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Chapter 1

Making Mathematics Work in School[1, 2]

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We begin with a well-known classroom episode. It takes place on a midwinter day in third grade. About 6 minutes into mathematics class, a boy named Sean[3] raises his hand and says he has been "thinking about the number six." He says he has been thinking it could be even or it could be odd. His is a peculiar claim, because these third graders already know that six is an even number. They learned that fact in second grade. In the ensuing minutes, his classmates, sure that six cannot possibly be odd, challenge him and try to show him the fallacy in his thinking. Still, significantly, no one laughs at this exchange as mathematical foolishness. And they do not simply contradict him, nor refer to some "fact" learned somewhere else. Instead, they press him to "prove" his idea. Listening to them, one can detect that "prove it" constitutes a serious intellectual request, not a social taunt.

Sean persists. He has noticed something special about six that he finds interesting: it is made up of two threes, but also of three twos. Since two is even and three is odd, six seems to him to have both even *and* odd structures. Sean thinks this composition is interesting. Sean's idea agitates many of his classmates, who are quite sure six is not odd. They argue with him, and Tembe challenges Sean to "prove it" to the class. At this point in the class, the teacher presses the students to consider their working definition for even numbers. One of Sean's classmates, Mei, listening intently, understands. "I think I know what he is saying," she says. "I think what he's saying is that you have three groups of two. And three is an odd number so six can be an odd number *and* a even number." Sean is pleased: Mei has grasped his idea. However, then Mei says she disagrees with this reasoning. According to Mei, whether a number is considered even or odd is not according to how many groups of two it has. To provide an example, she draws 10 circles on the chalkboard (see Figure 1.1).

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[2] We gratefully acknowledge our colleagues Hyman Bass and Edward Wall for their influence on the thinking and work reflected in this article.

[3] All names are pseudonyms, standardized across published analyses of these data, and selected to be culturally similar to the children's real names. For example, Sheena, an African American child, was given a name chosen from among other moderately common African American girls' names; Tembe was from Kenya, and his pseudonym was selected from among similar Kenyan boys' names.

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Figure 1.1. Mei's drawing of 10 circles as five groups of two.

Mei shows Sean that if he extended his observation of six to other numbers—in this example, ten—other numbers would be similarly "even and odd." Ten has five groups of two. She thinks he will see the error and retract his claim. Instead, Sean acknowledges that ten can be even and odd, too. This idea sends the class into pandemonium. Mei is amazed:

> What about *other* numbers?! Like, if you keep on going on like that and you say that other numbers are odd and even, maybe we'll end it up with *all* numbers are odd and even. Then it won't make sense that all numbers should be odd and even, because if all numbers were odd *and* even, we wouldn't be even having this *discussion!*

In the next 20 minutes, other students enter the fray, and eventually the class has realized that Sean's observation of six applies to other numbers as well: 14, 18, 2, 26, …, that is, all the odd multiples of two. The teacher eventually labels these special evens as "Sean numbers," properly distinguished from the terms *even* and *odd*. The pupils explore the properties of these new numbers.

This remarkable episode, briefly summarized in the foregoing, has served as the subject of many analyses and much discussion, in the research literature and beyond. The episode is one excerpt from a single mathematics lesson, and video clips from this lesson have been used to stimulate broad conversation about education, mathematics, teaching and learning, teacher development, and policy. Less examined, however, is the backdrop of these pupils' and their teacher's work. Sean's idea was percolating for several minutes before he brought it up in class. He and his classmates had also already been practicing ways to explore mathematical ideas and develop conjectures about them. This article makes that backdrop the focus. What constitutes doing such mathematical work in school?

Lampert (2001) argues that learning and teaching mathematics in school are constructed interpersonally, socially, and intellectually across time.

> The different units of time in which teaching happens are one source of its complexity. Teaching problems are solved in particular moments of interaction, and in the larger scale of the lesson as a whole. . . . The teacher also acts across groups of lessons, ranging from a pair of lessons connected across two days to the totality of all lessons across the year. Still other actions, tasks, or strategies may need to be performed to maintain longer-term connections between students and mathematics. The complexity of teaching in time does not end with considering different units, however, because teaching acts are not uniquely situated in single units of time. (p. 36)

To gain perspective on this oft-discussed episode, we examine the 6 minutes of class that lead up to Sean's announcement that six "could be an odd and an even number, both."[4] We use this segment to explore the nature and demands of doing

mathematical work of the kind in which Sean and his classmates were engaged. Our purpose is to unpack the work of teaching in ways that make it accessible for scrutiny, analysis, and development. We argue that a central task of teaching is making mathematical work for students, that intellectually honest mathematical work can be done in school, and that much of the work of teaching is itself mathematical work.

MATHEMATICAL WORK

Teacher: Um, there's a lot of disagreement about this issue, right? And you saw that the fourth graders who have been thinking about this for a long time also disagree about it, don't they? (turns 20B–20C)[5]

Is zero even or odd? Or is it "special"? What is the answer to 0 minus 9? Is 4/4 smaller or larger than 5/5, or are those quantities the "same"? Questions such as these fill the days of elementary school teachers as students wonder and puzzle about mathematics. Even if many adults can answer these questions for themselves, how eight-year-olds might sensibly reason about them is not immediately obvious.

The challenge for the elementary-grades teacher grows from the imperative that answers be based on mathematical reason in teaching and learning mathematics. After all, zero is even because the definition of an even number is an integer multiple of two, and zero is 0 times 2. But even if eight-year-olds *have* a definition, it does not come in this form: In fact they likely do not know either of the terms *integer* or *multiple*. Adults know that the solution to 0 – 9 is (–9). But eight-year-olds inhabit a mathematical world comprised solely of positive whole numbers. In this world, is 0 – 9 impossible? Or is the answer zero?

Adding to the challenge for the elementary-grades teacher is not just answering reasonably but figuring out what is required for children to reason about mathematics in school (Ball & Bass, 2003; Lampert, 1990, 2001; O'Connor, 2002; Sfard, Nesher, Streefland, Cobb, & Mason 1998; Yackel & Cobb, 1996). Facing this challenge requires consideration of what mathematics is, offers, and requires. What constitutes mathematical reasoning? What even demands a reason? Facing the challenge also entails respect for the fact that the people involved are *children,* with mathematical knowledge that is both underdeveloped and under development. This reality raises the question, What version of mathematical reasoning is appropriate for children? Furthermore, this reasoning occurs *in school,* where norms exist for how students and teachers relate, and about what, and with what purposes and constraints. So in part this challenge demands consideration of mathematics, of students, and of school as a setting for learning mathematics. Additionally, and often less visibly, it also involves *teaching*.

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[4] This class period has been written about elsewhere (e.g., Ball, 1993).

[5] Quotations of classroom talk from the episode under study are referenced throughout this chapter to the transcript found in Appendix 1 to this volume.

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In our research, we seek to uncover and make visible what the *work of teaching* mathematics is; what makes the teaching, itself, *mathematical;* and what teachers do to *make* their work. To ground our inquiry, we analyze data from elementary school classroom teaching of mathematics. A primary resource for us is a large base of "records of practice" collected in Ball's third-grade public school classroom during the 1989–1990 school year.[6] A second source of material consists of analogous, albeit less extensive, materials from other teachers' classrooms and records from subsequent years of Ball's teaching.

The title of this chapter, "Making Mathematics Work in School," frames a central problem of our research. Using perspectives drawn from multiple disciplines—psychology, organizational theory, philosophy, linguistics, anthropology, sociology, and mathematics itself—we seek to uncover and analyze the work that teachers do. In particular, we do so with an eye to a fundamental question about how to treat the discipline of mathematics with integrity in the context of helping pupils learn. Following Dewey (1902/1990), Bruner (1960), and Schwab (1961/1974), we ask the meaning of being "intellectually honest" about the subject matter, of representing within the curriculum "fragments of the discipline," and of engaging pupils in fundamental elements of knowing and doing mathematics in school. As we study the work of teaching, we seek to understand its demands and the resources useful to its practice.

The words of our title evoke aspects of what teachers *do* in several senses. In one sense, the title refers to what teachers give students to *do* so that they will learn mathematics. Teachers define and *make the mathematical work* in which students engage: tasks, activities, and questions (Doyle, 1983). We investigate what that "work" is: What do teachers ask students to do that we would call the "mathematics work" of a lesson or a sequence of lessons? Certainly it is more than the assigned tasks (Arendt, 2000). Surely it includes the time, the materials, and what is done with them. It also includes the *talk* that carries and surrounds the tasks. That talk, and the work that constitutes it, is the focus of this chapter.

In a second sense, our title calls attention to the teacher's challenge of making it possible to *do mathematics in school.* We study what is required to maintain the integrity of the mathematics that students do in school. School mathematics is often remote from what counts as mathematics in the discipline. Its scope distorts the structures of the discipline, representing ideas as rules, problems as mere exercises, and solutions as fact-based answers (Goodlad, 1984; Lampert, 1990; Schwab, 1961/1978; Stodolsky, 1985). In this second interpretation, then, we explore what it might mean for mathematics to "work" in school. We consider what distinguishes a distortion of the discipline from a properly scaled, "mathematically honest" version, appropriate for 6-year-olds or for 10-year-olds.

