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The Mathematical Understandings That Prospective Teachers Bring to Teacher Education

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Abstract

This article focuses on the subject matter knowledge of preservice elementary and secondary mathematics teachers. In order to examine what teacher candidates understand about mathematics as they enter formal teacher education, results from questionnaires and interviews with 252 prospective teachers participating in a large study of teacher education are discussed. The results reveal the mathematical understandings that these elementary and secondary teacher candidates brought with them to teacher education from their precollege and college mathematics experiences, understandings that tended to be rule-bound and thin. Based on these data, the article challenges 3 common assumptions about learning to teach elementary or secondary mathematics: (1) that traditional school mathematics content is not difficult, (2) that precollege education provides teachers with much of what they need to know about mathematics, and (3) that majoring in mathematics ensures subject matter knowledge. These assumptions underlie current teacher education practices as well as proposals to reform the preparation of teachers.

> I am really worried about teaching something to kids I may not know. Like long division—I can *do* it—but I don't know if I could really *teach* it because I don't know if I really *know* it or know how to *word* it. [Cathy, elementary teacher education student]

> Teaching the material is no problem. I have had *so* much math now—I feel very relaxed about algebra and geometry. [Mark, mathematics major, secondary teacher education student]

> I'm not scared that kids will ask me, you know, a *computational* question that I cannot solve; I'm more worried about answering *conceptual* questions. Right now, my biggest fear—and I'm going to have to confront this on the third of February—

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> is what I am going to do if they ask me some kind of question like, "Why are there negative numbers?" [Cindy, mathematics major, secondary teacher education student]

Cathy, Mark, and Cindy (all names are pseudonyms) are preservice teachers who differ in what they think they need to know in order to teach mathematics. While Mark has confidence in the sufficiency of his mathematics knowledge, both Cindy and Cathy suspect that they may come up short when they try to teach. The comments of these three teacher candidates exemplify alternative points of view about the subject matter preparation of teachers. Cathy's view—that she understands the mathematics herself but needs to learn to teach it—is the basis for traditional formal preservice teacher education, where students are taught methods for teaching but rarely study content. Mark expresses a view that undergirds many of the current proposals to reform traditional teacher education: that is, that people who have majored in mathematics are steeped in the subject matter and have thus acquired the subject matter knowledge needed to teach. Cindy's fear that, although she can do the mathematics, she may not have the kind of mathematical understandings she will need in order to help students learn, is not shared by many who consider the preparation and certification of teachers.

This article focuses on the subject matter knowledge of prospective mathematics teachers. Although no one would disagree that subject matter is a critical component of what teachers need to know, it has not been the focus of research on teacher learning (Ball & McDiarmid, in press). Moreover, even when it is discussed, there is little agreement about what is meant by "subject matter knowledge for teaching"—what would count as adequate understanding of a domain and how to know if someone has that kind of understanding. Discussion about prospective teachers' subject matter knowledge tends to center on suitable requirements, such as grade point averages, major fields of study, test scores, or courses. Rarely are the understandings of prospective teachers explored: How do they understand the subjects they plan to teach? What do they know and how do they think about it?

In this article, I present a perspective on what subject matter knowledge for teaching mathematics entails and analyze the mathematical understandings of a sample of prospective elementary and secondary teacher education students. The understandings that these students—both the elementary candidates and the mathematics majors—brought with them to teacher education raise serious questions about their subject matter preparation for mathematics teaching. The data highlight the need to reexamine common assumptions about what prospective teachers need to know and how they can learn it, assumptions that underlie current teacher education as well as, paradoxically, proposals to reform teacher preparation. I conclude the article with a closer look at three of these assumptions.

Method

This article draws on data from the Teacher Education and Learning to Teach Study (TELT) currently under way at the National Center for Research on Teacher Education (NCRTE) at Michigan State University. The study examines what teachers are taught and what they learn in 11 diverse preservice, induction, in-service, and alternative-route programs around the country, combining case studies of programs with longitudinal studies of participants' learning (see Ball and McDiarmid [1988] and NCRTE [1988] for additional information about the program and learner components of the TELT study).

This article focuses on one part of the large study: the understandings of mathematics held by 252 preservice teacher candidates at the point at which they entered formal teacher education in five preservice program sites: Dartmouth College, Univer-

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sity of Florida, Illinois State University, Michigan State University, and Norfolk State University. This sample includes 217 elementary education majors and 35 mathematics majors who plan to teach high school. The sample was not selected to be representative of the population of prospective teachers. Still, the demographic data indicate that the subjects are very similar to the general population of teacher education students in terms of characteristics such as ethnicity, gender, social class, and age.

The study design is longitudinal. At repeated intervals, we administer a questionnaire to all college students in the sample. We also interview and observe a smaller "intensive" sample of students whom we follow more closely throughout their preservice program and into their first year of teaching.

Both the questionnaire and the interview were designed to explore participants' ideas, feelings, and understandings about mathematics and writing, about the teaching and learning of mathematics and writing, and about students as learners of these subjects. Many of the questions are grounded in scenarios of classroom teaching and woven with particular subject matter topics. Among the mathematical topics are rectangles and squares, perimeter and area, place value, subtraction with regrouping, multiplication, division, fractions, zero and infinity, proportion, variables and solving equations, theory and proof, slope and graphing.[1]

We have tried to help the students who are participating in this longitudinal study understand the purposes of our research. Participants know that we are interested in what they think about a host of issues related to teaching and that we are trying to track whether and how any of their ideas change over the course of their teacher education program. We explain that many of the questions we ask are puzzling and that people's responses vary widely, that if participants are puzzled or confused about a particular question, we are interested in learning what puzzles them about it. Despite these explanations, we realize that participants' responses cannot be equally reliable or valid.

