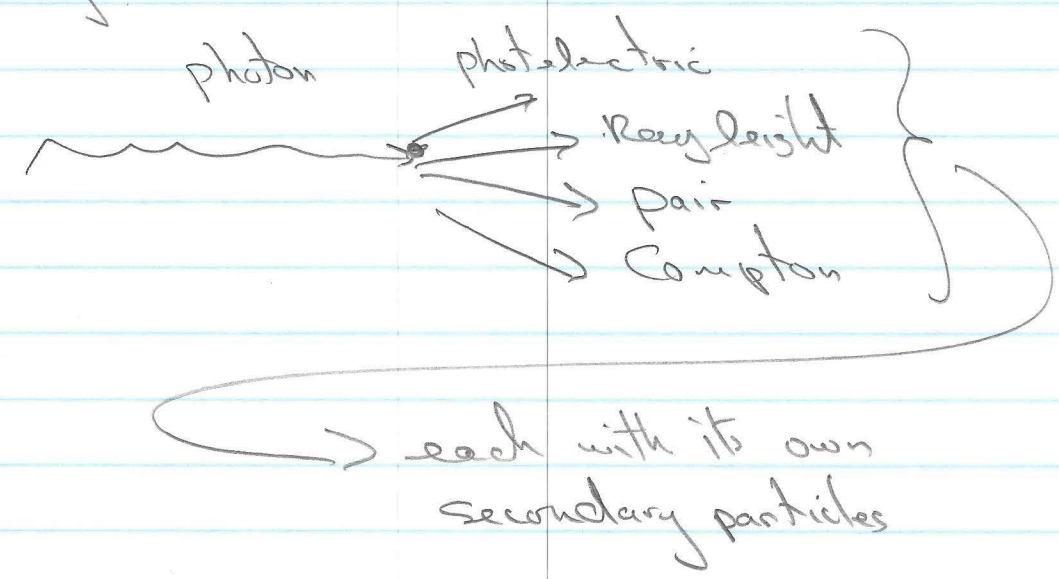
($\frac{\mu_i}{\rho_i}$: this is what is usually tabulated)

8.6 Branching Ratios

When $\mu(E)$ is used to determine the distance to an interaction, via

$$p(z) = e^{-\mu(E)z}$$

$\mu(E)$ may encompass several different interactions.



The fractional probability for each channel is:

$$P_i = \frac{p_i(z)}{p(z)} = \frac{\mu_i(E)}{\mu}$$

ith channel

Sampling procedure (Matlab)

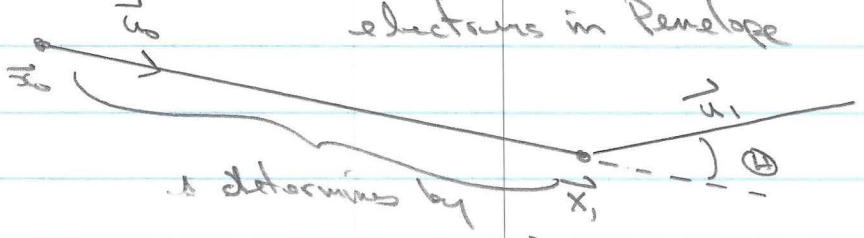
```
p = [ , , , ];  % vector of branching ratios
% if p can be sorted by largest element first,
% the procedure is more efficient
r = rand;
for i = 1:length(p)
    if (r <= sum(p(1:i)))
        interactionType = i;
        break   % breaks out of the for loop
    end
end
% Sample interaction type
```

See test code "branch.m"

8.7 Pathlength selection schemes

Pure analog Monte Carlo

photons in MCNP, EGSx, GEANT, Penelope ...
neutrons " MCNP "
electrons in Penelope



