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4

Interweaving Content and Pedagogy in Teaching and Learning to Teach: Knowing and Using Mathematics

Deborah Loewenberg Ball and Hyman Bass

INTRODUCTION[1]

Suppose you posed four numbers—7, 38, 63, and 90—to a class and asked the students to identify which of the numbers were even. And suppose, further, that you got this paper back from one of the students, with none of the numbers circled:

7 38 63 90

What would you make of this? Is this answer surprising or predictable? What might this student actually know? What number or numbers would you pose next to find out with more precision what the student thinks? Why would that selection be useful?

Thinking about this and figuring out what to do next this is one of many examples of the kind of mathematical problem solving in which teachers regularly engage. Although no teacher we have ever met could not correctly identify which numbers are even in the preceding list, understanding what there is to know about even numbers goes beyond being able to do this oneself and is critical to teaching well.

Identifying any even number entails knowing a definition for even numbers and being able to use that definition for any number. Viable definitions include:

Fair share: A number N is even if it can be divided into two (equal) parts with nothing left over (algebraically, $N = 2 \times k$; i.e., $k + k$).

Pair: A number N is even if it can be divided into twos (pairs) with nothing left over (algebraically, $N = k \times 2$; i.e., $2 + 2 + 2 + \ldots + 2$ [k terms]).

Alternating: The even and odd numbers alternate on the number line.

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Units digit: A number is even if its units digit is even (e.g., 712 is even because 2 is even).

But these definitions are not enough in themselves. One would need to know, first of all, how to use them to consider and determine the status of specific numbers. This is particularly salient with the units digit definition. This criterion can be easy to use, for, because one does not have to worry about any digit other than the units, one can deploy it routinely. However, it is more subtle to explain. Justifying it requires understanding place value, for this definition is based on understanding the decomposition of a number represented in decimal form—that is, that $712 = 7 \times 10^2 + 1 \times 10^1 + 2 \times 10^0 = 700 + 10 + 2$. If one does not understand that this is the basis on which the definition is founded, one may get confused if one forgets the algorithm; 712 can look ambiguous, for the 7 and the 1 are both odd.

Another important understanding is to know the domain to which these definitions are usefully and conventionally applied. (Even the appreciation that this is a fundamental mathematical question about a definition is an important sensibility.) For example, are fractions typically categorized as even or odd? Is zero?

Third, one should have good sense of when each definition might be useful. For example, the units digit definition is useful for large numbers; the alternating definition is cumbersome for any but very narrow intervals in which one already has an established referent (e.g., with small numbers, or with large numbers where some neighboring number is already known to be even).[2]

Finally, one would want to understand how the four definitions compare: Why do they each work to identify the same set of numbers? How might one explain these correspondences *mathematically?*

Knowing and being sensitive to all these things, and being able to use them in the context of the student's response, can equip one to consider plausible reasons why a child might not mark any of the numbers. Seven is not even, and, like the even/odd status of each of the digits, can be simply memorized as such. Thirty-eight includes an odd digit as well as an even one, and one might consider it "mixed." Sixty-three is not even, and a child might consider it mixed (as 38) or might use one of the other definitions to establish it as odd. Ninety packs a double mathematical issue: 9 is odd, and for the same reason as 38, might present difficulties. Moreover, 0 might be considered odd, or neither even nor odd.

Knowing and being sensitive to all these kinds of things and being able to use them is also critical to be able to manage other kinds of situations that might arise. A child might ask why the units digit definition works. Another might ask whether 1/2 (or 2/3) is even. Children often wonder about the status of zero. Managing these real situations demands a kind of deeply detailed knowledge of mathematics and the ability to use it in these very real contexts of practice. This chapter draws from work we have been doing to understand the mathematical knowledge entailed by teaching (e.g., Ball, 1999; Ball & Bass, 2000). We begin by looking backward, acknowledging that this question is far from new and that our work builds on substantial recent progress to address it.

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CHASMS IN KNOWING AND LEARNING PRACTICE

At the turn of the 20th century, John Dewey (1904/1964) articulated a fundamental tension in the preparation of teachers—that of the "proper relationship" of subject matter and method. At the turn of the 21st century, this tension endures. In fact, many of the same questions persist. On the one hand, to what extent does teaching—and hence, learning to teach—depend on the development of knowledge of subject matter? On the other, to what extent does it rely on the development of pedagogical method?

Clearly the answer must be, "it depends on both." Yet, across the century, this tension has continued to simmer, with strong views on both sides of what is unfortunately often seen as a dichotomy. Policymakers debate whether teachers should major in education or in a discipline. Others argue that what matters is caring for students as well as skills at working effectively with diverse learners. Dewey's (1904/1964) conception of the relationship of subject matter knowledge and method was sophisticated and subtle—so much so that 100 years later, his idea is still elusive. He wrote:

> Scholastic knowledge is sometimes regarded as if it were something quite irrelevant to method. When this attitude is even unconsciously assumed, method becomes an external attachment to knowledge of subject matter. (p. 160)

This separation of substance from method, he argued, fundamentally distorted knowledge. How an idea is represented is part of the idea, not merely its conveyance.

Dewey also believed that good teachers were those who could recognize and create "genuine intellectual activity" in students, and he argued that methods of such activity were intimately tied into disciplines. Subject matter, he believed, was the embodiment of the mind, the product of human curiosity, inquiry, and the search for truth. Teachers who were accustomed to viewing subject matter from the perspective of its growth and development would be prepared to notice nascent intellectual activity in learners. Such individuals would know subject matter in ways that prepared them to hear and extend students' thinking. To do this, he argued, teachers would need to be able to study subject matter in ways that took it back to it "psychical roots" (p. 162).