[6] These records were collected in Ball's third-grade mathematics class and Magdalene Lampert's fifth grade, with support from the National Science Foundation, for a project in which we set out to investigate the potential of using new technologies together with extensive records of practice to design new approaches to the pedagogy and curriculum of teacher education. For a discussion of this project and its results, see Lampert & Ball (1998).

In a third sense, the title points to teachers' putting mathematics to work so as to do the work of teaching. Mathematics can be thought of as a resource for teaching (Ball, 1993, 2000; Ball & Bass, 2000a; Lampert, 1990, 2001). As teachers work—to motivate students, to make sense of what students say, to provide a sense of closure for a lesson—mathematics offers ideas, language, and moves for carrying out that work. For example, asking a good mathematical question at the right moment can pique students' interest and draw them into the work. Knowing the "right" mathematics can help supply that question. In this interpretation of the title we ask, What does mathematics afford as a tool for carrying out the work of teaching? What mathematics is useful to know? And where in teaching is that mathematics likely to be helpful?

Finally, our title calls attention to teachers' core responsibility to ensure that what they do "works." No well-designed mathematical task or activity, nor amount of careful attention to the integrity of the mathematics, alone constitutes effective instruction. Rather, teaching works only if students develop mathematical proficiency, if they learn and grow, and if they become capable of doing and using mathematics and disposed to do so with confidence (Kilpatrick, Swafford, & Findell, 2001).

This chapter represents one part of our larger study of mathematics teaching. Like the companion chapters in this volume, we focus on this one day of mathematics class—in particular the short 6-minute segment from the beginning of the lesson. Across these 6 minutes, the teacher asks students to talk about their experiences of the meeting with the fourth graders. Although students might have chosen to talk about the dynamics, or what it was like to be with the older pupils, mathematics dominates their discussion. Students express ideas, offer mathematical explanations, and generate useful examples. They engage with one another, listen thoughtfully, and respond with interest. They agree and disagree. Observers of this episode often note students' serious engagement—with mathematics and with one another—and wonder what is required to produce it.

In this segment our attention was drawn to *talk:* to how the teacher's and students' work can be understood through a close examination of their talk—an examination that keeps its eye on both the discipline of mathematics and the practice of teaching. Although students do much of the talking during this segment, we sought in our analysis to uncover the teaching that structures students' mathematical talk. Our study of that talk led us to identify three aspects of the mathematical work in which students are engaged through that talk, and we used those three aspects to probe the work of the teacher. These three aspects are *naming and using names, making and interpreting claims,* and *evaluating mathematical assertions.* We show that they are, at once, both mathematical practices and teaching practices.

TALK IN MATHEMATICS CLASS

A 6-minute segment of classroom interaction can be studied in myriad ways. For example, one could study the written work done by individual students during

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a class segment, or pay close attention to the chalkboard work produced by the teacher and students for public record. Alternatively, one could obtain the curriculum and lesson plans for this particular day to analyze where the lesson conformed to, or deviated from, the intended plans. One could record which students participate, and document students' physical signs of engagement, making note of the socioeconomic status of each student. In this article, however, we turn our attention to the *talk* during this 6-minute segment while keeping our eye on the discipline of mathematics and the practice of teaching.

We made this analytic choice for several reasons. One simple reason is that talk pervades this particular slice of the day's lesson. Had we studied a different 6-minute segment from the same day's lesson, we might have chosen instead to analyze the individual written solutions children had produced when working at their desks. A second reason for our choice is that talk is thought to be a primary vehicle for student learning. Vygotsky's work (1978, 1986) and that of others (cf. Bakhtin, 1981, 1986; Wertsch, 1998) have emphasized the prominent role that speech plays in learning. A third reason grows directly from our own larger research agenda: Talk is a primary medium of teaching (Cazden, 1986). As one of teaching's major "technologies," talk is important to study. Particularly worth examining is teachers' own mathematical talk and how they elicit mathematical talk from their students. Significant analyses of classroom discourse (e.g., Au, 1980; Cazden, 1991) offer portraits of the discursive action and interaction of lessons. Still, less is known about the workspace in which teachers can deliberately cultivate and mediate classroom talk. A fourth reason for our interest in talk is our interest in equity. The facility with everyday talk that children bring to the classroom from home can be a platform from which to build mathematical talk. Perhaps more than other resources needed for mathematical work, children bring knowledge and experience of talk to the classroom. Elsewhere (Ball, Bass, Hoover, Lewis, & Wall, 2003), we have shown that the teacher's skillful design and ordering of questions can engage *all* children in the mathematical work of the class. With careful attention to the discontinuities as well as the connections between everyday talk and mathematical talk, teachers can use children's facility with everyday talk as a springboard to mathematical talk. Thus, although beyond the scope of this chapter, our decision to focus on classroom talk in our research is driven in part by our conjecture that although language can be a barrier to access, talk—used prudently—can also make mathematical work accessible to all children.

Our work joins a growing corpus of research on talk in elementary mathematics classrooms (see, e.g., Cobb & Bauersfeld, 1995; Kieran, Forman, & Sfard, 2002; Lampert & Blunk, 1998; Sfard, et al., 1998). For example, Yackel and Cobb (1996) consider how "sociomathematical norms" shape opportunities to learn mathematics for individual learners. They contend that "individuals' reasoning and sense-making processes cannot be separated from their participation in the interactive constitution of taken-as-shared mathematical meanings" (p. 460). Thus, their focus is on the extent to which conversation enables learning for the individual student. O'Connor (2002) begins instead with the mathematical content that one teacher is likely to encounter, and from that base considers what might be needed for the teacher to plan, carry out, and review a "position-driven discussion" that will help students learn the target content. For O'Connor, the primary focus is the work of the teacher in orchestrating conversation among students that will help them learn specific mathematical content.

Also concerned with talk, Nesher takes a different view of its role (in Sfard et al., 1998). She distinguishes among mathematical competencies engendered by conversation in mathematics class: are those conversations *about* mathematics, or are they *talking mathematics?* For Nesher, *talking mathematically* means describing "the world's situations with the formal models of mathematics" (p. 43). Similarly, Sfard (2001) argues that mathematical thinking *is* communication, so mathematics *is* mathematical talk. Nesher, Sfard, and others distinguish natural language about mathematical ideas from actual mathematical talk that employs the symbols and language specific to the discipline. Their work calls attention to the sense in which talking mathematics is central to *knowing,* not just learning, mathematics.

Common to many of these analyses of classroom talk is attention to the social dimension of conversations in mathematics classrooms. Weingrad (1998), for example, uses politeness theory (Brown and Levinson, 1987) to analyze a fifth-grade mathematics lesson. She argues that students are engaged in a kind of social positioning and maneuvering that affects the academic work in which they are thought to be engaged. Many researchers see classroom conversations about mathematics as socially risky, in which children can, in Goffman's (1955) terms, "lose face." From this perspective, a significant problem is the dynamics of the social relations that underlie classroom transactions (Lampert, Rittenhouse, & Crumbaugh, 1996).

Our investigations have led us to appreciate the sense in which social and intellectual purposes can be productively intertwined in classroom talk. Drawing on Lampert's work (1990), we argue that the terms of participation can be reappropriated to "produce lessons in which public school students would exhibit—in the classroom—the qualities of mind and morality that Lakatos and Pólya associate with doing mathematics" (p. 33). Because classrooms are group settings, we see also the extent to which the collective nature of mathematical practice depends on and uses talk as a primary medium of mathematical work. Through talk, mathematical ideas are aired and examined, and then ratified, revised, or discarded. In sum, talk provides the connective resources for making mathematics work in the collective settings of school classrooms.

ANALYTIC FRAMING

To probe how classroom mathematical talk shapes the work of teaching, we began by examining the students' and teacher's talk across this 6-minute segment. Three questions framed our initial analysis:

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1. What features stand out about the teacher's and students' talk? What do they talk about, with what tools, and to whom, and for what purposes?
2. What are the social and mathematical features of their talk?
3. What is the role of the teacher in prompting, supporting, and teaching students' mathematical talk?

We viewed and reviewed the video segment, and coded the transcript. We observed, in slightly over 6 minutes of videotape, five distinct chunks of student talk. Each chunk is punctuated by the teacher asking the students for comments about the meeting of the day before, and each request by the teacher is followed by a student-initiated comment that precipitates an exchange among two or more classmates (see Figure 1.2).

Turn number	Length of turn (in minutes)	Brief description
1	0:51	Teacher asks for comments about the meeting.
2–8	0:47	Sheena comments on the meeting.
9	0:02	Teacher returns to ask for comments about meeting.
10–20a	0:51	Sheena and Sean exchange ideas about whether zero is even.
20b	0:17	Teacher asks for comments about the meeting.
21–23	0:23	Mei comments on how the meeting changed her thinking.
24	0:03	Teacher asks for comments about the meeting.
25–58	2:26	Nathan describes how and why he changed his mind about zero's being even, and the class reacts.
59	0:08	Teacher asks for comments about the meeting.
60–66	0:52	Sean says that he thinks six is neither even nor odd.

Figure 1.2. Summaries of chunks of student talk in 6-minute segment from mathematics lesson.