For the analysis reported in this article, frequencies were calculated for relevant questionnaire responses. Careful substantive analyses of the interview questions led to the creation of a set of response categories for each question. These categories were modified in the course of data analysis to better accommodate teacher candidates' responses. Most questions were cross-analyzed on several dimensions: subject matter understanding, ideas about teaching, learning and the teacher's role, and feelings or attitudes about mathematics, pupils, or self.

Prospective Teachers' Knowledge of Division with Fractions

This section opens with a close analysis of the prospective teachers' understanding of one particular mathematics topic—division with fractions—and then moves to a more general discussion of the qualitative dimensions of their mathematics knowledge. This close look illustrates the level of analysis needed, both of the teacher candidates' ideas and ways of thinking as well as of the content itself, in exploring and appraising prospective teachers' subject matter knowledge. Examining a specific topic also makes more vivid the contrast between some key characteristics of what preservice teachers have learned as students and what they need to know as teachers.

Explanation of the Concept

Division is a central concept in mathematics at all levels and figures prominently throughout the K–12 curriculum. When studying division, students can learn about rational and irrational numbers, place value, the connections among the four basic operations, as well as about the limits and power of relating mathematics to the real world. Because students often have difficulty learning division, teachers should un-

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derstand it well. I discuss division briefly here in order to help the reader appraise the prospective teachers' knowledge in the section that follows.

At its foundation, division has to do with forming groups. Two kinds of groupings are possible:

1. *Forming groups of a certain size.* For example, the teacher might take a class of 28 students and form groups of four. The problem is, How many groups of that size can be formed? This is sometimes referred to as the *measurement* model of division.

2. *Forming a certain number of groups.* For example, the teacher takes a class of 28 students and forms four groups. The problem is to determine the size of each group. This model is sometimes referred to as the *partitive* model of divison.

Consider a typical division statement with whole numbers, such as $7 \div 2$, which may represent one of two kinds of situations:

> 1. I have 7 slices of pizza. If I want to serve 2 slices per person, how many portions do I have? (*Measurement* interpretation—answer: 3½ portions.)
>
> 2. I have 7 slices of pizza. I want to split the pizza equally between 2 people. How much pizza will each person get? (*Partitive* interpretation—answer: 3½ slices.)

Because these two situations represent different meanings of division, the referent for 3½ is different in the two. In the first situation, the answer is a number of *portions* (the size of the portion was already decided); in the second, the answer is a number of slices *per portion* (the question was how large each portion would be). In each case, multiplying the result (3½) by the number used to divide the original total yields that total ($3½ \times 2 = 7$).

What about the specific topic at hand, division with fractions? Remembering to "invert and multiply"—that is, to invert the divisor and multiply it by the dividend—is one way of understanding division by fractions. A typical eighth-grade textbook introduction to division of fractions states simply, "Dividing by a fraction is the same as multiplying by its reciprocal" (Bolster et al., 1978, p. 147). Little or no attention is given to the meaning of division with fractions, and no connections are made between division with fractions and division with whole numbers. Each is treated as a special case.

Dividing by fractions, however, is not different conceptually from dividing by whole numbers. Suppose, for example, that you owe a friend $100 and must repay the money, although without interest. You can explore how long it would take to repay this debt, given different payment amounts. If you pay $2 per week, it will take you 50 weeks. This can be formulated mathematically as $100 \div 2 = 50$. Now, consider how long it will take you if you repay at a rate of 50¢ per week. Fifty cents = ½ dollar, so this option can be expressed as $100 \div ½$. If you understand division and have a feel for amounts of money, you know that 200 weeks is a reasonable answer to this (although 4 years to pay back $100 will probably jeopardize your friendship). If you are uncertain about your calculations, you can verify them by multiplying the number of weeks by the amount of the payment. For example, $200 \times .50 = \$100.00$ (or $200 \times ½ = 100$). Both of these repayment plans model the first meaning of division above—that is, take $100 and form "groups"—first of $2 and then of one-half dollar. The problem in both cases is how many "groups"—or payments—of the given amount there are in $100, a measurement representation of $100 \div ½$.

To see a partitive model of the same expression, try the following. Imagine that you are packing apples in crates. You have 100 apples and find that they fill exactly half a crate. How many apples will it take to fill a crate? In this case, the number ½ represents the number of groups already on hand. The question is, How many will be in each whole group? Notice that this rep-

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resentation corresponds directly to the algorithm "invert and multiply": in fact, one would think about it in terms of doubling the number of apples it takes to fill half a crate. Worth noting too is that the procedure "invert and multiply" is not uniquely suited to dividing with fractions. Understanding why "invert and multiply" is a plausible strategy entails understanding division as the inverse of multiplication, with integers and fractions. For instance, $6 \div 2$ yields the same result as $6 \times \frac{1}{2}$ (where one has inverted the 2 and multiplied the result by the dividend, 6). Yet rarely, if ever, is this made explicit to pupils. Since division with fractions is most often taught algorithmically, it is a good site for examining prospective teachers' understanding of the *meaning* of division.

Exploring Prospective Teachers' Understandings

We used two approaches to learn how preservice teachers understood division with fractions: a questionnaire item and an interview task.

Questionnaire item. This item asks respondents to select from among a set of four story problems those that represent a given division statement, for example:

> Which of the following is a good story problem to illustrate what $\frac{1}{4} \div \frac{1}{2}$ *means*? Choose all that apply.
> *a*) A recipe calls for $4\frac{1}{4}$ cups of milk. How much milk is needed for half a batch?
> *b*) It takes $4\frac{1}{4}$ hours to drive 200 miles. How far will we have gone in half an hour?
> *c*) Jim needs $4\frac{1}{4}$ pounds of lentils. How many half-pound bags should he buy?
> *d*) None of these. Instead: _____ .
> *e*) I'm not sure.

Although only one of the options (*c*) is a mathematically appropriate representation of the division, respondents are directed to mark all that apply.

