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CHAPTER VII

Making Believe: The Collective Construction of Public Mathematical Knowledge in the Elementary Classroom

DEBORAH LOEWENBERG BALL AND HYMAN BASS

Near the end of a class, third graders are concluding a discussion with their teacher.

Riba: (*to the class*) So, Sean is saying that some even numbers, in a pattern, can be even *and* odd and some can't. Four can't, because it's two groups. Six can. Eight can't. Ten can. (*Pointing at the number line above the chalkboard, she uses a pointer to mark off consecutive even numbers.*) Can't. Can. Can't. Can . . .

Ofala: Well, I just think that just because twenty-two is eleven groups, that doesn't mean it's an odd number. My conjecture, I think it's always true, is that if all twos are circled in a number, then it's an even number.

Sean: What conjecture?

Ball: Ofala, tell him what you're talking about when you talk about your conjecture. He's not sure what you're referring to.

Ofala: That conjecture I already . . .

Sean: That's not a conjecture. That's a *definition*.[1]

These children and their classmates are struggling with concepts of evenness and oddness as a consequence of one child's claim that six could be even and it could also be odd. His argument has challenged the group's previous complacency about definitions of even and odd, and they are reasoning about his claim. In the process, they are solidifying their understanding of the definitions. And they are *constructing* mathematical knowledge.

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Perspectives on the Construction of Mathematical Knowledge

In this chapter, we examine the construction of mathematical knowledge in classroom teaching and learning. Much has been written in recent years about constructivist theories of learning and their implications for instruction. Indeed, "constructivism" has been arguably one of the most influential—and most multiply interpreted—ideas in mathematics education. From one perspective, learners are thought to construct their own understandings of content, based on their prior knowledge and experience.[2] What they bring shapes the sense they make of new experience. This perspective derives principally from cognitive theory and focuses generically on individual learning. A second increasingly significant theory of learning takes a sociocultural point of view.[3] On this view, individual learning takes place as a function of engagement in cultural activity within communities of practice. The focus of this work is on how participation in such communities, in the company of more experienced others, enables the development of competent practices and knowledge.

D. C. Phillips has written extensively about various interpretations of constructivism and has analyzed contemporary perspectives on how individuals construct knowledge, individually and as members of communities. These different views of constructivism, he points out, embed more than a perspective on how to teach mathematics or science—they involve particular epistemologies, theories of learning, and even interpretations of the philosophy and history of particular fields.[4] Worrying a bit that discussions of constructivism veer toward relativism, Phillips argues for preserving some focus on the external world and on attempts toward justification and truth. He writes that ". . . constructivists of both types—psychological and social—have to find room for the fact that our knowledge is *about* something. And whatever it is, it has to be granted a role in influencing our constructions . . ."[5] Phillips's is a call for a focus on *knowledge* in contemplations on its construction.

In our work, constructivism enters from a perspective distinct from either the individual cognitive or sociocultural views. We scrutinize classroom mathematics learning and teaching in light of ideas about the construction of knowledge that are rooted in mathematics as a discipline.[6] When students are at work in a *mathematics* class, we see them as constructing *mathematical* knowledge. Looking at the development of students' knowledge in this way illuminates some inherently mathematical dimensions of their work. The ways in which they seek to justify

claims, convince their classmates and teacher, and participate in the collective development of publicly accepted mathematical knowledge have powerful resonances with mathematicians' work. As students explore problems, make and inspect claims, and seek to prove their validity, we see that even young children engage in forms of mathematical reasoning and make use of mathematical resources.[7] This mathematical perspective makes visible some critical aspects of mathematics teaching and learning that are hidden when viewed from a cognitive or sociocultural perspective. In particular, this analysis allows for and explores a subject-specific view of learning.