Displaying the segment in this way, and examining each chunk of talk, led us to some observations. The teacher's purpose at the beginning of this class was to give students a brief opportunity to reflect on the meeting. We observed that students seemed to understand the task, and they responded to it, for the most part, by raising important mathematical points that had attracted their attention and that they were still pondering. Central among those points were the nature of the number zero, and, in particular, its status as even or odd. We noted that students who spoke were engaged with their classmates, listening to their comments and questions, and responding. We saw that the teacher spoke relatively less than the students, and yet she seemed to be substantially shaping their talk. The students' talk did not seem "natural"—that is, it did not seem like talk probable in a typical

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third-grade classroom. We wondered what lay beneath our observations. If such talk was indeed not automatic, then we suspected that it was being deliberately cultivated. We sought to uncover evidence of the teaching and learning that might be shaping the students' mathematical talk—and, hence, work.

Our further analysis led us to posit three essential elements that undergird the nature of the mathematical talk in the segment. One is the available lexicon for mathematical talk: what words and phrases are being used, and to name and label what things. We refer to this element as "naming." A second is the students' evident orientation to making and understanding mathematical assertions. Third are methods for evaluating such assertions—that is, for judging assertions to be true or untrue and for revising or discarding them. We turn next to examine each of those elements, exposing its role in the segment. Following this discussion, we consider the role played by the teacher in these three arenas and its impact on the cultivation of useful mathematical talk.

NAMING AND USING NAMES IN CLASSROOM TALK

Names shape the discursive space of classroom talk. Names specify what is being talked about, what ideas or processes are important, and who is to do what. When Sean introduces his observation about six to the rest of the class, he needs a way to name what he has noticed, and so he says that six can be "even and odd." Although his observation is not, strictly speaking, the conventional use of the terms *even* and *odd*, we can see in his talk the important need to name ideas. And, later, the coining of the term "Sean numbers" facilitates the students' consideration of properties of even numbers, odd numbers, and Sean numbers.

Observers of this third-grade class often note the vocabulary employed by both the teacher and students. They are surprised, for some common terms in use seem far from ordinary for young pupils. The students regularly inquire whether two *representations* are the *same,* make *conjectures* and *arguments,* request *proof, comment* and *disagree, confer* with classmates, and *revise* their thinking. Why do these students, over half of whom are learning to speak English as a second language, use such terms? What role does this lexicon play in their mathematics work?

Consider the opening of the segment in which Ball frames the task for the pupils (turn 1G):

Ball: I'd like you to be thinking back to yesterday and to the meeting that we had on even and odd numbers and zero. And I have a few questions. First—my first question is, I'd just like to hear some comments about what you thought about the meeting, what you noticed about the meeting, what you learned at the meeting, just what kinds of comments you have about yesterday's meeting? And could you listen to one another's comments so that we can um, benefit from what other people say? See what you—what you think about other people's comments?

This single turn is replete with terms that signal the objects of attention, the tools to be used, and the ways to work. One lexical set—*even and odd numbers and zero*—is explicitly mathematical. These three terms are used to identify the

mathematical ideas under discussion. Additionally important for the pupils' focus is a *meeting* at which these mathematical objects were discussed. The term "meeting" identifies a forum in which many pupils came together to discuss even and odd numbers and zero. Other words specify what students are to do: Their assignment is to make *comments* and to *listen* to others' comments. "Comments" labels a kind of discursive turn in which observations or reflections may be shared. Making a "comment" is not to give an answer. Rather, it is to contribute a view, a perspective, or a reaction, for others' consideration—in this example, what was "noticed" or perhaps "learned." Hence, asking pupils for "comments" is to point to a particular sort of work. Similarly, they are to "listen" to others' comments. Wall's (2003) analyses of "listening" in these data show that student listening is an explicitly developed practice with perceptual, behavioral, anticipatory, and conceptual dimensions. He traces, from the first classes, how *listening* to one another's ideas was guided and encouraged. Wall's analysis shows that although listening involved courtesy, it was an intellectual—not merely social—practice.

A final interesting term in this single teacher turn is the word *benefit* (turn 1F):

Ball: Could you listen to one another's comments, so that we can um, *benefit* from what other people say?"

This term is not likely familiar to eight-year-olds, particularly among whom such a high proportion are just learning English. It sounds "grown-up" and important and, in context, suggests important reasons for listening to others' ideas. Whereas the others seem to be well-defined terms, *benefit* seems less so. Instead, the teacher's use of this unfamiliar word may suggest something ineffable—a sense of respect and seriousness about the work in which the students are engaged.

Figure 1.3 displays the terms used in this single teacher turn, classifying them by the aspect of mathematical work that they name.

	Mathematical content	Mathematical practice	Learning activity	Attitude, stance
Even numbers				
Odd numbers				
Zero				
Meeting				
Comments				
Listen				
Benefit				

Figure 1.3. Aspects of terms used in Ball's beginning-of-class turn.[7]

7 By "mathematical content" we mean ways of doing that generate or work on mathematics. By "learning activity" we mean an instructional activity in this classroom that may or may not be mathematical in nature. By "attitude, stance" we mean a feeling or posture in the classroom.

Although all these terms were from one turn of the teacher's talk, similar patterns can be found in the students' talk. They use terms not likely to appear spontaneously in eight-year-olds' talk. And those terms shape the mathematical work they appear to be doing as reflected in what they say in class. Consider one of Mei's turns in this segment (turns 21–23):

Mei: Um, I h— I thought that zero was always going to be a even number, but from the meeting I sort of got mixed up because I heard other ideas I agree with and now I don't know which one I should agree with.

Ball: Um-hm. So what are you going to do about that?

Mei: Um, I'm going to listen more to the discussion and find out.

Figure 1.4 displays Mei's use of terms, which mirror her teacher's although she also uses terms not used by her teacher: *agree* and *discussion*.

	Mathematical content	Mathematical practice	Learning activity	Attitude, stance
Even numbers				
Odd numbers				
Zero				
Meeting				
Comments				
Listen				
Benefit				
Agree				
Discussion				

Figure 1.4. Aspects of Mei's use of terms in one turn sequence.

What is Mei doing, as reflected in her "comment"? First, she seems to understand what is called for in making a *comment* in this class. She reflects on a new confusion she has experienced as a result of what she heard in the meeting. Second, her comment also reveals what she was doing the day before in class: Her confusion grows out of the work she did during the *meeting,* in which she apparently *listened* to other' ideas about the parity of zero, and those ideas challenged her own prior notions. Third, her comment contains evidence that she knows that *agreeing* is a mathematical practice, something for which evidence is needed, and she is aware that she can gain evidence by *listening* to a *discussion*. For Mei, a turn labeled "comment" provides her a space for reflecting on and articulating what she did and thought about zero. And throughout her "comment," Mei refers to significant objects and activities as she talks about her mathematical work. The names—*listening, agreeing, meeting*—offer her labels for those objects and activities.

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Whereas Mei's use of terms reflects her teacher's, Ball also helps Mei make her comment at mid-turn, asking in relation to her fresh confusion about zero, "So what are you going to do about that?" This addition extends observation and reflection to a natural consideration of implications for the mathematical work at hand. In doing so, the teacher scaffolds other students' work in making comments as well, helping them fill out what they have to say. One striking example of supporting students in giving comments is in the first turn of the segment, turns 2–7, when Sheena is the first to respond to the teacher's request for comments:

Sheena: I—I—I liked it because, well, I like talking to other classes and, and when you talk to other classes sometimes it helps.

She stops, but Ball probes, "In what way?" and Sheena continues, "It helps you to understand a little bit more." Ball probes again, "Was there an example of something yesterday that you understood a little bit more during the meeting?"

Sheena: Well, I didn't think that zero was—zero, um—even or odd until yesterday they said that it could be even because of the ones on each side is odd, so that couldn't be odd. So that helped me understand it.

The teacher underlines and labels Sheena's comment:

Ball: Hmm. So y— so you thought about something that came up in the meeting that you hadn't thought about before? Okay.

Sheena: (*nods*)

In this sequence, the teacher seems to be helping Sheena understand and perform the work of "making a comment." Ball asks her to fill in more detail by asking for an "example" and points her toward reflecting on what she might have learned. Her interaction with Sheena provides a scaffold for what counts as a "comment" and what sort of work might be done to produce one, out of the child's experience of the meeting. In this exchange, the teacher's use of terms (*example, meeting, understand*) helps to show the sort of work involved in "making a comment." [8]

In addition to the work of making *comments,* which is explicitly named and taught by the teacher, the students' talk reflects their active engagement in figuring out how to think about whether zero is even or odd. They listen to ideas, make assertions, and assess arguments to determine whether they *agree* or *disagree* with them. And as they do so, the teacher labels what they are doing with terms designed to provide linguistic structures to support their work.

One example of this active naming occurs when Sheena shares that she used to think that zero was neither even nor odd until the meeting, but when she listened to another student's explanation using the number line, she thought differently: "It helped me understand it." The teacher labels what Sheena did (turn 20E):

Ball: Sheena commented that it was good to have the two classes together because she heard an idea that she hadn't thought about and it made her think about and even *revise* her own idea when she was in the meeting yesterday.

[8] This kind of teaching work, in which a teacher sensitively scaffolds students' participation in the doing, talking, and learning of the discipline, is typically more familiar to teachers when teaching reading and writing, but it applies as well in teaching mathematics.

Offering the term *revise* as a label for changing one's mind on the basis of new evidence or insight accords significance to this important sort of thinking. With a name to label it, the practice is made visible and is accorded value.