Despite these prescient ideas that intimately interweave knowledge and learning, teacher education across the 20th century has consistently been severed by a persistent divide between subject matter and pedagogy. This divide has many traces. Sometimes it appears in institutional structures as the chasm between the arts and sciences and schools of education, or as the gulf between universities and schools (Lagemann, 1996). Sometimes the divide appears as fissures in the prevailing curriculum of teacher education, separated into domains of knowledge, complemented by "experience"—supervised practica, student teaching, practice itself. In all of these, the gap between subject matter and pedagogy fragments teacher education by fragmenting teaching.

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In recent years, in yet another peculiar fragmentation, commitments to equity and concerns for diversity have often been seen as in tension with a focus on content in teacher education. Courses in multiculturalism contend with subject matter courses for space in the professional curriculum. Yet subject matter understanding is essential in listening flexibly to others and hearing what they are saying or where they might be heading. Knowing content is also crucial to being inventive in creating worthwhile opportunities for learning that take learners' experiences, interests, and needs into account. Contending effectively with the resources and challenges of a diverse classroom requires a kind of responsibility to subject matter without which efforts to be responsive may distort students' opportunities to learn (Ball, 1995). Moreover, the creativity entailed in designing instruction in ways that are attentive to difference requires substantial proficiency with the material.

The overarching problem across these many examples is that the prevalent conceptualization and organization of teachers' learning tends to splinter practice, and leave to individual teachers the challenge of integrating subject matter knowledge and pedagogy in the contexts of their work. We assume that the integration required to teach is simple and happens in the course of experience. In fact, however, this does not happen easily, and often does not happen at all.

QUESTS TO BRIDGE THE CHASM

These chasms in our ways of thinking about content and pedagogy have plagued researchers, teacher educators, and policymakers. And although perhaps not in these forms, these issues have plagued teachers as well, for our incomplete understanding of how content matters in practice has often left practitioners under-prepared for their work, challenged by the problems and mysteries that arise with distressing regularity.

That teachers' own knowledge of the subject affects what they teach and how they teach seems so obvious as to be trivial. However, the empirical support for this "obvious" fact has been surprisingly elusive.[3] And although conceptions of what is meant by "subject matter knowledge," as well as valid measures thereof, have been developing, we lack an adequate understanding of what and how mathematical knowledge is used in practice.

What are the weaknesses in current widely shared ideas about teacher content knowledge? First, subject matter knowledge for teaching is often defined simply by the subject matter knowledge that students are to learn—that is, by the curricular goals for students. Put simply, most people assume that what teachers need to know is what they teach. Many would also add to the list, arguing that teachers must know more in order to have a broad perspective on where their students are heading. Nothing is inherently wrong with this perspective. However, to assume that this suffices is to assume that the enactment of the curriculum relies on no other mathematical understanding or perspective.

Furthermore, the use of mathematical knowledge in teaching is often taken for granted. The mathematical problems teachers confront in their daily work—such

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as the simple case at the beginning of this chapter—are left unexplored, the occasions that require mathematical sensitivity and insight unprobed. Hence, the content and nature of the mathematical knowledge needed in practice is insufficiently understood. Moreover, the role played by such knowledge is also left unexamined.

In 1986, Lee Shulman, Suzanne Wilson, Pamela Grossman, and Anna Richert introduced "pedagogical content knowledge" to the lexicon of research on teaching and teacher education (Shulman, 1986). The term called attention to a special kind of teacher knowledge that links content and pedagogy. In addition to general pedagogical knowledge and knowledge of the content, argued Shulman and his colleagues, teachers needed to know things like what topics children find interesting or difficult or the representations most useful for teaching a specific content idea. These scholars identified and named a unique kind of knowledge that intertwines aspects of teaching and learning with content.

The introduction of pedagogical content knowledge brought to the fore questions about the content and nature of teachers' special subject matter understanding. Consider the following example. As an experienced classroom teacher, Ball knows that figuring out what her fifth graders know about decimals depends in part on her knowledge of number systems and in part on her understanding of the kinds of errors that 10–year-olds typically make. For example, she knows that they will often confuse .5 with .05 and that they draw this confusion, in part, from their prior conviction that 5 and 05 *are* the same number. This means that a fifth-grade teacher needs to understand a lot about the base 10 number system and about positional notation. When a fifth grader asks, "Where is the 'oneths' place?," a teacher needs to be able to hear that this likely emanates from a 10–year-old's reasonable expectation that if there is a ones place to the left of the decimal point, and a tens place to the left of that, there should be a symmetry to the right of the decimal. In other words, why is the place immediately to the right the tenths place, and not a "oneths" place? But being able to hear this student is not enough. Why *isn't* there a "oneths" place? Answering this for oneself requires a certain explicit understanding of place value and of the multiplicative structure of the base 10 system that goes beyond being able to name the places (ones, tens, hundreds, etc.) or read numbers. And then, beyond being clear about the mathematics, helping a fifth grader understand the missing "oneths" requires an intertwining of content and pedagogy, or pedagogical content knowledge.

This kind of understanding is not something a mathematician would have, but neither would it be part of a high school social studies teacher's knowledge. It is special to the teaching of elementary mathematics. Pedagogical content knowledge—representations of particular topics and how students tend to interpret and use them, for example, or ideas or procedures with which students often have difficulty—describes a unique subject-specific body of pedagogical knowledge that highlights the close interweaving of subject matter and pedagogy in teaching. Bundles of such knowledge are built up by teachers over time as they teach the same topics to children of certain ages, or by researchers as they investigate the teaching and learning of specific mathematical ideas.