Interview task. We first asked the prospective teachers how they were taught to divide fractions and asked them to show us by doing $1\frac{3}{4} \div \frac{1}{2}$. Then we told them that many teachers try to come up with pictures, models, stories, or real world representations to help pupils understand mathematical ideas and asked them whether they could think of something to represent the statement $1\frac{3}{4} \div \frac{1}{2}$. If they were able to describe a representation, we probed their understanding of its correspondence to their paper-and-pencil calculations. If they were unable to come up with anything, we told them that many people find this difficult and asked them why they thought it was hard to do.

What is entailed in answering these questions? To use "invert and multiply," the mixed number must first be converted to an improper fraction—that is, $1\frac{3}{4} \div \frac{1}{2}$ becomes $\frac{7}{4} \times \frac{2}{1}$. Multiplying the numerators and denominators produces $\frac{14}{4}$, which should be expressed as $3\frac{1}{2}$. What does the answer $3\frac{1}{2}$ *mean*? Herein lies the essence of the algorithm's conceptual background. As in any division, $1\frac{3}{4} \div \frac{1}{2}$ has to do with groups—in this case, groups of a certain size: $\frac{1}{2}$. How many groups of that size can be formed out of $1\frac{3}{4}$?

Asking the prospective teachers how they learned to solve $1\frac{3}{4} \div \frac{1}{2}$ allowed us to see whether they knew the algorithm "invert and multiply" or any other procedure for dividing by fractions. Asking them to generate a representation gave us a view of their understanding of division.

Results

The elementary candidates as well as the secondary students (who were majoring in mathematics) had significant difficulty "unpacking" the meaning of division with fractions. These results fit with evidence from other parts of the interviews and questionnaires that suggest that the teacher education students' substantive understanding of mathematics was both rule bound and compartmentalized.

Selecting a representation. Selecting an appropriate representation of division from

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among an array of options is easier than generating one. Still, only 30% of the preservice teachers chose the story problem that represented 1¾ ÷ ½. These results are further tempered by the fact that many of those who selected the correct response also selected one or more *incorrect* responses (they were instructed to "circle all that apply"). For example, 30% of the respondents selected the story that modeled 4¼ ÷ 2 as well the one that represented 4¼ ÷ ½, despite the fact that differentiating between these two is the key conceptual issue. In addition, even when presented with four choices, 10% of the elementary and 6% of the secondary teacher candidates selected the "I don't know" option. Table 1 shows the distribution of responses to the item among elementary and secondary candidates.

Generating a representation. The interview results help us understand reasons for responses to the questionnaire item. Virtually all of the teacher candidates were able to *calculate* 1¾ ÷ ½ during the interviews. They remembered to invert and multiply—the rule they had been taught—and although a few were momentarily unsure about which number to invert, most were able to carry out the procedure.

However, very few secondary teacher candidates and no elementary candidates were able to generate a mathematically appropriate representation of the division. Table 2 shows the distribution of responses across these three categories among elementary and secondary teacher candidates.

Appropriate representations. Only four out of 35 teacher candidates in the intensive sample (11%) were able to describe a completely appropriate representation of 1¾ ÷ ½. Correct responses did not come easily, however.

For example, Connie, a mathematics major, said she did not have any problems "doing it" (i.e., inverting and multiplying to get the answer) but that she was not good at creating story problems. She thought for a moment and then said she might use "representations on paper, on the blackboard, things like that. . . . I guess I would, I could use a number line. You know, let's mark off where it is; here's one-and-three-quarters and there's a half. Now how many times does a half count there? Then you get this far and then you have to talk about that. How much of a half is that?" Connie explained that this representation would be good because it showed that the question was: How many halves are in 1¾?

Although the other three prospective teachers who gave correct representations understood the meaning of division with fractions, they had more difficulty developing representations. Their problems had a somewhat forced quality. For example, Tony first said he could not "think of anything specific." Then he said he would use pizza:

> If you took the pizza and took ½ of a pizza and you took a whole pizza and ¾ of a pizza (that would be 1¾). You put the ½ of the pizza on top of each piece. So first you'd take the whole pizza and

TABLE 1. Preservice Teachers' Representations of 1¾ ÷ ½ on Questionnaire

Response	Teacher Candidates (N = 252)		Totals
	Elementary (%) (n = 217)	Secondary (%) (n = 35)	
Appropriate representation	30.3	40.0	31.1
Inappropriate representation	60.2	54.3	59.2
Unable to generate a representation	9.5	5.7	9.7

NOTE.—Respondents were instructed to "mark all that apply" from among the alternative representations; 30% of those who selected the appropriate representation also marked one or more of the inappropriate representations.

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TABLE 2. Preservice Teachers' Representations of 1¾ ÷ ½ during Interviews

Response[a]	Teacher Candidates $N = 35$		Totals
	Elementary ($n = 25$)	Secondary ($n = 10$)	
Appropriate representation	0	4	4
Inappropriate representation	10	2	12
Unable to generate a representation	15	4	19

[a] Data are frequencies.

> you'd put it on top of it. Then you'd take that off, whatever it fits on, and you'd do it again. Only take if off if it fits the whole thing. If . . . both pieces are equal. Then you go through the ½ a piece and do the same for that. Take that off. Then you get that last piece and you . . . well, that's the way I'd explain it.

Tony then explained what the answer (3½) meant in this context: "You'd take the ½ and the answer would be how many times you got a whole half (if you want to say that). Of the . . . whatever's left over, what part of it is *of* the half. I guess you could say. You'd have a ¼ left, which is ½ of a half."

This makes sense, although laying pieces of pizza on top of whole pizzas forces the imagination a bit and suggests that it was not easy for Tony to devise a realistic story. All four teacher candidates' examples appropriately represented division with fractions. Their answers, while a bit awkward, indicated that they understood the problem in terms of division, that they thought about how many halves there are in 1¾.