Lampert (2001) argues that names accord importance and that they make particular ideas, skills, and habits usable by labeling them. In her analysis of a year of fifth-grade teaching, she identifies three mathematical practices among those worth naming and teaching. She describes how she sought to teach these practices to her fifth graders, and how she also named them to give them special status in the public discourse:

- finding and articulating the "conditions" or assumptions in problem situations that must be taken into account in making a judgment about whether a solution strategy is appropriate;
- producing "conjectures" about elements of the problem situation, including the solution, which would then be subject to reasoned argument; and
- revising conjectures on the basis of mathematical evidence and the identification of conditions.

These activities were important to teach *deliberately*. They represent the essence of mathematical activity in a way that makes it doable by 10-year-olds (Lampert, 2001, p. 66).

Lampert's book offers close detail of the ways in which she worked to establish a classroom culture that would support mathematical reasoning. The mathematical practices shaped the ways in which she formulated the tasks she gave to students, and she also used them to name and draw attention to important moves students made as they began working. They structured the mathematical work in which she engaged her students.

In the next sections of this chapter, we examine more closely what the students and teacher do as they talk to make, evaluate, and decide on the truth of mathematical claims.

MAKING AND INTERPRETING CLAIMS

In mathematics, the making of claims is central. In the well-known episode with which we began this article, Sean made a claim about the number six. Provisional claims are, in essence, the raw material for building justified mathematical knowledge. Once proposed, claims are revised, refined, justified, or refuted. We see the third graders engaged in those activities with the guidance of their teacher. And justified claims, or theorems, are the primary products of mathematical work. Hence, mathematical talk in classrooms should involve the making and reworking of claims. This lens offers another perspective from which to examine the interactions in our 6-minute classroom segment.

In this segment, 34 mathematical claims are made, 30 by the pupils and 4 by the teacher. For example, Sheena asserts that, on the basis of what she heard at the meeting, zero is an even number. Sean claims that it cannot be so. Later,

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Nathan claims that zero is "special." Another twenty-some claims are made about mathematics, roughly half by the teacher and half by the students. For example, the teacher says the fourth graders have been thinking about zero for a long time and still disagree, suggesting that disagreement is to be expected and that some mathematical issues are not easy matters and require thoughtful, long-term consideration. And Mei says she is going deal with her confusion by listening more, suggesting that mathematics is something one figures out by listening and thinking; it is an ongoing endeavor. Implicit in those claims is an image of what doing mathematics means.

Evident in the pupils' talk is recognition that both making mathematical claims and making claims *about* mathematics are central mathematical activities. When Sheena reports on her experience of the meeting, she recounts her reactions to claims that were made. She explains, "Well, I didn't think that zero was—zero, um—even or odd until yesterday they said that it could be even because of the ones on each side is odd, so that couldn't be odd. So that helped me understand it." Sheena recalls that before the meeting, she thought that zero was neither even nor odd, but that during the meeting, she heard an argument that caused her to change her mind. Revised, her new claim is that zero is even—or at least that it is not odd. What Sheena attended to at the meeting reveals her sense of the importance of formulating statements about mathematical objects, processes, and relationships.

Sean is similarly attentive to the role played by mathematical claims. He hears in Sheena's comment a claim that he doubts. To ascertain what she is saying, he suggests that he disagrees, and speaking directly to her, poses a question to probe what she has claimed: "What two things could make it?" Sheena is not sure she understands his question, and turning around to look at him, asks graciously, "Could you repeat what you said, please?" These two pupils' interactions in turns 10–19 are directed toward sifting through the various claims on the table. In putting out a mathematical claim, Sheena makes a mathematical move, creating space for further mathematical work. On this occasion she may not have meant to initiate a line of work, but her claim has that effect anyway. Sean picks it up and works with it. This process of making claims and responding to them lies at the heart of mathematical work.

In particular, across the 6-minute segment, we see multiple examples of children generating mathematical claims. Typically, in school mathematics, teachers and textbooks make claims and children try to remember them (Stodolsky, 1985). As Lampert. Rittenhouse, and Crumbaugh (1996) argue, for children to learn to formulate mathematical claims in school, the terms and structure of discourse will need to be reconfigured. And beyond that, a sense of what counts as mathematical work in school will need to be reconfigured as well. The students in this classroom have learned something about *what counts as a mathematical claim,* about *how to express mathematical claims so they are usable by others,* and about *how to evaluate and respond to mathematical claims.* Together, those activities do, indeed, reconfigure the mathematical work of mathematics class.

But what is involved in making and evaluating mathematical claims? Claims are part of everyday life. We make claims about the world around us, about how

we feel, and about other people and the things they do. Claims represent our appraisals of the world. However, making *mathematical* claims in the classroom is often new territory for learners and teachers. Making mathematical claims requires developing a mathematical sensibility or mathematical eye. In many ways, mathematical claim-making launches mathematical work, but to launch productive mathematical work and productive learning, students and teachers need to understand what counts as a mathematical claim, what claims are worth making, and how, in a group setting, to express and respond to claims respectfully.

Of the 34 mathematical claims made in this 6-minute segment, roughly a third are claims about zero, a third are about making and combining even numbers, and a third are about the number six. For example, Figure 1.5 lists the claims proffered about zero.

Mathematical Claims About Zero
Zero is not even or odd.
Zero could be even.
Zero is not odd.
Zero has to be an even.
Zero is not an even number.
Zero is always going to be an even number.
Zero is not always going to be an even number.
Zero is even.
Zero is special.

Figure 1.5. Mathematical claims about zero.

This list highlights two central features of mathematical claims—the importance of precise language and the need to carefully clarify the meaning of claims. More than in other disciplines, mathematics demands precision. Precision is one of its hallmarks. Reading through the list of claims about zero, asking what is meant by each of those different claims, one can see why that precision is important. The differences are subtle, yet significant. "Is," "could be," "has to be," "is always going to be," "is not always going to be"—each of those expressions means different things. Each involves different lines of argument. When Sheena says, "It could be even because of the ones on each side is odd, so that couldn't be odd," what is she claiming? Is she saying zero is even, or that it is not odd, or does she see those statements as the same? And what is the role of the "could" and "couldn't" in her statement? To argue that zero *could* be even, one might focus on consistency (i.e., considering zero to be even fits with other agreements already made). To argue that zero *is* even, one might return to the definition for even numbers. Quantification and negation play prominent roles in formal

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mathematical logic as well as in less formal mathematical reasoning. And, as we see in these exchanges, the students are exploring this important mathematical terrain.

In part, the apparent breakdown in the exchange between Sheena and Sean stems from having reasoned unwittingly from different definitions, but the difference between "could be" and "has to be" also confounds their exchange. Their confusion may result from different interpretations of mathematical quantification, from competing conceptions of mathematical truth, or from their tentative sense of conviction. Whatever the reason, careful attention to the exact wording and meaning of the claim opens up an important line of mathematical work. As the pupils attempt to make and defend the claim, important tacit assumptions emerge. Some students think that even and odd are mutually exclusive categories, for instance, yet this idea is not shared by all.

With these claims about zero, we see that precision shapes and is shaped by reasoning. Precision matters, even in third grade. In turn, the need for precision heightens the need to clarify claims. And we see that as the students in this class make claims, they seem to engage in a process of clarifying those claims, whether on their own or with the support of the teacher. The major chunks of student talk in this 6-minute segment are, in large part, occasions of making and clarifying claims. Sean asks Sheena for clarification. Ball, along with other students, asks Nathan for clarification. And several members of the class ask Sean for clarification. Much of the mathematical work we see going on in the segment (a kind of work that the teacher scaffolds from the first day of class) is about clarifying, testing, and revising (further clarifying) the claims being made.

This dynamic is particularly visible in the next group of claims, in which Nathan says he thinks zero is "special" because "even numbers, like they they *make* even numbers" (see Figure 1.6). He claims that zero is "special," but he also claims something about the class of even numbers rather than about a specific number. The teacher articulates two possibilities for what he might be saying, "Were you saying that when you put even numbers together, you get another even number, or were you saying that all even numbers are made up of even numbers?" Betsy and other classmates continue the process of asking for clarification, and the variations they generate begin to clarify what Nathan is claiming.

In this progression of claims we see a process of claim-making and claim-clarifying, with proposed rewordings and revisions, with examples and counter-examples, with students' taking on Nathan's idea enough to put it in their own words and give it respectful audience. Claims like these, about classes of numbers, create their own challenges, and examples often play a large role. Nathan constantly turns to examples to indicate what he means, "like two plus two is four and four plus four is eight." Giving examples seems to be something these students have learned to do. Betsy counters Nathan with an example of her own, "You can't with six." These examples help clarify claims and help make them precise. By the end of the exchange, Betsy summarizes Nathan's thinking and Nathan agrees, "Yeah, I'm not going by every single number."

Mathematical Claims About Combining and Making Even Numbers
Even numbers make even numbers.
When you put even numbers together, you get another even number.
All even numbers are made up of even numbers.
Two even numbers, just that same.
All even numbers can be made up of two identical even numbers.
You can't with six.
Six is two odd numbers to make an even number.
You need three twos to make six.
Three is odd.
I'm just talking about, like two plus two is four, and four plus four is eight, and I just skipped six, so I just added the ones that, that add.
Six can be an odd number.
What you're doing is you're going by twos and then what two equals from then you go from—all the way up.
Yeah, I'm not going by every single number.

Figure 1.6. Mathematical claims about combining and making even numbers.