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Liping Ma (1999) describes the "knowledge packages" that are part of the knowledge of the 72 Chinese elementary teachers whom she interviewed. These packages constituted a refined sense of the organization and development of a set of related ideas in an arithmetic domain. The teachers in her study had clearly articulated ideas about "the longitudinal process of opening up and cultivating such a field in students' minds" (Ma, 1999, p. 114). Their knowledge packages consisted of key ideas that "weigh more" than other ideas in the package, sequences for developing the ideas, and "concept knots" that link crucially related ideas. Ma's notion of "knowledge packages" represents a particularly generative form of and structure for pedagogical content knowledge.

Our work builds on pedagogical content knowledge by complementing what it offers for practice. Pedagogical content knowledge is a special form of knowledge that bundles mathematical knowledge with knowledge of learners, learning, and pedagogy. These bundles offer a crucial resource for teaching mathematics, for they can help the teacher anticipate what students might have trouble learning, and have ready alternative models or explanations to mediate those difficulties. Because one big challenge of teaching is to integrate across many kinds of knowledge in the context of particular situations, the fact that there are patterns in and predictability to what students might think, and that there are well-tried approaches to develop certain mathematical ideas, can help manage this challenge. However, a body of such bundled knowledge may not always equip the teacher with the flexibility needed to manage the complexity of practice. Teachers also need to puzzle about the mathematics in a student's idea, analyze a textbook presentation, consider the relative value of two different representations in the face of a particular mathematical issue. To do this, we argue, requires a kind of mathematical understanding that is pedagogically useful and ready, not bundled in advance with other considerations of students or learning or pedagogy.

Although pedagogical content knowledge provides a certain anticipatory resource for teachers, it sometimes falls short in the dynamic interplay of content with pedagogy in teachers' real-time problem solving. No repertoire of pedagogical content knowledge, no matter how extensive, can adequately anticipate what it is that students may think, how some topic may evolve in a class, the need for a new representation or explanation for a familiar topic. Moreover, more than one mathematical issue or goal may be at play at once, requiring simultaneous consideration of different content within the pedagogical context. That is, as they meet novel situations in teaching, teachers must bring to bear considerations of content, students, learning, and pedagogy. They must reason, and often cannot simply reach into a repertoire of strategies and answers. When teachers look at student work, choose a text to read, design a task, or moderate a discussion, they must attend, interpret, decide, and make moves. Their thinking depends on their capacity to call into play different kinds of knowledge, from different domains. An endless barrage of situations—of what we are beginning to understand as mathematical problems to be solved in practice—entails an ongoing use of mathematical knowledge. It is what it takes *mathematically* to manage these routine and nonroutine problems that

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has preoccupied our interest as we seek to build on the groundbreaking research on pedagogical content knowledge.[4] It is to this kind of *pedagogically useful mathematical* understanding that we attend in our work.

This chapter draws on work that we—Bass, a professional mathematician, and Ball, an educational researcher and elementary school teacher—began in 1996. We have been using our distinct disciplinary perspectives to probe the interplay of mathematics and pedagogy in practice.[5] The problem on which we have been working is one that is central to both professional education and instructional improvement: What mathematical knowledge is needed to teach elementary school mathematics well? How must it be understood and held so that it is available for use? Working with primary records of teaching and learning—videotapes, student work, curriculum materials, teacher notes[6]—we have been trying to analyze and articulate ways in which mathematical insight, sensibilities, and knowledge are entailed by the practice of teaching mathematics.[7]

Our research turns the usual approach to this problem on its head. Rather than identifying the mathematical knowledge needed for teaching by examining the curriculum, or by interviewing teachers, we begin instead with an examination of practice itself. Examining the curriculum, although useful, is incomplete for it fails to anticipate the mathematical demands of its enactment in classrooms. Interviewing teachers, though also valuable, is incomplete because it infers teaching's mathematical demands from teachers' accounts of what they think or would do. Without knowing whether the teachers interviewed are actually able to help all students learn mathematics well, what they report remains in some significant ways unwarranted. In any case, neither of these approaches bridges the gap between knowledge and practice, except indirectly through inference or report.

We seek to complement the examination of curriculum and of what experienced teachers know with a mathematical analysis of core activities of mathematics teaching. We intend with the phrase "core activities" to include such things as figuring out what students know; choosing and managing representations of mathematical ideas; appraising, selecting, and modifying textbooks; deciding among alternative courses of action; steering a productive discussion—and we seek to identify the mathematical resources entailed by these teacher activities.

In this work, we see teaching as a practice embedded with both regularities and endemic uncertainties. For example, some topics—such as arithmetic with integers, probability, and fractions—are quite often difficult for students. Certain ways of approaching these topics—particular representations and methods of development—can help mediate these difficulties. Oft-used mathematical tasks can be mapped by the range of typical approaches used by students of a given age (Stigler & Hiebert, 1999). Being prepared for these regularities of practice is enabled by what we think of as "pedagogical content knowledge," clusters that embed knowledge of mathematics, of students, and of pedagogy. However, no amount of pedagogical content knowledge can prepare a teacher for all of practice, for a significant proportion of teaching is uncertain. Many others have written about the uncertainties of teaching (Ball, 1996; Cohen, in preparation; Lampert, 1985; Lampert &

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Ball, 1999; Lortie, 1975; McDonald, 1992), citing numerous sources of uncertainty and providing analyses of the consequences for teachers and their work. Sources of uncertainty in teaching derive in part from its foundations: the impossibility of knowing definitely what students know, and the necessarily incomplete nature of knowledge of teaching, and even the inherent indeterminacy of mathematical knowledge itself that is germane to a given instructional context (Ball, 1996). Because teaching practice is constructed in the interplay of mathematics, students, and pedagogy, considerable parts of teachers' work are embedded with uncertainties. Acknowledging the uncertainty of teaching does not mean that teachers cannot be prepared to know in practice. Quite the contrary: Knowing mathematics for teaching must take account of both the regularities and the uncertainties of practice, and must equip teachers to know in the contexts of the real problems they have to solve.