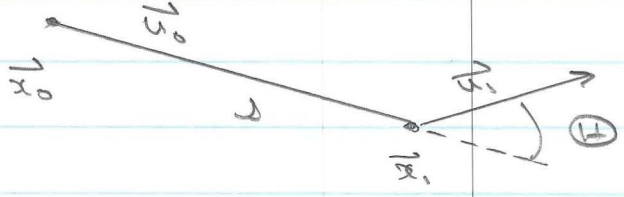
"catastrophic" events for charged particles

$$p(s)ds = \Sigma(E)\, e^{-\Sigma(E)s}\, ds$$

Σ not dependent on position within a geometrical element

- secondary particles can be created at $\vec{x}_1$

Condensed history Monte Carlo

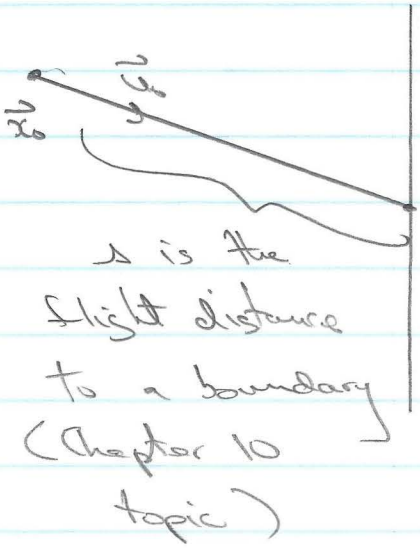


s is chosen small enough so that analytic theories for (1) or ΔE can be applied accurately at $\vec{x}_1$

Charged particles in $\vec{E}$ or $\vec{B}$ fields

s must be small enough so that transport in $\vec{E}$ + $\vec{B}$ fields can be done accurately

Geometry element interfaces



s is the flight distance to a boundary (Chapter 10 topic)

8.8 Model Interactions

Model interactions are used as theoretical benchmarks in Monte Carlo codes, and in this course, a lot!

8.8.1 Isotropic scattering

$$p(\theta,\phi)\,d\theta\,d\phi = \frac{1}{4\pi}\sin\theta\,d\theta\,d\phi \qquad (8.42)$$

$$\frac{1}{4\pi}\int_0^{\pi}\sin\theta\,d\theta\int_0^{2\pi}d\phi = 1 \quad \checkmark$$

Sampling $\hat{\phi} = 2\pi\xi_1$

where ξ_i is a uniform random # $0 \le \xi_i \le 1$

cdf $$c(\hat{\theta}) = \frac{1}{2}\int_0^{\hat{\theta}} d\theta\sin\theta$$

$$= \frac{1}{2}\int_{\cos\hat{\theta}}^{1} d\mu \qquad \mu = \cos\hat{\theta}$$
$$d\mu = -\sin\hat{\theta}\,d\hat{\theta}$$

$$= \frac{1}{2}(1-\cos\hat{\theta})$$

Sampling

$$\xi_2 = \frac{1}{2}(1-\cos\hat{\Theta})$$

$$\cos\hat{\Theta} = 1 - 2\xi_2 \qquad (3.43)$$

$$\text{or } \hat{\Theta} = \cos^{-1}(1-2\xi_2)$$

3.3.2 Semi-isotropic scattering

$$p(\Theta,\underline{\phi})\,d\Theta\,d\underline{\phi} = \frac{1}{4\pi}(1+a\cos\Theta)\sin\Theta\,d\Theta\,d\underline{\phi}$$

$|a| \le 1$ (why?)

Note: separable

$$p(\underline{\phi})\,d\underline{\phi} = \frac{1}{2\pi}\,d\underline{\phi}$$

$$\Rightarrow \quad \hat{\underline{\phi}} = 2\pi\xi_1$$

$$p(\Theta)\,d\Theta = \frac{1}{2}(1+a\cos\Theta)\,d\Theta$$

normalized

cdf?

$$c(\Theta) = \frac{1}{2}\int_0^{\Theta} d\Theta'\,(1+a\cos\Theta')\sin\Theta'$$

$$c(\theta) = \frac{1}{2}\int_{\cos\theta}^{1} d\mu\,(1+a\mu)$$

$$= \frac{1}{2}(1-\cos\theta) + \frac{a}{4}(1-\cos^2\theta)$$

Sampling?