The last group of claims evident in this 6-minute segment, about the number six, returns to considering a specific number but then uses this example to define a class of numbers of which six is merely one. In part, these claims form a natural outgrowth of the work done across the first two sets of mathematical claims. The students' consideration of how six might be thought of as both even and odd engages them beyond this segment, eventually leading them to define "Sean numbers," those that consist of an odd number of groups of two (2, 6, 10, 14, …).

In addition to making mathematical claims, the students and teacher in this segment make numerous claims *about* mathematics, as in Figure 1.7.

These claims reveal another layer of mathematical work. Making a mathematical claim is to state something about mathematical content. We note that the claims in Figure 1.6 are about odd and even numbers, the composition of numbers, and the methods for deriving conclusions about the status of different numbers. In contrast, the claims in Figure 1.7 are reflections on the process of working in mathematics, the characteristics of mathematics as a field, and the nature of experience in mathematics class. Making a claim about mathematics is to reflect on the practice of doing mathematical work, or the nature of the discipline of mathematics. Implicitly and explicitly, students and the teacher comment on mathematics and what it means to do it. When the teacher begins mathematics class by saying, "There's nothing to take notes about" and asks the students to think back to the meeting, she implicitly claims that a discussion at a meeting is legitimate mathematical activity and that reflecting on it and commenting on it are, as well. She communicates that usual

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Explicit and Implicit Claims *About* Mathematics and the Doing of Mathematics
When you talk to others, it helps you understand a little more.
So you thought about something that came up in the meeting that you hadn't thought about before.
I disagree that zero has to be an even number.
Could you repeat what you said, please?
I disagree because if it was an even number, what two things can make it?
I could show you it.
That doesn't mean that it always is even.
You said it was [even].
This is still a problem—we didn't settle it—there's still a lot of disagreement.
The fourth graders have been thinking about this a long time and still disagree about it.
Sheena commented that it was good to have the two classes together because she heard an idea that she hadn't thought about and it made her think about and even revise her own idea when she was in the meeting yesterday.
What other comments do other people have about the meeting and what happened yesterday?
I thought that zero was always going to be a even number, but from the meeting I sort of got mixed up because I heard other ideas I agree with and now I don't know which one I should agree with.
I'm going to listen more to the discussion and find out.
First I said that, um, zero was even, but then I guess I revised so that zero, I think, is special.
You said something like that yesterday.
I disagree.
So what you're doing is you're going by twos, and then what equals from then you go from—all the way up.
Um, I don't have anything about the meeting yesterday, but I was just thinking about six.
That doesn't necessarily mean that six is odd.
We need in the group to have an idea that we're working with.

Figure 1.7. Claims *about* mathematics and the doing of mathematics.

mathematical activity in this class includes taking notes and that thinking is an important, ongoing part of doing and learning mathematics.

Several of the claims about mathematics in Figure 1.7 characterize mathematics as an exchange and a testing of ideas. For example, Sheena claims that talking to others "helps you understand a little more." In contrast, Mei points out

that talking to others can also confuse: "I thought that zero was always going to be a even number, but from the meeting I sort of got mixed up." When the teacher asks her what she is going to do about becoming confused, she goes on to say, "I'm going to listen more to the discussion and find out." On the whole, students' comments imply that public testing of ideas is crucial to doing and learning mathematics.

What public accord means becomes more evident across the segment. At the outset, the teacher highlights the importance of hearing different comments from different people. She tells students to "listen to one another's comments" and to "think about other people's comments" so that they can "benefit from what other people say." Several times, she asks, "What other comments do other people have?" She expects different students to have different things to say and that those differences will contribute to the mathematical work at hand. The classroom segment gives evidence that students, too, value listening, thinking, and revising. Sheena, Mei, and Nathan explain how the meeting with the fourth graders was as an opportunity to hear and consider different ideas. In discussing the meeting, Nathan describes how he came to revise his ideas, "First I said that um, zero was even, but then I guess I revised so that zero, I think, is special."

On several occasions, students challenge one another's proposed claims. Doing so grows out of their listening and thinking. Such challenges often suggest the need for revision. And from their listening and thinking, students begin to "try on" other students' ideas and to express them in their own words. For instance, Betsy works hard to understand Nathan, raising challenges but also listening for what he is struggling to explain, "So what you're doing is you're going by twos…."

Across the students' claims about mathematics, we see the significant role mathematical reasoning plays in publicizing and testing of mathematical claims. For example, decisions are made on the basis of agreed-on definitions (as when Sean disagrees with Sheena's claim that zero is even, asking, "what two things could make it?" at turn 10); mathematical arguments are to be given in support of claims and to be interpreted by others (as when Sheena says she could "show you it," turn 14); and disagreement is to be expected and thoughtfully considered (as when the teacher explains that the class probably will not settle the issue and that "the fourth graders who have been thinking about this for a long time also disagree about it," turn 20).

These students are making claims about mathematics and how it is conducted. The point here is not so much whether their claims are correct but that they are engaged in discussing and commenting *about* mathematics. And they often do so by making claims about mathematics. Across the year, their ideas change and grow, but their claims about mathematics represent their maturing sense of what it means to do mathematics. Such claims reflect a significant component of the work this teacher does to engage her students mathematically.

The ability to make mathematical claims does not come into being magically. As we see in this segment, the teacher scaffolds the children's formulation of claims. The day before this one, the third graders and the fourth graders had had

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a meeting in which the topics of discussion were the number zero and even and odd numbers. The third-grade pupils observed older students making and evaluating mathematical claims, and this process constituted much of the mathematical "work" for that day. Part of the teacher's role in helping children formulate mathematical claims is to provide material that provokes thought. The meeting with the fourth graders offered such opportunities for thought.

In the segment we are studying, the teacher (a) provides mathematically engaging experiences for them to make claims about; (b) invites their claims about those experiences; (c) prompts them for sufficient elaboration of claims so that all can "dig into" them; (d) ensures that they hear one another's discussion of claims; and (e) solicits other students' reactions to the original claim. For example, in each of the four chunks of the segment, the teacher asks for comments. Each time, she queries the claims students make, even if briefly, first with Sheena, then with Mei, Nathan, and Sean. (And several times, students follow up with their own queries: first Sean probing Sheena, then several students probing Nathan, and finally, several other students probing Sean.) These clarifications allow students to "dig into" contributions and subsequently to size them up. When Sheena comments about the meeting, the teacher asks her to elaborate (turns 3–5): "In what way? … Was there an example of something yesterday that you understood a little bit more during the meeting?" These probes draw out what Sheena has in mind and make her thinking understandable and usable by others.

The teacher also places importance on children's hearing one another, an important element in constructing the work of making, elaborating, and refining claims collectively. At several junctures, Ball stops to make sure students can hear and follow the conversation. For instance across turns 30–36, she questions Betsy about whether Betsy had "said something like that yesterday, too." When the fact that Betsy had not been listening becomes clear, the teacher does not reprimand her but asks matter-of-factly, "Were you not listening to this just now?" and she then summarizes Nathan's contribution so Betsy can respond.[9]

This talk *about* mathematics is a prominent part of the mathematical work going on in this class. And although it often lies beneath the surface, it is pervasive and frequently comes fully into view. Together, students' mathematical claims and their claims about mathematics provide the raw material for mathematical work.

Next we examine how students learn to evaluate the claims that they hear. Doing so, in turn, makes mathematical work productive.

Evaluating Mathematical Claims

In the previous section, we saw multiple instances in which the teacher not only solicited students' comments but also solicited students' reactions to a claim. For example, when she opens the class, Ball ends her comments with the request "And could you listen to one another's comments, so that we can um, benefit from

[9] For a thorough examination of the role of listening and hearing, and the teacher's role in promoting listening, see Wall (2003).

what other people say? See what you think about other people's comments?" After Sheena makes a comment, one that Ball treats as a claim, she asks for other comments from other children in turns 9–10:

Ball: Other people's comments? Sean?

Sean: Um, I—I—I just want to say something to Sheena, when sh— what she said about um that, that one, um—zero has to be an odd, an even number bec— I disagree because, um, because what what two things can you put together to make it?

Notice that Sean responds to the teacher's request for more comments by responding to Sheena's claim, taking up, in a sense, the teacher's original invitation to think about, and comment on, other people's comments and claims. Later in the segment, at turns 26–29, Ball also models this type of response in her own engagement with Nathan's claim:

Ball: Can I ask you a question about what you just said? And then I'll ask people for more comments about the meeting. Were you saying that when you put even numbers together, you get another even number—

Nathan: Yeah.

Ball: —or were you saying that all even numbers are made up of even numbers?

Nathan: Yes, they are.

Mathematical knowledge is built from initial—and increasingly improved—mathematical claims. In *Proofs and Refutations* (1976), Lakatos provides a heuristic for the construction of mathematical knowledge. He identifies a pattern: generating a mathematical claim, mustering support for the claim through the provision of reason, generating possible counterclaims, and ultimately revising the claim.[10] This pattern offers a lens with which to scrutinize the mathematical work in which the students and their teacher were engaged. Given the prominence of mathematical claims in the discourse, we ask, What becomes of these productions? What makes them productive?