Inappropriate representations. Twelve out of 35 teacher candidates (34%) generated representations during interviews that did not correspond to the problem. The most frequent error was to represent division by 2 instead of division by ½. For example, Abby, a mathematics major, gave the following story: "If we had one and three-quarters pizzas left and there were two of us dying to split it, then how would we be able to split that?" She proceeded to give a careful conceptual explanation of how to split the pizza equally: "If there's three-quarters over here and first you want to tackle that part before we break into the nice pizza that's still there . . . and then if each of us eat half, ate half of a three-quarters for each of us, then each of us ate half of the whole, then we'll eat all the pizza." She said each person would get ⅞ of a pizza altogether.

Abby's was by far the most common response. Most of the prospective teachers used round food—pizzas or pies—and described two people sharing the food. The reasons for this are understandable but nonetheless troubling. Two are particularly striking: the students' preoccupations and confusions with fractions and their tendency to confound everyday and mathematical language.

Concerning preoccupations and confusions with fractions, the teacher candidates' comments showed that they saw the question as one about *fractions* instead of *division.* When asked, for example, what made the task difficult, most commented that it was hard (or impossible) to relate 1¾ ÷ ½ to real life because, as Lee said, "you don't think in fractions, you think more in whole numbers." Not only did their explanations reveal that they framed the problem in terms of fractions but also that many were uncomfortable with fractions as real quantities. Several commented that they did not "like" fractions.

Round models account for most pictorial representations of fractions in school textbooks. Focusing on fractions therefore in-

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clined the prospective teachers toward circular representations. Food was common because they assumed that it would be intrinsically more interesting to students than other divisible circular objects (Ball, 1988). In responding to our question, the prospective teachers tripped over their interpretation of the problem as essentially "about fractions" and their limited conception of and repertoire for representing fractions.

The respondents also confounded everyday language and mathematical language. Lee, a prospective elementary teacher, was one of only a few who departed from round food models—pizzas and pies. Thinking of food at first, she said that it was impossible to use food as a model "because it's hard to have ½ a person" sharing the food. Then it occurred to her to use a "time problem":

> I was just thinking that a plane ride, maybe, a plane flight takes an hour and three-quarters and you . . . were going to stop halfway through and pick up more passengers—so how much time would have passed from the time that you left the airport till the time you made it to your first stop or something like that. It's a really good example—the whole trip takes an hour and three-quarters and halfway through you're going to stop and pick up some more people and you left at, you know, 9:00 in the morning and, you know, you're supposed to get to your destination at 11:45 (*sic*). When you stopped to pick up the passengers, what time would it be then? It would be halfway through the flight.

Although Lee was able to come up with a novel context for her representation, she, like most others, represented division by *two* instead of division by *one-half*. The respondents tended to confuse dividing *in* half with dividing *by* one-half, and they did not seem to be aware of the difference, even though the answer they got to their story problems (⅞) differed from their calculated answer (3½).

The teacher candidates did not notice this discrepancy because it was masked by a change in the referent unit from wholes to fourths. Here is a typical example: suppose you have 1¾ pizzas to split equally between two hungry teenagers (1¾ ÷ 2). *Each pizza is divided into 4 pieces, so you have 7 pieces.* Therefore each person gets ⅞ of a pizza, which is 3½ pieces of pizza. However, to divide something *in* half means to divide it into two equal parts (÷ 2); to divide something *by* one-half means to form groups of ½ (see Fig. 1).

The teacher candidates' error resulted from a common but problematic confounding of everyday language with mathematical language. Orr (1987) discusses some of the mismatches between linguistic and mathematical use of prepositions. Awareness of such confusions is at the heart of what teachers must know if they are to help pupils understand mathematics.

No representation. Nineteen (15 elementary, 4 secondary) of 35 teacher candidates (54%) were unable to generate a representation for 1¾ ÷ ½. Most of them tried to use the numbers (1¾ and ½) but

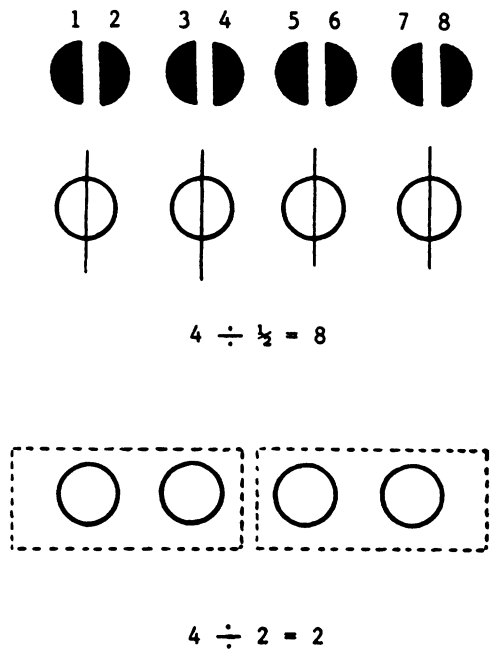


FIG. 1. An illustration of the difference between division by one-half and division by two.

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did not represent division. For example, Mary tried the following:

> Um, about a classroom on a field trip and one-and-three-quarters of the class went one way and one-half of the class went the other way, and we're solved—oh, wait a minute, that wouldn't work, would it? One-and-three-quarters divided by [pause], um, I think I would still use it as a group of people [pause], divided by half of it [pause], half of a wall or something that they walked by, and then how much of the class was left on the one side of the wall?

The interviewer asked, "Half of a wall? Is that what you said, or half of a *class*?" Mary tried again: "Half of the wall, like, that wouldn't work either. One-and-three-quarters [laughs], I guess I can't do it that way. I don't know. That would be really hard to do because you couldn't, you can't give like a half of something will divide the class, or, you know, something like that." Mary's story revealed confusions about and discomfort with fractions. Simply trying to represent the numbers themselves in some real context was difficult for many of the prospective teachers. One teacher candidate commented that it was hard to "[find] a model or a story to teach division by fractions. To me fractions are hard to begin with, and then the division part of it is confusing to the mind, I think; to *me* it is, it can be confusing." Respondents who did not generate a representation at all seemed to fall into two groups. Some recognized the conceptual problem. They initially proposed stories or models that represented division by 2 and then realized this. Others seemed to think that it was not a feasible task—that $1\frac{3}{4} \div \frac{1}{2}$ could not *be* represented in real-world terms.