This work finds company in recent advances in other fields. Samuel Wineburg, an educational psychologist who studies historical knowing, argues for a new approach to the study of the learning and teaching of history which he terms "applied epistemology."[8] This perspective entails the examination of the learning of history through the perspective of the discipline. Wineburg points out that this would depart from the traditional psychologist's generic perspective on knowing and learning. Suzanne Wilson, a researcher on the teaching and learning of history, elaborates what this might mean for the study of teaching and learning:

> An applied epistemology would . . . require that the historian and the psychologist listen to classrooms in new ways. As they do so, their own understanding might continue to unfold, as the words of a child or discourse of a classroom send them back to history and psychology in search of resonant ideas. They might see in a child's words central historical concepts like the role of human agency.[9]

Shari Levine Rose, in her study of fourth graders' learning of history in her own classroom,[10] distinguishes between what she was able to see in her students' work when she viewed it from the perspective of generic theories of learning and when she later began to view their work using a lens of historical sense-making. Initially, she explains, she was "influenced by constructivist theories of learning, . . . [believing] that children drew upon knowledge, values, and beliefs in actively making sense of new information." The generic perspective, however, she argues, did not help her see how the children were constructing meaning of historical events.[11] But with the historical lens, she was struck by the "historical nature" of children's sense-making. They repeatedly sought understanding through the construction of stories, much as historians fashion narratives, embedding meaning and interpretation in context.

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Rose writes about how the historical perspectives that she brought to bear in hearing and interpreting her students made visible how the children came to know the past and constructed meaning of historical events in ways that were much more rooted in the nature of historical sense-making.

Our work is similar. As we began to develop and use mathematical lenses to look in on third graders, we noticed them working empirically and generating conjectures from that work. We saw them constructing arguments intended to make their classmates believe that what they had noticed was true. We even saw them confronting the very nature and challenge of mathematical proof. For example, after working on several problems that involved patterns of sums, they had noticed that even numbers added to other evens seemed to always equal even numbers, and that evens added to odds equaled odds, and so on. They formulated conjectures, such as an even number plus an even number equals an even number, and set out to try to prove that these were always true. For the first time, in the course of this, it had occurred to some of the children that really *proving* something was something they had not done before, and that doing so presented challenges they had not previously recognized. Jeannie and Sheena, who were working on the conjecture that an odd number plus an odd number equals an even number, reported on their work:

> Jeannie: Me and Sheena were arguing together, but we didn't find one that didn't work. We were trying to prove that . . . you can't prove that Betsy's conjecture always works. Because um there's um like numbers go on and on forever and that means odd numbers and even numbers go on forever, so you couldn't prove that all of them aren't.

Ofala protested that she had tried "almost eighteen" of them and even some special ones, and they had all "worked," and so she thought that "it can always work." Mei pointed to several conjectures posted above the chalkboard that had been previously made. There had been collective public agreement about these conjectures and they were functioning almost like theorems, even though the class had not tested those with all numbers. Since numbers go on and on forever, she asked, "Why don't you say that those conjectures are not always true because we haven't tried every number there ever was?"

"I never said they were true all the time," replied Jeannie.

"Why didn't you disagree when like Lucy and everybody agreed with those conjectures?" continued Mei.

"She just thought about it," explained Sheena. "That's why we were trying to think about it today."

The disciplinary perspective from which we are viewing and interpreting classroom exchanges such as this one illuminates deeply mathematical qualities of children's work, just as Rose's historical perspective surfaces characteristically historical sense-making in fourth graders' efforts to understand the past. The children were encountering a central problem of constructing mathematical knowledge in the discipline: the challenge of proving that something is true when all cases cannot be checked. That this can be done is a fundamental achievement of mathematics, and these children were arriving at an appreciation of the need to be able to do so.

Before continuing, we comment briefly on the nature of the mathematics teaching and learning on which we focus. It is perhaps best characterized in terms of three fundamental commitments, not in terms of specific practices. First is the commitment to treat the discipline of mathematics with integrity; a second is to give serious respect to children's mathematical ideas, and a third commitment is to see mathematics as a collective intellectual endeavor situated within a community.[12] What it means to take seriously either mathematics or children's ideas, or to treat mathematical work as collective, is of course contestable. "Mathematics as discipline" is hardly a monolith; and "taking children's ideas seriously" can imply vastly different actions. Neither of these commitments provides clear guidance for practice, but together they function as twin imperatives for orienting the kind of substantive work children might do and how their learning might be fostered. Further, "collective intellectual work" has been too often distorted into a commitment to particular physical room layouts—group seating, for instance. Although not determinative of practice, these three commitments define a problem space for teaching and learning that shapes practical decisions, moves, interpretations, questions, and puzzles. This problem space also shapes the pedagogical and mathematical issues that arise.