$$\xi_2 = \frac{1}{2}(1-\cos\hat{\theta}) + \frac{a}{4}(1-\cos^2\hat{\theta})$$

$$\textcircled{1} \qquad \frac{a}{4}\cos^2\hat{\theta} + \frac{1}{2}\cos\hat{\theta} - \frac{a}{4} - \frac{1}{2} + \xi_2 = 0$$

$\frac{4}{a} \times \textcircled{1} \Rightarrow$

$$\cos^2\hat{\theta} + \frac{2}{a}\cos\hat{\theta} - 1 - \frac{2}{a} + \frac{4\xi_2}{a} = 0$$

Apply quadratic eq'n

$$\cos\hat{\theta} = -\frac{1}{a} \pm \sqrt{\frac{1}{a^2} + 1 + \frac{2}{a} - \frac{4\xi_2}{a}}$$

$$= \frac{\frac{1}{a^2} - \left(\frac{1}{a^2} + 1 + \frac{2}{a} - \frac{4\xi_2}{a}\right)}{-\frac{1}{a} + \sqrt{\frac{1}{a^2} + 1 + \frac{2}{a} - \frac{4\xi_2}{a}}}$$

$x - a$

top & bottom $= \dfrac{2+a-4\xi_2}{1+\sqrt{1+a^2+2a-4a\xi_2}}$

↑ this sign sets the correct $a=0$ limit

$$\cos\hat{\Theta} = \frac{2+a-4\xi_2}{1+\left((1+a)^2-4a\xi_2\right)^{1/2}}$$

Note: book equation (3.45) appears to be wrong.

3.3.3 Rutherford Scattering

$$p(\Theta)\,d\Theta\,d\bar{\phi} = \frac{a(2+a)}{4\pi}\,\frac{\sin\Theta\,d\Theta\,d\bar{\phi}}{(1-\cos\Theta+a)^2}$$

$0 \le a < \infty$

Sampling $\hat{\bar{\phi}} = 2\pi\xi_1$,

$$\cos\hat{\Theta} = 1 - \frac{2a(1-\xi_2)}{a+2\xi_2}$$

8.8.4 Small angle Rutherford

$$P(\Theta)\,d\Theta\,d\Phi = \frac{2a}{\pi}\,\frac{\Theta\,d\Theta\,d\Phi}{(\Theta^2+2a)^2}$$

Sampling

$$\hat{\Phi} = 2\pi\xi_1$$

$$\hat{\Theta} = \sqrt{\frac{2a\xi_2}{1-\xi_2}}$$

Chapter 9 Lewis Theory

Harold ("Hal") Warren Lewis
1923-2011

Lewis Theory is one of the most useful theories for describing the spatial evolution for situations where particle scatter more than once
i.e Multiple Scattering Theory
(see link To the paper)

It all starts with the transport equation (Lewis 1950)

$$(9.1)\quad \left[\frac{\partial}{\partial s} + \vec{\mu}\cdot\vec{\nabla} + \Sigma(E)\right]\psi(\vec{x},\vec{\mu},s)$$

$$= \int_{4\pi} d_{\mu'}\, \Sigma(\vec{\mu}\cdot\vec{\mu}',E)\,\psi(\vec{x},\vec{\mu}',s)$$

$\Sigma(\vec{\mu}\cdot\vec{\mu}',E)$: microscopic scattering cross section, no energy transfer, monochromatic, spatially uniform

Boundary condition

$$(9.3)\quad \psi(\vec{x},\mu,E,0) = S(\vec{x})\,\delta(\vec{z}-\tfrac{1}{\mu})\,\delta(E_0-E)$$

$s=0$; $S(\vec{x})$: at the origin; $\delta(\vec{z}-\frac{1}{\mu})$: starts along z axis; E_0: energy E_0