These responses reflect significantly different understandings. In one sense, those who saw the conceptual problem revealed a better grasp of the idea than those who represented division by 2. Still, despite this recognition, they were unable to figure out what "division by $\frac{1}{2}$" meant. Those who thought the task was impossible may have revealed a view of mathematics as a domain where algorithms, not meanings, are central.

Summary

Although few of the prospective teachers even mentioned division explicitly, the difficulties all of them experienced (including those who generated appropriate representations) suggest a narrow understanding of division. Although they worried about the fractions in the problem, they only considered division in partitive terms: forming a certain number of equal parts. This meaning (in this case, forming $\frac{1}{2}$ group out of $1\frac{3}{4}$ of something) corresponds less easily to division with fractions than does the measurement model (in this case, forming groups of $\frac{1}{2}$ out of $1\frac{3}{4}$ of something). In a study of preservice elementary teachers' understanding of division, Graeber, Tirosh, and Glover (1986) found that preservice teachers tended to think in terms of this partitive interpretation. Few of the preservice teachers in their study were able to write story problems that modeled a measurement interpretation of division. This finding offers another insight into why the task of making meaning out of $1\frac{3}{4} \div \frac{1}{2}$ was so difficult for the prospective teachers in our study.

Subject Matter Knowledge for Teaching

Underlying this analysis is the assumption that the goal of mathematics teaching is for students to develop mathematical understanding. On one hand, this implies that pupils should acquire knowledge of mathematical concepts and procedures, the relationships among them, and why they work. On the other hand, understanding equally implies learning about mathematical ways of knowing as well as about mathematical substance. The two are intertwined. In order for students to develop power and control in mathematics, students must learn to validate their own answers.

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They must have opportunities to make conjectures, justify their claims, and engage in mathematical argument, all of which depend upon and can extend pupils' understandings of concepts and procedures (Ball, 1988; Lampert, 1986, in press).

In order to facilitate this kind of understanding, teachers must understand mathematics deeply themselves. Neither a perfect textbook lesson nor a smooth procedure for calling on pupils will bail out the teacher who is confronted with students who want to know why, when they divide 1¾ by ½, the answer (3½) is "larger." Teachers must understand mathematics well themselves if they are to be able to respond to such a question—whether it is by directly answering the question or by reframing the question so that students can figure it out themselves. Teachers should understand the subject in sufficient depth to be able to represent it appropriately and in multiple ways—with story problems, pictures, situations, and concrete materials. They need to understand the subject flexibly enough so that they can interpret and appraise students' ideas, helping them to extend and formalize intuitive understandings and challenging incorrect notions.

What does subject matter knowledge for teaching entail, then? Hardly anyone would argue with the claim that teachers need *substantive knowledge of mathematics*—of particular concepts and procedures (rectangles, functions, and the multiplication of decimals, for example). This substantive knowledge of mathematics is what others most easily recognize as "subject matter knowledge." Three criteria characterize the kind of substantive knowledge teachers need. First, teachers' knowledge of concepts and procedures should be correct. Thus, they should be able to draw a rectangle, identify a function, and calculate $1\frac{3}{4} \div \frac{1}{2}$, for instance. Second, they must understand the underlying principles and meanings—what it means for $1\frac{3}{4} \div \frac{1}{2}$ to be $3\frac{1}{2}$ or why the numbers "move over" on each line of a long multiplication problem, for instance. Finally, teachers must appreciate and understand the connections among mathematical ideas: how fractions are related to division, how place value figures in multiplication computation, and the connections and distinctions among measurements of distance, area, and volume.

In addition to this kind of substantive knowledge, another critical dimension of subject matter knowledge for teaching is *knowledge about mathematics*. This includes understandings about the nature of mathematical knowledge and of mathematics as a field. What counts as an "answer" in mathematics? What establishes the validity of an answer? What is involved in doing mathematics? In other words, What do mathematicians do? Mathematical knowledge is based on both convention and logic. Which ideas are arbitrary or conventional and which are logical? What is the origin of some of the mathematics we use today and how does mathematics change? Similar understandings *about* subject matter are crucial to teaching in other areas, such as science, history, and literature.

Prospective Teachers' Substantive Knowledge

The assumption that people understand the meanings of the mathematical procedures they have learned to perform is a fallacy (Hatano, 1982). Many children and adults perform mathematical calculations without understanding the underlying principles or meaning. As one of the math majors we interviewed reflected, "I absolutely do it by the rote process—I would have to think about it." Although almost all the prospective teachers were able to calculate $1\frac{3}{4} \div \frac{1}{2}$ correctly, strikingly few were able to represent the meaning underlying the procedure they had learned.

Division of fractions is but one example of this phenomenon. Our interviews indicate that prospective teachers lack explicit understanding of concepts and principles even when they can perform the calculations involved. In order to help someone

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else understand and do mathematics, however, being able to "do it" oneself is not sufficient. Teachers must not only be able to describe the steps for following an algorithm but also discuss the judgments made and the meanings of and reasons for certain relationships or procedures. Teachers must be able to generate explanations or other representations, often on the spot in response to a student question. This is more than "metacognitive awareness" of the processes used in solving a mathematics problem or carrying out a procedure; it includes the ability to talk about and model concepts and procedures.

Few prospective teachers, including the math majors, seemed to have this kind of explicit conceptual understanding. As Ben, one of the math majors, reflected about division by zero, "I just *know* that. . . . I don't really know why. . . . It's almost become a fact . . . something that it's just *there*."