Mathematical reasoning within a community, central to the development of mathematical knowledge, is the focus of the analysis in this chapter. We choose mathematical reasoning because it is the core technology by which mathematical knowledge is constructed in the discipline. Mathematical reasoning can serve as an instrument of inquiry in the discovery and exploration of new ideas. It also functions centrally in the justification of mathematical claims. Many discussions of mathematical reasoning in school mathematics focus on the reasoning of

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inquiry; because of its important role in establishing the validity of claims, we focus here on the reasoning of justification. We begin by describing core elements of such reasoning in order to develop a framework for viewing classroom teaching and learning. We turn next to examine a series of classroom interactions in which ideas are developing, in which children are producing, seeking to convince others, and coming to believe particular mathematical claims, solutions, methods, and ideas. We use the processes of mathematical reasoning as an analytic lens to illuminate the construction of the children's mathematical knowledge. We conclude by considering what this disciplinary perspective on classroom learning implies for the practice of teaching.

Studying Mathematics Teaching and Learning from the Perspective of Practice

This chapter draws on an interdisciplinary study of the interplay of mathematics and pedagogy in elementary mathematics teaching and learning. One of us is an educational researcher who is also an elementary school teacher, the other a professional mathematician. We use our distinct perspectives to analyze where and how mathematical knowledge and ideas, skills and sensibilities, commitments and orientations may play a role for both teachers and students in the course of their work together. Our principal interest is to uncover less exposed mathematical entailments of the work for teachers, seeking useful answers to questions about what teachers need to know and be able to do in order to teach. To do this, we study elementary teaching and learning, seeking to uncover, identify, and understand the mathematical work in which teachers and students are engaged. Some of the mathematical elements underlying classroom work can clearly be viewed as knowledge, while others might be more appropriately labeled mathematical sensibilities or sensitivities, mathematical appreciation, or mathematical skills. Some of what we have begun to notice would not appear on most itemizations of the elementary curriculum, nor are they things that practicing mathematicians would readily name when asked to talk about what there is to learn. The things we see are recognizable to mathematically fluent people, who rely on and use these in their own mathematical thinking, but for whom these things are almost ineffable.

One example is a sensibility that we are currently referring to as the *imperative to reconcile multiplicities* (of interpretation, definition, representation, solution paths and methods, solutions). What we mean is that, in the practice of mathematics, when more than one representation is

offered for an idea or a problem, examining the extent to which these are mathematically isomorphic, equivalent, or similar is crucial. Doing so is part of the development as well as the verification of the work. The opportunity for this is actually ubiquitous in classrooms where students' ideas contribute substantially to the enacted curriculum.[13] For example, in one discussion in Ball's class, which we examine later in this chapter, three different definitions were offered for even numbers: as alternating whole numbers ("every other number" on the number line when you start with zero); as those numbers that can be divided into two equal groups of whole number size with no remainder (numbers of the form $2k$ where k is an integer); and as those numbers that can be divided into groups of two with none left ($k2$).[14] The mathematical imperative to reconcile the differences among these definitions would involve asking whether each of these produces the same set of numbers, and if so, whether and how the three versions map onto one another, or, at a more sophisticated level, asking whether they are all logically equivalent. Pursuing this leads to other mathematical ideas. Comparing the second and the third, for example, involves understanding commutativity, in this case understanding what it would take to establish that $2k = k2$. The issue of reconciling multiplicities in the course of classroom lessons has emerged repeatedly across our data, as well as in others' data on classroom teaching and learning. As children engage with interesting mathematical problems, they produce different solution strategies and solutions. Quite often, however, classes end with no comparative inspection of these. Instead, multiple representations, methods, and solutions are left to an uneasy coexistence, implying either a mathematically unverified equivalence, or the message that multiple solutions simply exist, much like multiple interpretations of text. But the disciplines of mathematics and literary interpretation differ, and the imperative to inspect and compare multiple ideas is central to mathematics. This sensitivity sometimes gets lost in the context of developing instructional approaches that are reactions against a "single right way" approach to teaching mathematics. Generating multiple solutions, representation, and explanations is often valued in its own right in "reform-oriented" classrooms, without the imperative to inspect and compare them. In mathematics, multiplicity can lead to illuminating relations or equivalence of representations or it can conceal mathematical contradiction, on which ensuing mathematical reasoning may founder. Unexpectedly, mathematical similarity, equivalence, and isomorphism, while often considered the province of advanced mathematical study, have repeatedly emerged in our analyses of elementary classroom lessons.