A cursory examination reveals that the students' talk is saturated with the evaluation of mathematical claims. They evaluate claims by agreeing and disagreeing—by supplying confirmatory examples and offering challenging counterexamples, by asking questions and trying out particular claims. When Sheena says that she now thinks zero "could be" even, Sean, on his own, poses a question that presses the implication of her claim and seeks to show that it is false: "What two things can make it?" When Nathan announces that zero is "special" because "even numbers make even numbers," his classmates press on the claim by asking questions, disagreeing, and raising problems with the idea. Over the next several minutes, his

[10] This rephrasing of Lakatos uses synonymous terms more prevalent in elementary mathematics education. Lakatos describes this pattern as (1) primitive conjecture; (2) proof ("a rough thought-experiment or argument, decomposing the primitive conjecture into subconjectures or lemmas"); (3) global counterconjectures; and (4) proof re-examined: "the 'guilty lemma' to which the global counterexample is a 'local' counterexample is spotted" (p. 127).

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claim is challenged, and ultimately sharpened: He is going "up by twos"—not "by every single number"—and what he is claiming is true even for zero.

Notable across these and the other examples is how entirely comfortable the children seem to be about receiving public disagreement and how attentively they listen to it and respond. Despite the fact that they are engaged in conversations that involve substantial disagreement and agreement, students' tone and manner are strikingly dispassionate and matter-of-fact. In this segment, students do not shy away from disagreeing with classmates' assertions. In fact, the practice is so pervasive that it appears to be standard practice for mathematics work in this classroom. How is it that students comfortably sustain mathematical argumentation despite the fact that disagreement is usually an uncomfortable or disallowed social position in school (Lampert, Rittenhouse, & Crumbaugh, 1996)? What makes this atmosphere possible?

Consider the ways in which students are actively engaged in evaluating mathematical claims during this 6-minute videotape segment. In the long stretch of talk that follows, almost every child's turn represents agreement or disagreement with Nathan's claim that "even numbers make even numbers." Nathan's assertion arises as part of an ongoing discussion about the parity of zero. Claiming that "even numbers make even numbers," he marks zero as "special." Implicit in his argument is that, whereas $2 + 2 = 4$, and $4 + 4 = 8$, and so on, in each instance producing a new even number, $0 + 0 = 0$. Therefore, zero is "special." He implies that this "specialness" exempts it from classification as either even or odd.

Following his assertion, and before other students comment, Ball first seeks to clarify his claim. She asks, "Can I ask you a question about what you just said? . . . Were you saying that when you put even numbers together, you get another even number, or were you saying that all even numbers are made up of even numbers?" He nods: He says he is arguing that even numbers are made up of other even numbers. Ball motions to Betsy, reminding her that she, too, had made a similar claim the day before.

This phase of clarification of claims marks an important aspect of the passage from Nathan's private idea to a public claim. Before the claim can be properly evaluated, the claim itself is clarified. Ball restates Nathan's clarified claim and presents it to the class to evaluate.

In this following segment, the teacher continues to help clarify the claim as the students discuss it:

34. *Ball:* Nathan said a minute ago that when you put even numbers together you get an even number.

35. *Betsy:* Mm-hm.

36. *Ball:* But he also said, I think, that all even numbers are made up of other even numbers.

37. *Mei:* I disagree.

38. *Sheena:* (*Says something to Mei.*)

39. *Ball:* Two even numbers just the same.

40. *Nathan:* Unh-uh.

41. *Ball:* The same even number?

42. *Nathan:* Yeah, like four.

43. *Ball:* Like eight is four plus four? Are all the even numbers—can you do that with all the even numbers? That they'd be made up of two identical even numbers?

44. *Sean:* Not—not—not—

45. *Betsy:* (*Looking toward Nathan*) You can't. Like six. Six is two, two, ... Six you can't get two.

46. *Sean:* Six is two odd numbers to make an even, to make an even number.

47. *Mei:* Three and three —

48. *Betsy:* (*Still looking toward Nathan*) You need three twos to make six. You can't put a four and a four or a . . .

49. *Sean:* Three twos???

50. *Betsy:* (*Looking toward Nathan*) Three's—Three is odd.

51. *Sean:* Or, um—

52. *Nathan:* I know that, but um, um I'm talking about like two plus two is four, and four plus four is eight and I just skipped the six so I just added the ones that, that add. Like the two plus two is four, and four is an even number, and I'm just talking about the things that, um, like—

53. *Sean:* Six can be an odd number.

54. *Nathan:* —what I just said—the um, like two is plus two is four and four plus four is eight and—

55. *Betsy:* So what you're doing is you're going by twos and then what two equals from then you go from—all the way up.

56. *Nathan:* Yeah, I'm not going by *every* single number. Like—

57. *Betsy:* Okay.

58. *Nathan:* —two, four, six, eight.

Although "I disagree" is uttered only once in this passage, 13 of the 25 turns in this passage imply agreement or disagreement with Nathan's claim. In turns 37, 45, 46, 47, 48, 49, 50, 51, and 53, students are disagreeing with Nathan's claim.

As students seek to evaluate claims, their moves to agree or disagree are based on three basic activities: (1) clarifying the claim or the reason underlying a claim, (2) providing examples to prove or disprove a claim, and (3) considering whether a claim would be true in *all* cases. The diagram in Figure 1.8 shows the structure of this work.

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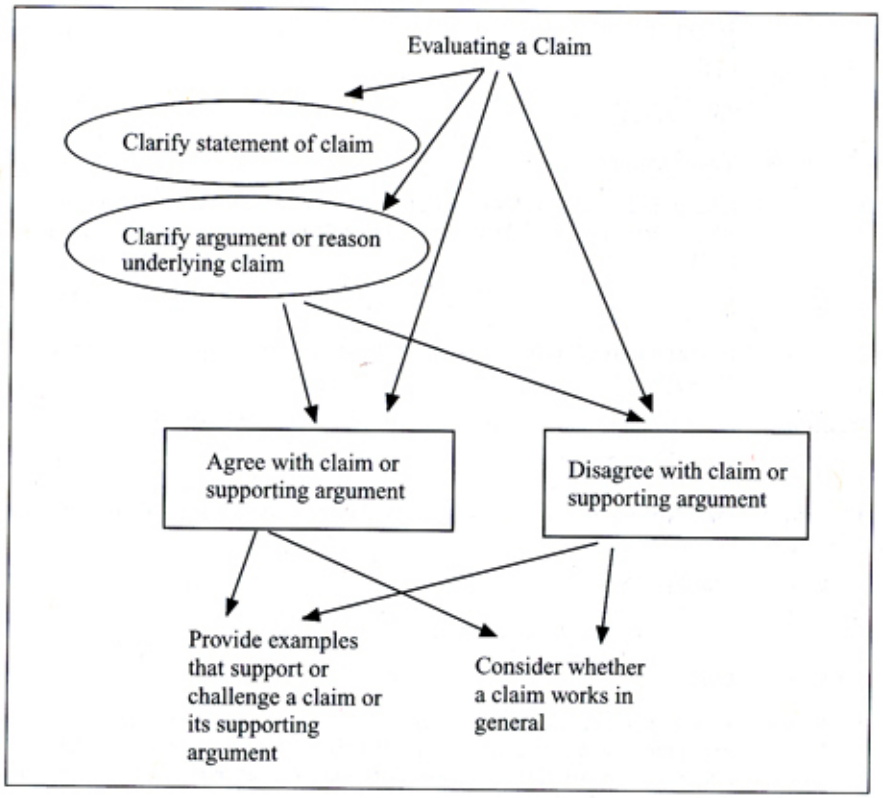


Figure 1.8. Structure of activities involved in evaluating claims.

Contrast this structure with the model of teacher decision-making offered in this volume by Schoenfeld (chapter 2), or the analyses of classroom talk offered by Horn in chapter 3 and by Posner in chapter 4. The structure offered here views teacher and student actions in the framework of a mathematical practice, the evaluation of a claim. In this model, the teacher's actions and the students' participation are framed by a mathematical practice that is social as well. It is social simply because evaluating a claim necessarily involves another person's idea and is conducted in the company of others. To compare this model with those in other chapters in this volume, we see that Schoenfeld's model is not specific to mathematics. Rather, his model of teachers' decision-making could ostensibly cross content domains. Horn's analysis concerns the learning of mathematics in the context of negotiating the social terrain; we suggest that social positioning drives Horn's argument. Posner's analysis foregrounds concerns for equity.

We return to our elaboration of Figure 1.8. In the short excerpt that follows, agreement or disagreement is bolstered by examples that prove or disprove the claim. The claim is considered at turns 45–50 by trying examples of numbers that either make the claim true or make the claim false.

Betsy: (*Looking toward Nathan*) You can't. Like, six. Six is two, two, ... Six you can't get two.

Sean: Six is two *odd* numbers to make an even, to make an even number.

Mei: Three and three—

Betsy: (*Still looking toward Nathan*) You need three twos to make six. You can't put a four and a four or a . . .

Sean: Three twos???

Betsy: (*Looking toward Nathan*) Three's—Three is odd.

A second path is to check for generalization. When Nathan says, "I'm not going by every single number," he takes a different path: a move to consider *all* cases. Beyond a single example that works to make the claim true, would the claim be true for *all* cases? This kind of move is central to mathematical reasoning, in which one checks the veracity of a claim for several cases and then seeks to ascertain that the claim is true for *all* cases.

A third pathway to agreeing or disagreeing with a claim is to clarify its underlying rationale. At several junctures, the teacher and the students clarify an idea previously offered in connection with the claim under consideration. Agreement or disagreement can be concluded through clarification of reason, as we see in this excerpt (turns 55–57):

Betsy: So what you're doing is you're going by twos and then what two equals from then you go from—all the way up.