With regard to connectedness of the prospective teachers' knowledge, few of the teacher candidates explicitly interpreted the division with fraction tasks as cases of *division*. When they were asked how they learned to do $1\frac{3}{4} \div \frac{1}{2}$, they were able to produce the specific rule for division of fractions and to use it. This specific algorithmic knowledge was inadequate, however, for representing the meaning of division of fractions. As students themselves, what they had to learn was the rule "invert and multiply." Knowing the rule and being able to use it had been sufficient to pass math tests; consequently, they had learned to divide fractions without focusing on the meaning of division of fractions as part of the larger concept of division.

Our interviews with prospective teachers showed that they varied in how clearly they remembered particular rules or procedures, but few connected particular ideas to larger concepts. For example, almost all of the prospective secondary teachers focused exclusively on the surface differences among three cases of division (division by zero, division by fractions, division in algebraic equations). They knew a different rule or procedure for each case and, even though we pressed them on how they could explain these, they were unable to go beyond these rules (Ball, in press–a). Similarly, although all could multiply large numbers, few prospective teachers mentioned the centrality of place value in the procedure for multiplying large numbers (why the numbers in each partial product "move over") (Ball, in press–b).

It is critical to note here that the standard school mathematics curriculum that most prospective teachers have studied treats the subject as though it consisted of discrete bits of procedural knowledge. As discussed above, for example, most mathematics textbooks introduce division of fractions as a new topic and do not show its relationship to division with whole numbers. Rarely are students encouraged or helped to make connections among the disparate ideas and procedures they encounter in their mathematics classes. These prospective teachers' responses to our questions were therefore not surprising, given the way in which they had probably been taught mathematics themselves.

Prospective Teachers' Knowledge about Mathematics

As discussed above, knowledge of concepts and procedures—substantive understanding of mathematics—is only part of knowing mathematics. In examining prospective teachers' understandings of mathematics, another critical area of inquiry and analysis is the way in which their ideas *about* mathematics influence their representations *of* mathematics. What do they think an explanation is? How do they sort out convention from logic with respect to particular principles or ideas? What do they think it means to "know" or to "do" mathematics?

The prospective teachers focused on procedures and rules for reasons that seemed to go beyond the nature of their substantive knowledge, reasons based in

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their ideas *about* mathematics. Some predominant assumptions reflected in their interviews included: doing mathematics means following set procedures step-by-step to arrive at answers; knowing mathematics means knowing "how to do it"; and mathematics is a largely arbitrary collection of facts and rules. Although people have many ideas about the nature of mathematics, these ideas are generally implicit, built up out of years of experience in math classrooms and from living in a culture in which mathematics is both revered and reviled. The prospective teachers seemed to take their assumptions about mathematics for granted. Unlike their understandings of the *substance* of mathematics, which some of them wished to increase or deepen, the preservice teachers did not focus on their understandings about mathematics. They did not seem dissatisfied with them, nor did they even seem to consider them.

The prospective teachers' assumptions about the nature and value of mathematical knowledge and what it means to know something in mathematics affected their responses to all of the interview questions. In this section, I focus on one specific idea that was widely shared by the teacher candidates: that mathematics is a collection of arbitrary rules to be memorized.

Knowing Something in Mathematics: Rules as Explanations

The questionnaire presented a set of mathematical ideas that are often represented as rules (e.g., "invert and multiply" or $a^n/a^m = a^{n-m}$). We asked the preservice teachers to indicate whether they thought each idea could be explained, must be memorized, or whether they were not sure. The answers to these questions illustrate the interaction of what the prospective teachers knew and what they believed it means to know something in mathematics.

Some interesting differences between prospective elementary and secondary teachers appear on this issue. The secondary candidates (mathematics majors) believed that they *knew* mathematics. All of them agreed strongly with questionnaire items about being good at math. As such, they were more likely to say that the mathematical ideas could be explained. The elementary candidates simply could not remember some of the content included in this section; consequently they did not know whether an idea could be explained or not.

One example was the statement, "The slope of a vertical line is undefined." Almost a third of the elementary candidates said they were not sure. In contrast, none of the secondary candidates was unsure, and over 90% said that this *could* be explained. However, in the interview, when we asked the secondary majors how they would explain this idea to their pupils, almost all of the explanations were simply restatements of rules. Most referred to division by zero, which they said was "undefined." "You can't ever do this. It's a rule you should never forget," explained Tony.

On the questionnaire, a third of the secondary candidates thought that most mathematics could not be explained, while twice as many elementary candidates thought it could not be explained. The interviews suggested that what respondents meant by an "explanation" was actually to give a rule, however. Almost all the prospective teachers in both groups agreed that remembering rules and facts was essential. The difference was that the secondary candidates remembered these rules better than the elementary majors and used them as explanations. In other words, the rules *were* explanations. For example, the most common explanation for division by zero was to say that "it's undefined." Although true, this does not explain what that means or *why* dividing by zero is undefined.

The prospective teachers' ideas about what it means to know something in mathematics generally centered on remembering rules and being able to use standard procedures. Obviously ideas about mathematics do not exist separately from substantive

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understandings of particular concepts or procedures. Many preservice teachers lacked explicit and connected conceptual understanding of mathematical ideas and procedures. As such, many of them could do little else but respond in terms of rules and procedures.

Feelings about Mathematics

Like their *understandings* about mathematics, the prospective teachers' *feelings* about math figured in the way they thought about and experienced mathematics and were tightly bound up with their substantive understandings of the discipline.

Many researchers have been interested in teachers' and prospective teachers' "attitudes" toward mathematics (e.g., Bassarear, 1986; Dutton, 1954; Ferrini-Mundy, 1986; Smith, 1964; Thompson, 1984). They have developed scales with which to measure teachers' interest in, enjoyment of, and confidence with mathematics, generally treating attitude as separate from understanding.