Because we are interested in the mathematical entailments of practice, we are interested not only in what teachers must know, but also how they must be able to use that knowledge (Cohen & Ball, 1999). "Knowing teaching is more than applying prior understandings. It also depends fundamentally on being able to know things *in the situation*" (Lampert & Ball, 1999, p. 38).

Our approach, a kind of "job analysis" of classroom teaching focused on the actual work, is rooted in these premises about practice and seeks to locate and analyze mathematics as it is used in practice. Such a mathematical perspective on the work of teaching can extend what we currently understand about the mathematical resources needed for teaching, the role of such resources in practice, and, by implication, what opportunities for teachers and prospective teachers need to be developed for them to be prepared to teach mathematics well.

KNOWLEDGE IN PRACTICE

We begin with two examples, each offering a closer look at a sliver of the work of teaching. Consider, first, the work of examining and preparing to teach a mathematics problem (Gelfand & Shen, 1993):

> Write down a string of 8's. Insert some plus signs at various places so that the resulting sum is 1,000.

At first glance, this problem may look trivial and uninteresting—one way of solving it entails simply adding 125 8's together. A closer look reveals that if several 8's are written together—888 or 88—many more solutions are possible. And working on the problem a little further reveals interesting and provocative patterns in the solution set. Figuring out how to organize the solutions is itself an interesting component of the work, and depending on how they are organized, different elements of the problem and its solutions are visible.

A teacher preparing to use this task must contemplate: Would this be a good problem for my students? What would it take to figure out the patterns and nu-

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ances? Is it worthwhile in terms of what students might learn? At least, it would be important to know what the problem is asking, whether it has one or many solutions, how the solutions might be found. How is it (or could it be) related to other parts of the curriculum? It seems obvious that the task entails some computation—for example, verifying any one solution—but what is the mathematical potential of the task? Are there important ideas or processes involved in the problem? What would it take to use this task well with students? It would help to know what might make the problem hard, and how students might get stuck, and anticipate what the teacher might do if they did. Would students find this interesting? What might it take to hook them on it?

Perhaps, on looking at this problem, a teacher would decide that it is interesting but a bit too difficult for her students. What would it take to make a mathematically similar problem that is a bit easier? At what grade levels would some mathematically equivalent but simpler version of this problem be accessible? How might one rescale the problem, for example, for third graders? For first graders—Could a similar problem structure be set up with Cuisenaire rods? Suppose, in contrast, the teacher worries that this problem is too easy. What would it take to make a more challenging, but again, mathematically similar task? What happens to the problem if one replaces 1,000 with other numbers, or 8 with some other digit? How might one modify the problem so that there are no solutions? Infinitely many solutions? This sort of analysis and preparation of a single math problem begins to reveal how much significant mathematical reasoning is entailed within the work of teaching.

We turn now to a second example. Unlike the preceding example, which provides a glimpse of the work of preparing to use a task with students, this example shows the work of using a task during class. In each example, we seek to remind the reader that the work of teaching, too often thought to be generic, is embedded with significant mathematical analysis and problem solving. Moreover, we seek to show that the mathematical resources entailed in such analysis and problem solving may not in fact be evident on the surface of the school curriculum. Simply looking at the math problem or considering the content on which students are working does not lead to a sufficient appreciation of the specific mathematical knowledge or sensibility that it takes to teach that problem or that content.

The following example, drawn from Ball's third-grade class, centers on the children's work on subtraction of multidigit numbers, learning the conventional place value algorithm, and also using other procedures. We drop in near the beginning of class. The students are discussing solutions to the simple problem:

> Joshua ate 16 peas on Monday and 32 peas on Tuesday.
> How many more peas did he eat on Tuesday than he did on Monday?

Several solutions are offered. Sean goes to the board and, counting up from 16 to 32 on the number line, explains,

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I went sixteen . . . , 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16 and I ended up on 32.

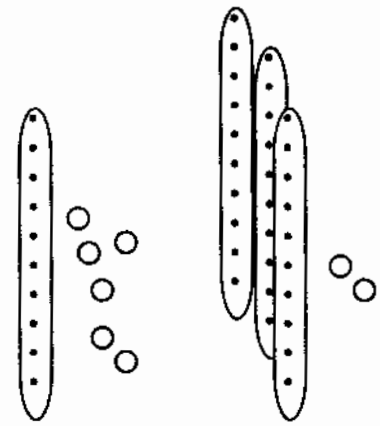
Lucy agrees with him, saying that she "got the same answer and did it the same way." Riba concurs, and offers to "prove that his answer is right." She explains:

Riba: Because a half of . . . a half of 32 would be 16.

Ball: Uh huh. And how does that prove that his answer is right?

Riba: I . . . because . . . it's . . . it's a half of 32. Sixteen is a half of 32. That proves his answer.

Ball, not sure what to do with Riba's idea, continues on. Betsy, speaking mostly to Sean, says that she used beansticks[8] to solve the problem and that she has gotten 15. She goes up to the overhead projector and lays out representations of 16 and 32:



She begins matching individual beans, and then trades a beanstick for 10 loose beans. She continues matching individual beans with others and then one beanstick from each group. Mei objects to Betsy's method of representing both 16 and 32 beans on the overhead:

If you do that you'll . . . if you want to do 32 take away 16 or something like that, you'll need to take away only 16 and . . . and you shouldn't be putting on 32 *and* 16 up there.