It is usually used for monochromatic situation

$$s = vt \qquad (9.4)$$

(velocity)

or, using the "Continuous slowing down" approximation - CSDA

$$s = vt = \int_{E}^{E_0} \frac{dE'}{L(E')} \qquad (9.5)$$

($L(E')$: stopping power)

9.1 The formal solution

$$\Phi(\vec{x},\vec{\mu},s) = \sum_{lm} \Phi_{lm}(\vec{x},s)\, Y_{lm}(\vec{\mu}) \qquad (9.6)$$

(Y_{lm}: spherical harmonics)

$\Rightarrow$ some work

Transport eq'n becomes

$$\left[\frac{\partial}{\partial s} + \kappa\right] \Phi_{lm}(\vec{x},s) = - \sum_{\lambda\mu} \vec{\nabla} \Phi_{\lambda\mu}(\vec{x},s) \cdot \vec{Q}^{\lambda\mu}_{lm} \qquad (9.7)$$

(κ: kappa)

where

$$(9.8)\quad \kappa_\ell(E) = \int_0^{2\pi} d\Phi \int_0^{\pi} d\Theta \sin\Theta\, \Sigma(\cos\Theta, E)\,(1 - P_\ell(\cos\Theta))$$

- assume scattering law is azimuthally symmetric (does not depend on Φ)

and

$$(9.9)\quad \vec{Q}^{\lambda\mu}_{\ell m} = \int_{4\pi} d\vec{\mu}\; \mu\, Y^*_{\ell m}(\vec{\mu})\, \vec{\mu}\, Y_{\lambda\mu}(\vec{\mu})$$

The boundary condition is: (error in notes)

$$(9.10)\quad \psi_{\ell m}(\vec{x}, 0) = \delta_{m0} f(\vec{x}) Y_{\ell 0}(\cos\Theta, \Phi)$$

$$= \sqrt{\frac{2\ell+1}{4\pi}}\, \delta_{m0} f(\vec{x})$$

This formal solution has now been solved!

If one integrates over all space, just to look at the angular part,

$$(9.11) \qquad \left[\frac{\partial}{\partial s} + \kappa_\ell \right] \psi_\ell(s) = 0$$

one obtains a result discovered by Goudsmit & Saunderson in 1940

$$(9.12) \qquad \psi(\vec{\mu}, s) = \frac{1}{4\pi} \sum_{\ell=0}^{\infty} (2\ell+1) P_\ell(\cos\Theta) k_\ell(s)$$

(Legendre polynomials)

where

$$(9.13) \qquad k_\ell(s) = \exp\left(- \int_0^s ds' \, \kappa_\ell(\varepsilon) \right)$$

$k_\ell(s)$: (K, kept) — $\kappa_\ell(\varepsilon)$: kappa

Note that this result holds true for any scattering law

Lewis's contribution was to find the spatial moments and the space angle correlations.

Namely

(9.14) $$\langle z \rangle = \int_0^s ds' \, k_1(s')$$

(9.15) $$\langle z \cos\theta \rangle = \frac{k_1(s)}{3} \int_0^s ds' \, \frac{1 + k_2(s')}{k_1(s')}$$

$$\langle x^2 \rangle = \langle y^2 \rangle = \frac{1}{2} \langle x^2 + y^2 \rangle$$

(9.16) $$\langle x^2 + y^2 \rangle = \frac{4}{3} \int_0^s ds' \, k_1(s') \int_0^{s'} ds'' \, \frac{1 - k_2(s'')}{k_1(s'')}$$

$$\langle x \sin\theta \cos\phi \rangle = \langle y \sin\theta \sin\phi \rangle$$

$$= \frac{1}{2} \langle x \sin\theta \cos\phi + y \sin\theta \sin\phi \rangle$$

$$\langle x \sin\theta \cos\phi + y \sin\theta \sin\phi \rangle =$$

(9.17) $$\frac{2 k_1(s)}{3} \int_0^s ds' \, \frac{1 - k_2(s')}{k_1(s')}$$

$$(9.18) \qquad \langle \vec{x} \cdot \vec{\mu} \rangle = k_1(s) \int_0^s ds' \frac{1}{k_1(s')}$$

$$(9.19) \qquad \langle z^2 \rangle = \frac{2}{3} \int_0^s ds' k_1(s') \int_0^{s'} ds'' \frac{1 + 2k_2(s'')}{k_1(s'')}$$

$$(9.20) \qquad \langle x^2 + y^2 + z^2 \rangle = 2 \int_0^s ds' k_1(s') \int_0^{s'} ds'' \frac{1}{k_1(s'')}$$