Yet, people's *understandings* of mathematics are interrelated with how they *feel* about themselves and about mathematics. The teacher candidates' knowledge, ways of thinking, beliefs, and feelings jointly affected their responses. Their approaches to figuring out problems were shaped by their self-confidence, their repertoire of strategies, what they were able to remember about related concepts, as well as what they believed about the fruitfulness of trying to figure out a problem in the first place. Together these comprise the way they participate in and experience mathematics.

We are using three approaches to explore prospective teachers' feelings about mathematics and about themselves in relation to math. The questionnaire includes a number of items designed to elicit respondents' feelings. A second approach, better suited to learning about the relationship between affect and cognition, is to pay attention to affective dimensions of the teacher candidates' responses to all the interview questions, especially the ones in which they have to deal with mathematical ideas. Giggles, sighs, exclamations, shaky voices, and comments all contribute information about the interaction between their feelings and understandings. Third, in the interviews we asked prospective teachers what they remembered about their experiences with mathematics throughout school—in elementary, middle, and high school, as well as in college. Their accounts give us a biographical perspective, helping us to understand how they feel about mathematics, how they feel about themselves in relation to mathematics, and how those feelings both affected and were influenced by their school experiences with mathematics.

I focus here on the questionnaire responses only. As one would expect, we found considerable variation among the teacher candidates regarding their feelings about mathematics. Some seemed to enjoy math and were eager to take additional courses; others were obviously tense about the subject. Some were confident in their own ability to do and understand mathematics; some believed that they were not good at math and were anxious about both studying and teaching it.

In this area, substantial variation was apparent between elementary and secondary candidates' responses. Only half of the elementary teacher candidates said they enjoyed and were good at mathematics; over a third of them felt that they were not good at math and said they tried to avoid it. In contrast, all of the secondary candidates enjoyed mathematics and thought they were good at it. The secondary candidates all believed they were capable of understanding even "advanced" math; almost half of the elementary majors believed they were not. The interviews suggest that this gap is even greater than it appears, for what the elementary majors meant by "advanced math" was often algebra and geometry, while the mathematics majors were thinking of advanced calculus and group theory.

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Preliminary analyses of the interviews show that the teacher candidates' feelings about mathematics and about their own mathematical capabilities were interwoven with how much mathematics they felt they knew, their views of the subject, and what they believed about mathematical ability in general. It is not surprising that the most anxious teacher candidates also had taken the fewest mathematics courses in high school and college and thought of mathematics as a collection of arbitrary facts. They generally also viewed mathematical ability as innate. This combination of ideas was accompanied by intense feelings of dislike, fear, and anxiety. In contrast, other teacher candidates thought they knew mathematics, thought mathematics was a body of knowledge, and thought mathematical ability was largely a matter of effort and desire. These ideas characterized prospective teachers who exhibited a calmer feeling of confidence and control. These teacher candidates were much surer of themselves and of their responses, despite the fact that many of their responses reflected significant misconceptions about particular mathematical ideas or about the discipline of mathematics itself.

These complex combinations of ideas and reactions that undergird the teacher candidates' feelings about mathematics and about themselves seemed to influence their responses during the interviews. Prospective teachers' feelings are *part of* the way they participate in and understand mathematics, *not* a separate affective dimension called "attitude," and are a critical area of focus for teacher education.

Rethinking Common Assumptions

Despite the fact that subject matter knowledge is logically central to teaching (Buchmann, 1984), it rarely figures prominently in teacher preparation. Constraints on formal preservice teacher education (e.g., too little time, too few credits) are often offered as the explanation for the fact that the subject matter preparation of teachers is left to precollege and "liberal arts" college mathematics classes. This typical arrangement reflects implicit and questionable assumptions about the understandings of mathematics that prospective teachers bring to teacher education, about what they need to know in order to teach mathematics, and where and when that can best be learned.

Traditional School Mathematics Content is Simple

This first assumption underlies the following three and is based on the notion that topics such as addition, decimals, graphing, and factoring are not very complicated. Implicitly, the message is: If you can "*do*" these topics, then you can teach them. Given this assumption, it is logical for teacher educators to leave subject matter preparation to precollege and college math classes and to concentrate on teaching methods and knowledge of children instead. This assumption is founded, however, on an unexamined conception of subject matter knowledge for teaching mathematics, one that accepts remembering and doing as indices of mathematical understanding.

Assuming that the content of first-grade mathematics is something any adult understands is to doom school mathematics to a continuation of the dull, rule-based curriculum that is so widely criticized. Throughout our interviews, many college students had difficulty working beneath the surface procedural level of so-called "simple" mathematics.

The discussion of division of fractions in this article shows the insufficiency of this conception of knowledge by illustrating what Duckworth (1987) refers to as the "depths and perplexities of elementary arithmetic." Lampert's work (1985, 1986, in press) also reveals in detail the mathematical complexity of the "simple" content of the school curriculum. However, without reviewing and explicating the meanings underlying this simple content, many prospective teachers will be unprepared to do

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more than teach rules such as "invert and multiply."

Elementary and Secondary School Mathematics Classes Can Prepare Teachers to Teach Mathematics

This assumption flows logically from the first. Whatever the contributions of college mathematics study to teachers' disciplinary knowledge, a large part of what they *teach* is material that they studied in elementary and secondary school. The fact that prospective teachers study very little school mathematics content as part of their formal teacher preparation implies that this second assumption is widely shared.

Our research sheds some light on the validity of this assumption. In order to respond to the questions and tasks on the questionnaire and in the interviews, the teacher candidates drew on what they had learned in school. Even the mathematics majors, most of whom had taken over seven college math courses, had to recall what they had learned in their elementary and high school classes in order to answer the questions. When they did this, seeking particular mathematical concepts, procedures, or even terms, they found loose fragments—rules, tricks, and definitions. Most did not find meaningful understanding or even the knowledge with which to figure out such understandings on the spot.