Betsy tries to explain. She counts out her beans and sticks, saying that the 16 "what he ate on Monday" and the 32 was what he ate on Tuesday. Then she tries to justify her method:

So, what I'm doing is I'm seeing how much more he ate by putting them together. And when you put them together, you're matching it up just like . . . just about the same way Sean would. But, see instead of adding them together, I'm putting them together like this. And then, since it has a match, I'm putting it down here. So that means you don't count these ones because those are the one that have a match. So, I keep . . . I did this

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> and then see I can't take 4 away from 10. So, what I do is take this in for 10 beans and then I match these together. Then, I counted how many I had.

Mei seems unconvinced. Betsy goes through her solution again. With the teacher's help, she narrates the placement of beansticks and what they represent. She explains the processes she is using to compare the two amounts:

> See, I'm taking these two beans and matching them with these two beans. I'm taking these two beans and matching them with these two beans. These two beans and matching them with them.

After doing it, slowly, with explanations, Betsy arrives at the correct answer, 16, which she recognizes is inconsistent with her original answer, 15. Experiencing, in front of the class, as well as in her own mind, the disequilibrium of this contradiction, she proceeds, with the invitation of her teacher and indulgence of her classmates, to reenact carefully the matching of the 16 beans with part of the 32 beans, and, once again, finds that 16 beans remain unmatched. At this point she places (a still slightly tentative) confidence in the answer, 16. Moreover she retracts her earlier notion that her solution is like the method of "counting up" on the number line used by Sean. The class goes on to see yet another solution, presented by Cassandra, hers using the conventional subtraction algorithm. This prompts Sean to offer

$$\begin{array}{r} {}^{1}16 \\ +16 \\ \hline 32 \end{array}$$

for another approach.

By the end of class, the children have seen six different methods, and worked back and forth between the symbolic representations and the concrete forms. They have discussed why some children used subtraction while others added, and they have tried to identify similarities and differences across the methods. This apparently simple word problem has taken the teacher and the children deep into some significant mathematical territory, invisible on the surface of the problem. How are subtraction and addition related, in both symbolic and concrete models? How are the comparison and "take-away" interpretations of subtraction related? How do the beanstick representations map onto the symbolic forms, and how do the processes used by each child map onto each of these? How, for example, does Betsy's method of matching compare with Sean's "counting up" method? Was Mei's objection that Betsy should not represent both the 32 and the 16 legitimate? What is Betsy doing, and how can one reconcile it with Mei's objection? What is Riba thinking when she seeks to "prove" Sean's answer by talking about 16 being "half of" 32?

When teachers hold class discussions, they make decisions about which (and whose) ideas to pick up and pursue and which (and whose) to suspend or let drop. The teacher formulates probes, pushes students, offers hints, and provides explana-

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tions. Students get stuck: What does one do to help them remobilize? None of these tasks of teaching can be carried out generically. No matter how committed one is to caring for students, to taking students' ideas seriously, to helping students develop robust understandings, none of these tasks of teaching is possible without making use in context of mathematical understanding and insight.

Herein lies a fundamental difficulty in learning to teach, for despite its centrality, usable mathematical knowledge is not something teacher education, in the main, provides effectively. Although some teachers have important understandings of the content, they often do not know it in ways that help them hear students, select good tasks, and help all their students learn. No prospective or practicing teacher we know is unable to solve the problem of Joshua and the peas. But the mathematical issues embedded in the enactment of this task in class are not trivial. Being able to help Riba develop her idea, for example, would require that a teacher be sensitive to the nature of mathematical reasoning and the need for the steps in an argument to be developed, at a level of granularity appropriate for the context (Ball & Bass, 2000). Riba's claim—that 16 is half of 32—is correct; the issue is not this, but rather how this can support a proof of Sean's claim. The teacher would need to "hear" (and guess) the unspoken elements of her proof and be able to think of what to ask or say that might help Riba develop her idea enough so that the other children could consider what she is saying.[9]

Thus, teachers need mathematical knowledge in ways that equip them to navigate these complex mathematical transactions flexibly and sensitively with diverse students in real lessons. Not providing this undermines and makes hollow efforts to prepare high-quality teachers who can reach all students, teach in multicultural settings, and work in environments that make teaching and learning difficult. Despite frequently heard exhortations to teach all students, many teachers are unable to hear students flexibly, represent ideas in multiple ways, connect content to contexts effectively, and think about things in ways other than their own. For example, in their study of a middle school teacher's attempt to teach the concept of rate, Thompson and Thompson (1994) highlight the crucial role played by language. They describe, vividly, the situation of one teacher who, although he understood the concept of rate himself, was restricted in his capacity to express or discuss the ideas in everyday language. Satisfied with computational language for his own purposes, when these did not help students understand, he was not able to find other means of expressing key ideas. In addition, teachers may not be able to size up their textbooks and adapt them effectively; they may omit topics central to students' futures or make modifications that distort key ideas. They may substitute student interest for content integrity in making subject matter choices.

A recent analysis provides a glimpse of the importance of the distinction between knowing how to do math and knowing it in ways that enable use in practice. This distinction is key to understanding how mathematics knowledge matters in good teaching. In general, astonishingly little empirical evidence exists to link teachers' content knowledge to their students' learning. One hypothesis has been that what is being measured as "content knowledge" (often teachers' course attain-

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ment) is a poor proxy for pedagogically usable subject matter understanding. However, in their 1997 *Sociology of Education* article describing their analysis of data from the National Education Longitudinal Study of 1988, Rowan and his colleagues report strong positive correlations between teachers' responses to items designed to measure the use of mathematical knowledge in teaching and their students' performance (Rowan, Chang, & Miller, 1997).[10] This analysis provides some confirmation that understanding the use of mathematics in the work of teaching is a critical area ripe for further examination. It is not just what mathematics teachers know, but how they know it and what they are able to mobilize mathematically in the course of teaching.[11] Though less easily quantified than other indices, such as courses taken, it is this pedagogically functional mathematical knowledge that seems to be central to effective teaching.