One can not overstate the importance of these results for the development of Monte Carlo methods (later lectures)

Application: Assignment 4, isotropic scattering

Application

9.2 Isotropic scattering in uniform media

(9.21) $$\sigma(\Theta,\underline{\Phi})\,d\Theta\,d\underline{\Phi} = \frac{\sigma_0}{4\pi}\sin\Theta\,d\Theta\,d\underline{\Phi}$$

(σ_0 ← constant — microscopic cross section)

$$\Sigma(\Theta,\underline{\Phi})\,d\Theta\,d\underline{\Phi} = \frac{\rho N_A}{M}\frac{\sigma_0}{4\pi}\sin\Theta\,d\Theta\,d\underline{\Phi}$$

(9.22) $$= \Sigma_0\,\frac{1}{4\pi}\sin\Theta\,d\Theta\,d\underline{\Phi}$$

From (9.8)

$$k_\ell = \int_0^{2\pi} d\underline{\Phi} \int_0^{\pi} \sin\Theta\,d\Theta\;\Sigma(\cos\Theta,E)\left[1-P_\ell(\cos\Theta)\right]$$

(9.23) $$= \Sigma_0\,(1-\delta_{\ell 0})$$

$$k_\ell = \exp\left(-\int_0^{\lambda} d\lambda'\,\Sigma_\ell(E)\right)$$

(9.24) $$= \delta_{\ell 0} + (1-\delta_{\ell 0})\,e^{-\Sigma_0 \lambda}$$ ← call this γ

i.e. $k_0 = \gamma$ $\quad \ell = 0$

$k_\ell = e^{-\gamma}$ $\quad \ell \geq 1$

From this it can be shown

full expression	$\lambda \to 0$ limit	$\lambda \to \infty$ limit
$\Sigma_0 \langle z \rangle = 1 - e^{-\lambda}$	λ	1
$\Sigma_0 \langle z \cos\theta \rangle = \frac{1-(1-\lambda)e^{-\lambda}}{3}$	λ	1/3
$\Sigma_0 \langle x \sin\theta \cos\phi \rangle =$		
$\Sigma_0 \langle y \sin\theta \sin\phi \rangle = \frac{1-(1+\lambda)e^{-\lambda}}{3}$	$\frac{\lambda^2}{6}$	1/3
$\Sigma_0 \langle \vec{x} \cdot \vec{j} \rangle = 1 - e^{-\lambda}$	λ	1
$\Sigma_0^2 \langle z^2 \rangle = \frac{2[1+\lambda-(1+2\lambda)e^{-\lambda}]}{3}$	λ^2	$2(\lambda+1)/3$
$\Sigma_0^2 \langle x^2 \rangle =$		
$\Sigma_0^2 \langle y^2 \rangle = \frac{2[\lambda-2+(2+\lambda)e^{-\lambda}]}{3}$ (error in notes)	$\frac{\lambda^3}{9}$	$2(\lambda-2)/3$
$\Sigma_0^2 \langle x^2+y^2+z^2 \rangle = 2[\lambda-1+e^{-\lambda}]$	λ^2	$2(\lambda-1)/3$

Note!

Discussion of assignment

NERS544: (Introduction to) Monte Carlo Methods

Assignment 4: **Lewis Theory** Two week assignment, single weightFall 2016

Revision: October 6, 2016 Alex Bielajew, 2927 Cooley, `bielajew@umich.edu`
Due: Thursday Oct. 20, 2016 before class

Submit your answers to 1.(a) on paper and 1.(b) as a Matlab script M-file via email.