While troubling, it is also unsurprising given the widespread criticism of the algorithmic understanding fostered in many math classrooms (e.g., Davis & Hersh, 1981; Good, Grouws, & Ebmeier, 1983; Goodlad, 1984; Madsen-Nason & Lanier, 1987; Stodolsky, 1988; Wheeler, 1980). Our findings suggest that what prospective teachers have learned in their precollege mathematics classes is unlikely to be adequate for teaching mathematical concepts and procedures meaningfully.

At the same time, the image of mathematics implicit even in good elementary and secondary mathematics classrooms (Schoenfeld, in press), as well as in our culture, is one of a linear, rule- and fact-filled body of knowledge in which perspective, interpretation, and argument are irrelevant. Prospective teachers are therefore unlikely to acquire an appropriate view of the discipline as a result of precollege mathematics experience. The view they do hold is likely to shape not only the way in which they teach mathematics once they begin teaching but also the way in which they approach *learning to teach* mathematics (Ball, 1988): A vicious cycle emerges.

Majoring in Mathematics Ensures Subject Matter Knowledge

Some people do not make the second assumption. In fact, some completely overlook prospective teachers' precollege mathematics instruction and assume instead that subject matter preparation for teaching occurs at the university. Others would remedy the shortcomings of precollegiate mathematics education through university coursework. Many recent proposals for reforming teacher education and certification (e.g., Carnegie Task Force, 1986; Holmes Group, 1986) recommend that elementary teachers specialize in an academic discipline. Other reforms propose to certify college graduates who have completed an academic major but have had no teacher education. Underlying such proposals is the assumption that the study entailed in a college major can equip a prospective teacher with a deep and broad understanding of subject matter.

In mathematics, however, we are finding less difference in substantive understanding between elementary and secondary teacher candidates than one might expect (or hope). Although the latter, because they are mathematics majors, had taken more mathematics, this did not seem to afford them substantial advantage in articulating and connecting underlying concepts, principles, and meanings.

Some propose that this is due largely to the poor academic caliber of teacher education students. However, our interviews with mathematics majors who are not plan-

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ning to teach (not reported in this article) do not support this suggestion. They, too, struggle with making sense of division with fractions, relating mathematics to the real world, and coming up with explanations that go beyond the restatement of rules. Furthermore, many of the secondary teacher candidates in our sample are good students, with impressive college entrance exam scores and high grade point averages in their college math courses. A more plausible explanation for the mathematics majors' problems is that even successful performance in traditional math classes does not necessarily develop the kinds of understanding needed to teach mathematics for understanding, for success in these classes often derives from memorizing formulae and performing procedures. Moreover, studying calculus does not usually afford students the opportunity to revisit or extend their understandings of arithmetic, algebra, or geometry, the subjects they will teach.

We are also seeing few differences in ideas *about* mathematics between the elementary candidates and the mathematics majors. Most of the prospective teachers in both groups tended to see mathematics as a body of rules and facts, a set of procedures to be followed step by step, and they considered rules as explanations. Clearly, some secondary candidates did not see mathematics this way, while all the elementary candidates did. However, the overlap between the two groups was larger than one might expect, given that the secondary candidates had had far more opportunity to be immersed in the discipline than had the elementary candidates.

Our data suggest that the biggest difference between the two groups lies in the area of feelings. Most of the secondary candidates were confident that they knew the mathematics and were less tentative in their responses. If they could not figure something out, they assumed they were "rusty." The elementary candidates were more anxious and more convinced that they did not know mathematics. Unlike the mathematics majors, they tended to blame their gaps in knowledge on the arbitrariness of the subject and on their own inadequacies.

What is not at all clear is the extent to which the experience of majoring in mathematics contributed to these differences, compared with, for example, differences that derive from other social, cultural, and individual influences. Furthermore, recency was a factor. Some of the elementary candidates had not had a mathematics class for over 3 years, while the secondary candidates were all currently enrolled in mathematics courses.

Clearly, being anxious about mathematics is a disposition not well suited to teaching math. Hilton (1980) writes about the cyle of poor mathematics education and math anxiety that results from having ill-prepared and anxious teachers teaching mathematics. However, confidence, coupled with thin and rule-based understanding, can pose a threat to student learning if teachers confidently proclaim wrong ideas or portray mathematics in misleading ways. In any case, requiring teachers to major in mathematics, or even increasing the mathematics course requirements for prospective teachers, will not necessarily ensure increases in their substantive understanding or changes in their ideas or feelings about mathematics.

Conclusion

In deciding on the content and focus of preservice teacher education, teacher educators must ask themselves: Of all the things teachers need to know, which do prospective teachers already know? Which can be learned on the job from experience? Our data suggest that the mathematical understandings that prospective teachers bring are inadequate for teaching mathematics for understanding. And, although teachers claim to learn subject matter from teaching it, this is an empirical question, and one as yet unanswered (Ball & McDiarmid, in press). Common sense would

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suggest that, while teachers may figure out the meaning of division with fractions from teaching it, they do not necessarily uncover the concepts and principles underlying all the many rules they have learned nor the connections among them, and they probably do not change their ideas *about* mathematics.

Although it is widely acknowledged as a central component of what teachers need to know, the subject matter preparation of teachers is rarely the focus of any phase of teacher education. Instead, everyone is willing to assume that it will happen somewhere else: prior to college, in liberal arts classes, from teaching. What we are learning about the understanding of mathematics that prospective teachers bring with them to teacher education suggests the danger of this assumption and points to the need to make it a central focus. Doing that requires not only changes in emphasis, since much mathematics teaching does not produce the kind of understanding of mathematics that teachers need. Attending seriously to the subject matter preparation of elementary and secondary math teachers implies the need to know much more than we currently do about how teachers can be helped to transform and increase their understanding of mathematics, working with what they bring and helping them move toward the kinds of mathematical understanding needed in order to teach mathematics well.

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1. Readers may obtain copies of the NCRTE questionnaire and interviews by writing to the National Center for Research on Teacher Education, 116 Erickson Hall, Michigan State University, East Lansing, MI 48824.

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