An important challenge for teacher education at the beginning of the 21st century is to bridge the chasm identified by John Dewey almost 100 years ago. Our schools are more diverse than ever and we ask more of both teachers and students. What would it take to bring the study of content closer to practice and prepare teachers to know and be able to use subject matter knowledge effectively in their work as teachers?

CLOSING THE GAP: DEVELOPING AND USING KNOWLEDGE IN PRACTICE

Three problems stand out; problems that we must solve if we are to meet this challenge to prepare teachers who not only know content but can make use of it to help all students learn. One problem concerns identifying the content knowledge that matters for teaching, a second regards understanding ways in which such knowledge needs to be held and a third centers on what it takes to learn to use such knowledge in practice.

What Mathematics Is Entailed by Teaching?

First, we would need to reexamine what content knowledge matters for good teaching. Subject matter knowledge for teaching has too often been defined by the subject matter knowledge that students are to learn. Put simply, many assume that what teachers need to know is what they teach—plus a broad perspective on where their students came from and are heading. Nothing is inherently wrong with this perspective. However, the lists of what teachers should know that are produced by analyzing the school curriculum are long and largely arbitrary. Little is known about how "knowing" the topics on these lists affects teachers' capabilities. The unexamined conviction that possessing such knowledge is all that teachers need to know has blocked the inquiry needed to bring together subject matter and practice in ways that would enable teacher education to be more effective.

Instead of beginning solely with the curriculum, our understanding of the content knowledge needed in teaching must start also with practice. We must under-

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stand better the work that teachers do, and analyze the role played by content knowledge in that work.

Consider, on one hand, the subtraction problem. We have what looks like a simple calculation, 32 – 16, embedded in a simple story problem. Viewed purely from the point of view of curriculum, this lesson entails some rudimentary knowledge of place value, and of how the algorithm for subtraction with borrowing works.

Consider, on the other hand, the mathematical themes and events encountered as the problem unfolded in the children's work. Sean counted the distance up from 16 to 32 on the number line, finding 16. Riba observed that 16 is half of 32, proposing that this "proved" the correctness of Sean's answer. Betsy used base 10 bean sticks to construct a physical matching between a collection of 16 beans with a part of a separate collection of 32 beans, and then counted the unmatched beans, which involved trading in a 10–stick for 10 individual beans. The result of this, 16, contradicted Betsy's original answer of 15, which she then tried to reconcile. Mei protested that Betsy should not have displayed a separate collection of 16 beans, but only the collection of 32 beans. The physical presence of the 16 beans looked to Mei as though they were being added, not taken away. Sean used the symbolic addition, 16 + 16 = 32, as the basis for another derivation of the answer 16. In the end, the students produced six mathematically distinct approaches to the problem.

What mathematical demands are created by this lesson, beyond knowledge of the symbolic algorithm for subtraction with borrowing, and of the underlying place value system? First are the several models or representations of the problem. Symbolically, the subtraction, 32 – 16 = ?, is equivalent to the missing addend problem, 16 + ? = 32. There are the two interpretations of subtraction, "take away" (if you take 16 away from 32, how many are left?) and "compare" (how many more is 32 than 16?). Second are the many representations of these. One is on the number line, counting 16 down from 32, or counting the distance up from 16 to 32 (as done by Sean). Other representations use bean sticks, either removing 16 from 32 beans (which entails trading a 10–stick for 10 individual beans) or matching 16 beans with some of the 32 beans, and counting what remains (as done by Betsy; which also entails trading a 10–stick for 10 individual beans). Finally, given the multiple approaches produced by the students, there is a profound mathematical imperative to inspect, analyze, and reconcile them.

Permeating this lesson is also a set of class norms for how to justify mathematical claims. This is another large domain of mathematical knowledge on which the teacher must draw, for example in assessing, by both teacher and students, the different student responses, and in evaluating and processing claims such as Riba's pretended "proof" of Sean's answer. What kind of functional knowledge of proof, of mathematical justification, is germane to elementary instruction?[12]

This kind of direct examination of practice seeks to uncover what teachers need to know and be sensitive to about content in order to teach well. This kind of analysis may bring some surprises. For example, in our research, we expected to see that concepts such as place value and decimal notation, the arithmetic of fractions, and so on, would be central—and they have been, as have operations and informal

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methods of reasoning. But beyond that, we have been struck by the unanticipated but recurrent prominence of certain mathematical notions. For instance, we have found that ideas about similarity, equivalence, mapping among representations, and even isomorphism emerge across many instances of ordinary and extraordinary teaching and learning. We have also uncovered salient issues involving mathematical language—symbolic notation and definitions of terms, their formation and expansion to larger mathematical domains (Ball & Bass, 2000). Similarly new notions are emerging from parallel work in the teaching and learning of history and science (Rose, 2000; Wilson, in press). Inquiries that begin with practice are revealing subject matter entailments of teachers' work that are not seen when we begin with lists of content to be taught that are derived from the school curriculum. These content demands emerge from analyzing the sorts of challenges with which teachers must contend in the course of practice, as they mediate students' ideas, make choices about representations of content, modify curriculum materials, and the like.

What Makes Mathematical Knowledge *Usable* for Teaching?

A second problem concerns *how* subject matter must be understood in order to be usable in teaching. We need to probe not just *what* teachers need to know, but to learn how that knowledge needs to be held and used in the course of teaching. Working on this problem requires examining the assumption that mathematically proficient people know the content sufficiently well to solve the mathematically implicated problems that arise in the course of teaching elementary students. We do not examine here the other sorts of knowledge they would need—of students, of teaching methods, of the contexts, of curriculum. We mean to refer here to how one must be able to understand mathematics in order to manage the deeply content-related issues that can arise.