1. 10,000 particles start out in an infinite, unbounded medium with initial position vector $\vec{x}_0 = 0$ and initial direction vector $\vec{u}_0 = \hat{z}$. The interaction probability is:

   $$p(s_i)\ \mathrm{d}s_i = \Sigma_0 \exp(-\Sigma_0 s_i)\ \mathrm{d}s_i$$

   where s_i is a measure of the pathlength of the particle along its current direction of motion starting from its current position to the i'th interaction point. Σ_0 is a constant in this case. Having interacted, the particle scatters isotropically and can scatter again, repeatedly, according to the same scattering law. This repeated scattering and transport results in a probability distribution function of the form

   $$F(\vec{x}, \vec{u}, s)\ \mathrm{d}\vec{x}\ \mathrm{d}\vec{u}\ \mathrm{d}s$$

   which you will develop stochastically through a Monte Carlo program. The variable s is a measure of total pathlength traveled, *i.e.* $s = \sum_i^{N(s)} s_i$. Note that the number of interactions $N(s)$ is a probabilistic quantity.

   (a) Consider the following moments:

   - $\langle x \rangle \pm \sigma_{\langle x \rangle}$ (*i.e.* average x)
   - $\langle y \rangle \pm \sigma_{\langle y \rangle}$ (*i.e.* average y)
   - $\langle z \rangle \pm \sigma_{\langle z \rangle}$ (*i.e.* average z)
   - $\langle x\vec{u} \cdot \hat{x} \rangle \pm \sigma_{\langle x\vec{u}\cdot\hat{x} \rangle}$ (*i.e.* average correlation of x with the x-axis direction cosine)
   - $\langle y\vec{u} \cdot \hat{y} \rangle \pm \sigma_{\langle y\vec{u}\cdot\hat{y} \rangle}$ (*i.e.* average correlation of y with the y-axis direction cosine)
   - $\langle z\vec{u} \cdot \hat{z} \rangle \pm \sigma_{\langle z\vec{u}\cdot\hat{z} \rangle}$ (*i.e.* average correlation of z with the z-axis direction cosine)
   - $\langle xy \rangle \pm \sigma_{\langle xy \rangle}$ (*i.e.* average correlation of x and y)
   - $\langle xz \rangle \pm \sigma_{\langle xz \rangle}$ (*i.e.* average correlation of x and z)
   - $\langle yz \rangle \pm \sigma_{\langle yz \rangle}$ (*i.e.* average correlation of y and z)
   - $\langle x^2 \rangle \pm \sigma_{\langle x^2 \rangle}$ (*i.e.* average x^2)
   - $\langle y^2 \rangle \pm \sigma_{\langle y^2 \rangle}$ (*i.e.* average y^2)
   - $\langle z^2 \rangle \pm \sigma_{\langle z^2 \rangle}$ (*i.e.* average z^2)

   Which moments would you expect to be zero? Why?

   Which moments would you expect to be equal to each other? Why?

   (b) For $\Sigma_0 = 1$ and $\lambda \equiv \Sigma_0 s = 0.001$, 0.002, 0.005 0.001, 0.02, 0.05, 0.1, 0.2, 0.5, 1, 2, 5, 10, 20, 50, 100, 200, 500, 1000, tally the above moments, their estimated errors, alongside their theoretical predictions.

   Plot the results divided by λ^n as a function of $\log(\lambda)$. n is some power of n such that the correct small and large pathlength asymptotic forms (according to Equations (9.32)–(9.45)) divided by this λ^n, is a constant. (*e.g.* $\langle z^2 \rangle / \lambda^2$ would be expected to go to the constant 1 for small λ and $\langle z^2 \rangle / \lambda$ would be expected to go to the constant 2/3 for large λ.) If you expect the moment to be zero (on average), set $n = 0$. Verify that both the $\lambda \longrightarrow 0$ and the $\lambda \longrightarrow \infty$ limits discussed in this chapter are reached.