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Our work turns the usual approach to the study of teacher knowledge on its head. Instead of asking what do teachers need to know, and making lists of topics based on the curriculum for which teachers are responsible, we ask, "What is teaching, and what does it take to teach mathematics?"

In order to ground our inquiry, we analyze data from elementary classroom teaching of mathematics. Primary is a large base of "records of practice" collected in Ball's third grade public school classroom during the 1989-90 school year.[15] Daily mathematics lessons were videotaped using two cameras each day, and audio tapes were made as well. The students' written work was photocopied and Ball's teacher journals and notes were preserved. There were tape-recorded semi-structured interviews with all the students, as well as informal conversations. Most lessons have been transcribed from the video and audio, and some have been closely improved to a point where the quality and faithfulness of the written transcript is high. These materials are catalogued and accessible, some supported with software designed for rapid access and navigation as well as manipulation of the material. They constitute a large primary collection of data on elementary mathematics teaching and learning. A second source of material consists of analogous materials (though not as extensive) from other teachers' classrooms and records from subsequent years of Ball's teaching.

One central analytic task we face is to probe the particulars of the cases we are examining: to uncover mathematical issues that can be seen to figure in particular moments of teaching practice, to seek connections with other moments, and to consider the role such mathematical ingredients play in teaching. The other is to identify what is generalizable and what is specific to particular approaches to teaching, or to the specific cases which we are studying. This is done both through logical analysis (focused on identifying elements of mathematical knowledge or insight that would be important in any approach to teaching) and through comparisons derived from further empirical analyses of other classroom data. Combing through records of classroom activity, records that document that activity from multiple perspectives (e.g., videotape, teacher's written notes) and represented in multiple forms (e.g., transcripts of classroom lessons, videotapes of those same lessons), we look for signs of mathematical activity, places where mathematical issues appear salient. Using a mathematical eye as a tool of the inquiry, we seek to annotate and index mathematical issues that shape an account of what is happening in the class. One form this takes is as interlinear annotation on the "texts" of classroom activity. Another is

commentaries on segments of classroom activity, supported with particular evidence from those "texts." We seek to identify patterns, themes, mathematical issues and lacunae, and to support the identification of those with evidence in the records. In interaction with other members of the research group, our written and oral exchanges over the mathematical analyses enable us to meld mathematical issues to pedagogical patterns, issues, and events that we identify as intersecting with them. These emanate from analyses of teaching and learning from the perspective of practice.[16]

We turn now to examine mathematical reasoning as a lens for understanding the construction of mathematical knowledge in classroom teaching and learning.

MATHEMATICAL REASONING IN PRACTICE

Viewed from the perspective of the practicing mathematician, reasoning is one of the principal instruments for developing mathematical understanding and for the construction of new mathematical knowledge. Such reasoning rests on two foundations. One is a body of public knowledge on which to stand as a point of departure, and which defines the "granularity" of acceptable mathematical reasoning within a given context or community. The second foundation of mathematical reasoning is language—symbols, terms and other representations, and their definitions—and rules of logic and syntax for their meaningful use in formulating claims and the networks of relationships that are used to justify them.