Ma (1999) describes what she calls "profound understanding of fundamental mathematics" in terms of the depth, breadth, and thoroughness of the knowledge teachers need. "Depth," according to Ma, refers to the ability to connect ideas to the large and powerful ideas of the domain, whereas "breadth" has to do with connections among ideas of similar conceptual power. Thoroughness is essential in order to weave ideas into a coherent whole. In addition to the premium she places on connections, Ma also emphasizes flexibility as held in a multiplicity of representations and approaches. Drawing on Bruner's (1960) ideas about the "structure" of a discipline, Ma stresses the importance of teachers knowing and attending to the "simple but powerful basic concepts and principles of mathematics" (p. 122), and developing "basic attitudes" (p. 122)—for example, to seek to justify claims, to seek consistency in an idea across contexts, to know how as well as why. How such profound understanding of fundamental mathematics (PUFM) is used in practice is both dynamic and situated in contexts. Ma argues that teachers' knowledge of mathematics for teaching must be like an experienced taxi driver's knowledge of a city, whereby

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one can get to significant places in a wide variety of ways, flexibly and adaptively (p. 123).

It is to the question of use that we have been drawn. Looking at knowledge from the perspective of practice, and the actual work of teaching, we have been increasingly intrigued by the many moments in teaching when mathematical insight, knowledge, and sensibility matters. In the wide variety of mathematical issues, problems, and tasks that arise, we are struck with the variety of ways in which mathematics is entailed by practice.

Flexibility and adaptiveness are clear requirements of teaching. As Ma (1999) argues, teachers must be able to reorganize what they know in response to a particular context. To do this, one needs to be able to deconstruct one's own mathematical knowledge into less polished and final form, where elemental components are accessible and visible. We refer to this as *decompression*. Paradoxically, most personal knowledge of subject matter, which is desirably and usefully compressed, can be ironically inadequate for teaching. In fact, mathematics is a discipline in which compression is central. Indeed, its polished, compressed form can obscure one's ability to discern how learners are thinking at the roots of that knowledge. Knowing flexibly in and for teaching requires a transcendence of the tacit understanding that characterizes much personal knowledge (Polanyi, 1958). Because teachers must be able to work with content for students in its growing, not finished, state, they must be able to do something perverse: work backward from mature and compressed understanding of the content to unpack its constituent elements (Cohen, in preparation).

For example, they must be ready to hear students' ideas, and to hypothesize about their origin, status, and direction. And, in order to ascertain the opportunities for learning embedded in the examples and work that they assign, teachers must be able to decompose a mathematics task, considering its diverse possible trajectories of enactment and engagement. Teaching mathematics entails work with microscopic elements of mathematical knowledge, elements invisible that were, for someone with mature mathematical fluency, long ago covered up—or perhaps never even known. Speculating on why a six-year-old might write "1005" for "one hundred five," and not reading it as a mistaken count—"one thousand five"—requires the capacity to appreciate the elegance of the compressed notation system that adults use readily for numbers but that is not automatic for learners. After all, Roman numerals follow precisely the same structure as the young child's inclination, each element with its own notation—CV for "one hundred five"—without the "place value" core of our system. Being able to see and hear from someone else's perspective, to make sense of a student's apparent error or appreciate a student's unconventionally expressed insight requires this special capacity to unpack one's own highly compressed understandings that are the hallmark of expert knowledge. Even producing a comprehensible explanation depends on this capacity to unpack one's own knowledge, for an explanation works only if it is at a sufficient level of granularity—that is, if its logical steps are small enough to make sense for a particular

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learner or a whole class, based on what they currently know or do not know (Ball & Bass, 2000).

Being able to use mathematical knowledge involves using mathematical understanding and sensibility to reason about subtle pedagogical questions. What are the advantages and disadvantages of particular metaphors or analogies? Where might they distort the subject matter? For example, both "take away" and "borrowing" create problems for students' understanding of subtraction. These problems cannot be discerned generically, for they require a careful mapping of the metaphor against critical aspects of the concept being learned and against how learners interpret the metaphor. And knowing that subtraction is a particularly difficult idea for students to master is not something that can be seen from knowing the "big ideas" of the discipline. This kind of knowledge is quite clearly mathematical, yet formulated around the need to make ideas accessible to others.

These aspects of content knowledge help to illuminate the territory to which Dewey called attention almost a century ago, bridging the divide between content and pedagogy. However, teaching is a practice. It is, in Lampert's terms, "a thinking practice"—that is, it integrates reasoning and knowing with action (Lampert, 1998). Our tendency to focus either on its cognitive demands (teachers' knowledge, reasoning, decision making, reflection) or on its actions (teacher behavior, classroom management) is yet one more recent form of fragmentation in teacher education, and in particular in our efforts to help teachers acquire usable content knowledge.

How Might Teachers Develop Usable Mathematical Understanding?

Hence, a third problem we would have to solve is how to create opportunities for learning subject matter that would enable teachers not only to know, but to learn to use what they know in the varied contexts of practice (Ball & Cohen, 1999). Even with more grounded analyses of what there is to know and a more finely tuned conception of the nature of the understanding needed to teach, simply teaching such content may not solve the problems of use. How do teachers use content understanding in the context of practice to carry out the core activities of their work? How can opportunities for learning be designed that are aimed at helping teachers learn to use subject matter knowledge to figure out what their students know, to pose questions, to evaluate and modify their textbooks wisely, to design instructional tasks, to manage class discussions, to explain the curriculum to parents?

Some such work along these lines is already underway. One promising possibility is to design and explore opportunities to learn content that either simulate or are situated in the contexts in which subject matter is used—core activities of teaching.