Nathan: Yeah, I'm not going by every single number. Like—

Betsy: Okay.

Noteworthy is that the teacher's voice is nearly absent in the long passage—once she has launched this discussion by clarifying Nathan's two-part claim. After Ball clarifies Nathan's claim, Betsy immediately agrees ("Mm-hm."). The teacher adds that Nathan also thinks that all even numbers are composed of other even numbers, and Mei immediately says she disagrees. In the next more than twenty turns, the teacher's voice is not heard, and students offer examples and counterexamples to prove or disprove Nathan's two-part claim, and implicitly agree or disagree with his idea. Different compositions of the number six are offered as examples and counterexamples: three twos, three plus three. Students evaluate Nathan's claim on the basis of each example. Nathan is prompted by these examples and counterexamples to clarify his claim by bringing forth yet more examples, and mentioning that he does mean all cases ("I'm not going by every single number.")

How is it that students carry on this extensive evaluation of Nathan's claim without the teacher's interjection? This lesson is taken from a day in January, and students likely learned this form of mathematical argumentation over the course of the school year. But even within this short 6-minute segment of the lesson, we can see the teacher shaping the discourse toward this pattern of argumentation. Looking back to the beginning of the 6-minute videotape, recall that Ball had asked the class

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for reactions to the meeting with the fourth-grade class on a previous day. Note the pattern of Ball's queries, italicized in the text of turns 1E–7 that follows:

Ball: First—my first question is, I'd just like to hear some comments about what you thought about the meeting, what you noticed about the meeting, what you learned at the meeting, just *what kinds of comments you have about yesterday's meeting?* And could you listen to *one another's comments,* so that we can um, benefit from what other people say? See what y— what you think about other people's comments? Sheena, do you want to start?

Sheena: I—I—I liked it because, well, I like talking to other classes and, and when you talk to other classes sometimes it helps.

Ball: *In what way?*

Sheena: It helps you to understand a little bit more.

Ball: *Was there an example of something yesterday that you understood a little bit more during the meeting?*

Sheena: Well, I didn't think that zero was—zero, um—even or odd until yesterday they said that it could be even because of the ones on each side is odd, so that couldn't be odd. So that helped me understand it.

Ball: Hmm. So y— *So you thought about something that came up in the meeting that you hadn't thought about before?* Okay.

We note that the teacher begins with a question that anyone can answer, regardless of the student's mathematical knowledge. Responses to her request for reactions to the meeting can take almost any form, comprise any content. This opening question makes participation available to all.

Although the teacher is asking for general reactions to the meeting with the fourth graders, Sheena's reaction ("I liked the meeting") is treated as a kind of claim, and the teacher's questions that follow serve to clarify the reasons for her "claim," ("In what way?") and to provide examples ("Was there an example of something yesterday that you understood a little bit more during the meeting?"). Here the teacher establishes the pattern for evaluating claims: offer a claim, clarify the reasoning behind it, and offer examples in its support.

Sean then takes up Sheena's "claim" by disagreeing with her in turns 10–15, stating that her example does not work for all cases:

Sean: Um, I—I—I just want to say something to Sheena, when sh— what she said about um that, that one, um—zero has to be an odd, an even number bec— I disagree because, um, because what what two things can you put together to make it?

Sheena: Could you repeat what you said, please?

Ball: (*Speaks to Betsy and asks her to listen to this exchange.*)

Sean: Okay, um, I disagree with you because, um, if it was an even number, how—what two things could make it?

Sheena: Well, I could show you it. (*Moves toward the chalkboard and points to the number line above the chalkboard.*) Um, I forgot what his name was—but yesterday

he said that this one (*points to the 1 on the number line*) and each—this one is odd and this one (*points to the –1 on the number line*) is odd, so this one has to be even.

Sean: But, that doesn't mean it always is even.

Consider closely the structure of the two students' arguments shown in Figure 1.9. Each student makes and evaluates a specific—and competing—claim about the parity of zero:

	Sean	Sheena
Claim	Zero is not even (but not that it is odd).	Zero is even.
Definition	An even number is a number that can be made up of two (equal) things.	Even and odd numbers alternate on the number line.
Prior knowledge		One and negative one are odd numbers (fourth graders said so).
Argument	What two *things* can make it (zero)?	Zero is situated between negative one and one.
	Implied: No two *things* can make it.	
Conclusion	Therefore zero is not even.	Therefore zero is even.

Figure 1.9. Mathematical structure of Sean's and Sheena's arguments.

Whereas Sean claims that zero is not even, Sheena claims that it is. Sean relies for his argument on a definition of even numbers that it can be "made up of two things." Sheena uses a different definition based on alternation on the number line. Although these two definitions are in fact equivalent, they are currently unreconciled in the classroom discourse. Consequently, the two students' arguments reach opposing, contradictory conclusions.

With two different definitions, other mathematical work is needed to resolve the difference. Until the class has a shared definition for what an even number is, arguments such as this one cannot be reconciled. Ball does not attempt to engage in this significant piece of work on the spot. One does not always need to appease by resolving disagreements by fiat, without the proper tools to do so (Lampert et. al., 1996).

DISCUSSION OF MAKING MATHEMATICS WORK

In this 6-minute segment, the class engages in a mathematically sophisticated discussion, with students providing much of the mathematical substance. We analyzed what contributes to this outcome—what students do, what they have learned to do, and what the teacher does to teach them. We argue that a large part

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of what the teacher does is to reconfigure the talk that goes on and that this reconfiguring is directed at essential elements of mathematical reasoning—naming and using names, making and interpreting claims, and evaluating mathematical assertions.

Two symmetric acts combine to reconfigure talk in the mathematics classroom: On the one hand, the teacher's role is to ready mathematical content so that students can engage in it. On the other, the teacher readies students to be doers and learners of mathematics. We see in this 6-minute segment how the teacher weaves back and forth between readying the content for her students' use and readying her students to engage in it. On the surface mathematical content might seem to be "readied" by textbooks and curricula, but in fact teachers do a good deal to transform curricular documents into usable material for classroom work.

By way of example, consider the brief exchange, turns 26–28, between Nathan and the teacher:

Ball: Can I ask you a question about what you just said? And then I'll ask people for more comments about the meeting. Were you saying that when you put even numbers together, you get another even number—

Nathan: Yeah.

Ball: —or were you saying that all even numbers are made up of even numbers?

One can see in this excerpt how the teacher helps articulate Nathan's mathematical claim. To do so, the teacher has transformed a formal mathematical idea (that even numbers are composed of other even numbers) into a statement that third graders can manage ("when you put even numbers together, you get another even number"). The version she has readied for children lacks the rigor and precision of a formal mathematical statement. In mathematics, "putting numbers together" is not used to describe addition. One also does not "get" numbers. The teacher uses language to pose mathematical ideas in ways that are accessible to children.

At the same time, teachers also ready students to make profitable use of the mathematical material at hand. The videotape segment under discussion is replete with examples of the teacher doing so. In this same exchange, the teacher supports Nathan in practices that allow for mathematical work: she asks him to confirm his ideas with precision and clarity. She asks for his agreement, or for his adjudication. She models actively listening to his ideas, and publicly rendering them so that others can dig into them as well. In the segment we have studied for this chapter, *talk* is the agent through which the teacher attempts to mold these habits.

Talk in the mathematics classroom is a primary tool for doing this shaping of both the content and the students' work on it, whether it is rewording a problem midstream or recognizing the reasons for a student's halting statement. Thus, readying the mathematics for students' engagement, as well as readying the students to be doers and learners of mathematics, are largely a matter of designing and reconfiguring talk.

CONCLUSION

Mathematical reasoning is the foundation for the construction of mathematical knowledge. This chapter examines in detail the work entailed for teachers to engage students in this kind of fundamental mathematical work. Exhorting teachers to engage students in mathematical reasoning is inadequate as a support for their practice. Parsing the work of teaching makes instructional practice visible, and hence potentially learnable.

This article identifies three components of mathematical work: naming and using names, making claims, and evaluating claims. Evaluating claims, usually the focus of classroom analyses, is a complex and dynamic practice. Our analysis suggests that the "naming" of terms and definitions, and of ways of thinking and working together in the classroom, are foundational. We also suggest that such terms and labels are crucial for the formulation of claims. Teaching children to formulate a mathematical claim is not a straightforward endeavor: this article unpacks what is involved in such work. We then treat the evaluation of claims separately. Making analytic distinctions about the fostering of reasoning is essential to making teaching practice available for inspection and discussion.

The three practices—naming and using names, making and interpreting claims, and evaluating mathematical assertions—make visible what is involved in making mathematics work in the classroom. With these three practices in focus, teachers can reconfigure classroom talk for the purpose of engaging students in mathematical work. These practices offer ways of laying out the mathematical work to be done. They provide ways to focus on who talks, what gets talked about, and how it gets talked about. Our choice of them is strategic, serving both the discipline of mathematics and the social activity of teaching and learning. We close this chapter by describing three benefits of this resulting framework: epistemological benefits, relational benefits, and benefits for collective work.

Naming and defining, making claims, and agreeing or disagreeing are essential practices of the discipline of mathematics and foundational underpinnings to building understanding. As Ball and Bass (2001) argue, definitions and agreed-on language provide the base from which mathematical justification occurs. Reconfiguring these underlying features of talk from nondisciplinary discourse practices (such as typical patterns of teacher questions and student responses or justifications, for example, "My mother said …") to disciplinary ones (such as "I disagree because …" or "Nathan said a minute ago that …" or "What's our working definition?") provides a solid foundation for efforts to attend to teaching for understanding and the learning of concepts. It maintains the mathematical integrity of the talk and positions students to understand the mathematical basis for what they learn. The point here is not that students invent mathematics for themselves but that they come to see the mathematical basis for what is being taught and learned. For instance, what is the basis for knowing that six is not odd? As Mei concludes later in this lesson, if all numbers were both odd and even, we would have no reason to name or talk about odd numbers or even numbers. As students

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develop a sense for what is in a mathematical name and a mathematical justification, they become better positioned to understand the mathematics they learn.

Second, the framework we propose here provides resources for helping students see themselves as doers and learners of mathematics as well as for shaping relationships among students. In referring to Nathan's idea, Sean numbers, or "our working definition," students begin to identify themselves as people who do mathematics. Since names and claims can be connected, one with another—"Is that like what so-and-so said the other day?"—students' ideas get set in relationship with one another, and thus so do the students themselves. Likewise, agreeing and disagreeing gives students positions in the mathematical landscape and places students in relationship with one another. In these approaches, a teacher can take the words or ideas of students and make them central, putting students into the work in ways that are likely to help them see themselves as people who can do mathematics, perhaps against expectations and labels. These mathematical practices can be used to shape how students see themselves and how they are seen by others. Although mathematical argumentation is often seen as risky, in fact, arguing from a mathematical basis, together with revising, can be a resource for shaking up existing social dynamics (or setting up a world independent of those dynamics), and it provides mathematical criteria for respect in the mathematics classroom.

Finally, the framework furnishes resources for engaging students with one another in collective mathematical work. Names and definitions build a common language that can support collective work. They provide a base of common knowledge crucial for mathematical reasoning (Ball & Bass, 2000b), for instance, in establishing working definitions for even and odd numbers. Making claims is a kind of call to the group about work to be done. It sets the group agenda, focusing attention. By making a claim, one is in effect saying, "I take a stand, what do you think of this?" Making a claim invites people to engage in the mathematical work of evaluating mathematical assertions. This work provides a basis for knowing and learning, engages every student in the mathematics of other students, and engages them with other students.

Doing mathematical work in school depends on a reconfiguration of the talk, ubiquitous in classrooms, that is the medium of instruction. Sean did not serendipitously come up with his infamous claim about the number six; his classmates, in turn, did not engage seriously with him naturally. For students to learn to engage with mathematics and with one another, and for classrooms to nurture such engagement, requires, of teachers, work too often left invisible. By making the effort to unpack the work involved for teachers and their pupils, such practices can be learned and such mathematical work can become more the norm.

REFERENCES

Arendt, H. (2000). *The portable Hannah Arendt.* New York: Penguin Books.

Au, K. (1980). Participation structures in a reading lesson with Hawaiian children: Analysis of a culturally appropriate instructional event. *Anthropology and Education Quarterly, 11*(2), 91–115.

Bakhtin, M. M. (1981). *The dialogical imagination: Four essays.* Austin: University of Texas Press.

Bakhtin, M. M. (1986). *Speech genres and other late essays.* Austin: University of Texas Press.

Ball, D. L. (1993). With an eye on the mathematical horizon: Dilemmas of teaching elementary school mathematics. *Elementary School Journal, 93*(4), 373–397.

Ball, D. L. (2000). Bridging practices: Intertwining content and pedagogy in teaching and learning to teach. *Journal of Teacher Education, 51,* 241–247.

Ball, D. L., & Bass, H. (2000a). Interweaving content and pedagogy in teaching and learning to teach: Knowing and using mathematics. In J. Boaler (Ed.), *Multiple perspectives on the teaching and learning of mathematics* (pp. 83–104). Westport, CT: Ablex.

Ball, D. L., & Bass, H. (2000b). Making believe: The collective construction of public mathematical knowledge in the elementary classroom. In D. Phillips (Ed.), *Constructivism in education: Opinions and second opinions on controversial issues,* a Yearbook of the National Society for the Study of Education (pp. 193–224). Chicago: University of Chicago Press.

Ball, D. L., & Bass, H. (2003). Toward a practice-based theory of mathematical knowledge for teaching. In B. Davis & E. Simmt (Eds.), *Proceedings of the 2002 Annual Meeting of the Canadian Mathematics Education Study Group* (pp. 3–14). Edmonton, AB: CMESG/GCEDM.

Ball, D. L., Bass, H., Hoover, M., Lewis, J., & Wall, E. (2003). *Inattention to equity in teaching elementary school mathematics.* Paper presented at the Annual Meeting of the American Educational Research Association, Chicago.

Brown, P., & Levinson, S. C. (1987). *Politeness: Some universals in language usage.* New York: Cambridge University Press.

Bruner, J. (1960). *The process of education.* Cambridge, MA: Harvard University Press.

Cazden, C. (1986). Classroom discourse. In M. C. Wittrock (Ed.), *Handbook of research on teaching* (3rd ed.). New York: Simon & Schuster Macmillan.

Cazden, C. (1991). *Classroom discourse: The language of teaching and learning* (2nd ed.). Portsmouth, NH: Heinemann.

Cobb, P., & Bauersfeld, H. (Eds.) (1995). *The emergence of mathematical meaning: Interaction in classroom cultures.* Studies in Mathematical Thinking and Learning series. Hillsdale, NJ: Erlbaum.

Dewey, J. (1902/1990). *The school and society; and, The child and the curriculum/John Dewey.* Chicago: University of Chicago.

Doyle, W. (1983). Academic work. *Review of Educational Research, 53*(2), 159–199.

Goffman, E. (1955). On face-work: An analysis of ritual elements in social interaction. *Psychiatry, 18,* 213–231.

Goodlad, J. (1984). *A place called school: Prospects for the future.* New York: McGraw-Hill.

Kieran, C., Forman, E., & Sfard, A. (Eds.) (2002). *Learning discourse: Discursive approaches to research in mathematics education.* Dordrecht, Netherlands: Kluwer.

Kilpatrick, J., Swafford, J., & Findell, B. (Eds.) (2001). *Adding it up: Helping children learn mathematics.* Washington, DC: National Academy Press.

Lakatos, I. (1976). *Proofs and refutations: The logic of mathematical discovery.* New York: Cambridge University Press.

Lampert, M. (1990). When the problem is not the question and the solution is not the answer: Mathematical knowing and teaching. *American Educational Research Journal, 27,* 29–63.

Lampert, M. (2001). *Teaching problems and the problems of teaching.* New Haven, CT: Yale University Press.

Lampert, M., & Ball, D. L. (1998). *Teaching, multimedia, and mathematics: Investigations of real teaching.* New York: Teachers College Press.

Lampert, M., & Blunk, M. (1998). *Talking mathematics in school: Studies of teaching and learning.* Cambridge, MA: Cambridge University Press.

Lampert, M., Rittenhouse, P., & Crumbaugh, C. (1996). Agreeing to disagree: Developing sociable mathematical discourse. In D. R. Olson & N. Torrance (Eds.), *The handbook of education and human development: New models of learning, teaching, and schooling* (pp. 731–764). London: Blackwell.

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O'Connor, M. C. (2002). Can any fraction be turned into a decimal? A case study of a mathematical group discussion. In C. Kieran, E. Forman, & A. Sfard (Eds.), *Learning discourse: Discursive approaches to research in mathematics education* (pp. 143–185). Dordrecht: Kluwer.

O'Connor, M. C., & Michaels, S. (1996). Shifting participant frameworks: Orchestrating thinking practices in group discussion. In D. Hicks (Ed.), *Discourse, learning, and schooling* (pp. 63–102). Cambridge: Cambridge University Press.

Schwab, J. (1961/1978). *Science, curriculum, and liberal education: Selected essays.* Chicago: University of Chicago Press.

Sfard, A. (2001). There is more to discourse than meets the ears: Looking at thinking as communicating to learn more about mathematical learning. *Educational Studies in Mathematics, 46*(1–3), 13–57.

Sfard, A., Nesher, P., Streefland, L., Cobb, P., & Mason, J. (1998). Learning mathematics through conversation: Is it as good as they say? *For the Learning of Mathematics, 18*(1), 41–51.

Stodolsky, S. S. (1985). Telling math: The origins of math aversion and anxiety. *Educational Psychologist, 20,* 125–133.

Vygotsky, L. S. (1978). *Mind in society: The development of higher psychological processes.* Cambridge, MA: Harvard University Press.

Vygotsky, L. S. (1986). *Thought and language.* Cambridge, MA: MIT Press.

Wall, E. S. (2003). *Making sense: Listening, remembering, and facilitating in the elementary mathematics classroom.* Unpublished doctoral dissertation, University of Michigan, Ann Arbor.

Weingrad, P. (1998). Teaching and learning politeness. In M. Lampert & M. L. Blunk (Eds.), *Talking mathematics in school: Studies of teaching and learning* (pp. 213–237). Cambridge: Cambridge University Press.

Wertsch, J. V. (1998). *Mind as action.* New York: Oxford University Press.

Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentations, and autonomy in mathematics. *Journal for Research in Mathematics Education, 27*(4), 458–477.

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