Consider, for example, what is entailed in preparing and using academic tasks. As teachers construct or select a task, they analyze the nature and territory of the task and consider the curricular learning goals its engagement might support. They appraise its accessibility and challenge: For example, they examine whether it has

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multiple entry and exit points, whether it admits multiple solution strategies, and multiple solutions or levels of solution. They also size up whether it supports collective class work or is better suited to individual (home) work. At times, they seek ways of scaling the problem up or down in difficulty, linking the problem to other domains of the class work, and so on. In either teacher education or professional development settings, these deliberate opportunities for analysis and design could be used as sites for an integral part of the teacher's learning of mathematical content.

For example, with the 8's problem discussed earlier, mathematical analysis might start with construction of a solution, inspecting the methods used, then looking for other solutions, and further trying to find them all (seeing that there are only finitely many), trying to organize (or give structure to) the solution set, and contemplating ways of proving that one has all solutions. How many terms (addends) or how many digits does each solution involve? What are the patterns of these numbers? Further analyses could probe how these features are affected when various terms of the problem are varied, such as replacing 1,000 by another number, or 8 by another digit, or allowing other operations than +. These variations might produce versions that would challenge college students. On the other hand, one could try to model a "mathematically similar" version of this problem accessible to first graders. In each instance one could consider the design of enactment of the task with a given level of students, anticipating the likely results of student engagement, possible readings or misreadings of different formulations of the problem, and so on. Each of these analyses embeds crucial mathematical work, and as such, could be wielded to be critical points for teachers to learn mathematics.

As another example, some teacher educators use student work as a site to analyze and interpret what students know and are learning and, in so doing, to work on the content itself.[13] Another example lies in the use of videotape of classroom lessons or cases of classroom episodes (Lampert & Ball, 1998; Stein, Smith, Henningsen, & Silver, in press). Here the moves made by the teacher could be analyzed to consider the impact on the course of the lesson, the trajectory of the class's work, and the opportunities for learning for particular students and for the group. In both instances (using student work, using videotapes or cases of classroom lessons) teachers or prospective teachers might engage in content-based design work—developing a possible next assignment in response to their analysis of students' work, or planning a next instructional segment based on analysis of the classroom episode. Each of these activities takes a task of teaching that entails content knowledge and creates a possible site for teachers' learning of and using that content in authentic contexts.

But much more work is needed to contend with this endemic problem of use. Working in specific contexts might run the risk of limiting the generality of teachers' learning of content and their capacity to use it in a variety of contexts. How can teachers be prepared to know content sufficiently flexibly such that they are able to make use of content knowledge with a wide variety of students, across a wide range of environments? How could teachers develop a sense of the trajectory of a topic

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over time, how to develop its intellectual core in students' minds and capacities, so that they eventually reach mature and compressed understandings and skills?

Solving these three problems—what teachers need to know, how they have to know it, and helping them learn to use it—by grounding the problem of teachers' content preparation in problems and sites of practice, could help to close the gaps that have plagued progress in teacher education. But we should realize the challenges that doing this would pose. After all, Dewey thought his vision at the turn of the 20th century was imminently realizable. He thought that what he was describing was "nothing utopian." He suggested that, "the present movement . . . for the improvement of range and quality of subject matter is steady and irresistible" (Dewey, 1904/1964, p. 170). One hundred years later, as we stare at university and college catalogs that divide "methods" courses from disciplinary studies from practica, or at professional development offerings that are devoid of content or chock full of activities for kids, we should understand that bridging these strangely divided practices will be no small feat.

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NOTES

1.The ideas in this chapter about subject matter knowledge in teaching—its nature, uses, and how it might be acquired—have benefited from and drawn on Ball's work and discussions with colleagues David Cohen, Magdalene Lampert, Suzanne Wilson, and Joan Ferrini-Mundy. Members of the Mathematics Teaching and Learning to Teach Group have also contributed significantly to the development of our ideas: Mark Hoover, Jennifer Lewis, Ed Wall, Raven Wallace, Merrie Blunk, Deidre LeFevre, Geoffrey Phelps, Katherine Morris, Heather Lindsay.

2. This definition helps also for general claims, such as the fact that any product, N (N + 1), (N a whole number) is even, which is one explanation of why the binomial coefficient, N (N + 1) / 2, is an integer.

3. See, for example, Ball (1999) and Ball, Lubienski, and Mewborn (in press).

4. See, for example, Shulman (1986, 1987); Wilson, Shulman, and Richert (1987); Wilson (1988); Grossman (1990); and Ma (1999).

5. Ball (1999).

6. These main data comprise a year's worth of primary records of teaching and learning gathered in Ball's third-grade class during 1989—1990, under a grant from the National Science Foundation to Magdalene Lampert and Deborah Ball. In addition, we study records from other elementary classrooms, as a means to compare the mathematical entailments across classrooms.

7. See, for example, Ball (1999); Ball and Bass (2000).

8. Beansticks are a base 10 model, constructed with 10 dried beans glued to a popsicle stick to represent tens, and loose beans to represent units. Ten 10–sticks can be glued side

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by side on a cardboard square to represent hundreds. The children were working only with tens and ones in this lesson.

9. One plausible line of reasoning might go as follows: Sixteen is half of 32. So Joshua ate half as many peas on Monday as he ate on Tuesday. So the other half was how many more peas he ate on Tuesday than on Monday. In other words, half of 32, or 16, is how many more peas he ate on Tuesday than on Monday.

10. These items were developed at the National Center for Research on Teacher Learning, Michigan State University. See Kennedy, Ball, and McDiarmid (1993).

11. These ideas about the use of mathematics knowledge in teaching draw on Ball's work with David K. Cohen. See, for example, Cohen and Ball (1999).

12. We explore this in Ball and Bass (in press).

13. Several professional development curricula in mathematics are built on this idea. See, for example, Schifter; Barnett; and Stein, Smith, Henningsen, and Silver (in press).

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