We first discuss the *base of public knowledge.* We have struggled to come up with a term to describe this knowledge. Erna Yackel and Paul Cobb use the label "taken-as-shared" to refer to the meanings, norms, and ideas that are negotiated and used as common within a classroom.[17] Derek Edwards and Neil Mercer also write about the development of "common knowledge" in teaching and learning, and focus particularly on the discourse patterns whereby teachers establish such common knowledge.[18] Our focus is more than social or what Yackel and Cobb call *sociomathematical.*[19] Whereas Yackel and Cobb focus on the normative aspects of mathematics discussions specific to students' mathematical activity, such as agreements about what counts as "mathematically different" solutions or what counts as an "acceptable mathematical explanation," we focus on the *mathematical knowledge* that is available for public use by a particular community in constructing mathematical claims and for seeking to justify them to others. It is knowledge already assumed or developed, part of the record of the children's prior experience or the

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class's past work. We do not wish to imply that each member of the community *knows* it individually in the same way; this is an empirical question beyond the scope of our analysis. We mean to call attention to that knowledge that can comfortably be assumed and used publicly without additional explanation. We contrast such knowledge with ideas or procedures that are not shared, and must therefore be justified before they can be used in the collective discourse of a community.

This base of public knowledge is defined relative to a particular community of reasoners. In the case of professional mathematicians, it might consist of an axiom system for some mathematical structure (such as euclidean geometry or group theory), simply admitted as given, plus a body of previously developed and publicly accepted mathematical knowledge derived from those axioms. University instructors listing prerequisites for a course are prescribing part of the common knowledge upon which they wish to rely in developing the scope of the course's mathematical work. We argue that this idea is useful in understanding the work of a class of elementary school students as well, where this base of public knowledge comprises the expanding set of publicly established ideas and shared knowledge which can be used by the class in explanation or justification.

Take a simple example. On one day, early in September, in Ball's third grade class, the children were working on the problem, "Write number sentences for 10." In the early stages of their work, most students were writing simple equations: $4 + 6 = 10$, $3 + 7 = 10$, $8 + 2 = 10$. Pressing them gently into more complex solutions, Ball asked:

> I wonder if someone can think of a number sentence that uses more than two numbers here. Just so we have a bunch of ideas of how we could do this. Who can make a number sentence that equals 10, but has more than two numbers adding up to 10?

Quickly, Tembe[20] began:

> One plus one plus one plus one—

Ball wrote on the board as he rattled off: "One plus one plus one plus one plus one plus one plus one. Plus three."

"Why does that equal 10?" she asked, and called on Haroun to justify Tembe's claim. Haroun, repeating, said, "That's just one plus one plus one plus one . . ." Ball asked the students, "How do we know that that equals 10?" Riba, eager to respond, explained, "Because one plus one plus one plus one plus one plus one and plus three equals 10." Ball, still not satisfied, said, "You're just sort of reading it. How could you *prove* it to somebody who wasn't sure?"

> Riba: Because I counted it.
> Ball: What did you count? What did you find out?
> Riba: There's one and the next one is two and the next one is three, next one is four, next one is five, next one is six, next one is seven, next one is—seven, and then three more, eight, nine, ten.

In this simple segment, Tembe's offered solution, the first that included more than two terms, was not something automatically presumed to be within the common knowledge of the class and Ball asked for justification. If this had been a fifth grade class, one might presume differently. A young child's initial sense of addition comes from counting, which is adding one at a time. Adding many terms at once, or adding, in a single step, two numbers larger than one, is a higher order operation, not only for young children, but also mathematically. When Riba is called upon to explain why the string of terms equals 10, she first just recites the equation. When Ball presses her, "But how could you *prove* it to somebody who wasn't sure?" Riba replies that she "counted," and Ball then encourages her further to make this counting public. In response, Riba expands her explanation:

> There's one and the next one is two and the next one is three, next one is four, next one is five, next one is six, next one is seven, next one is—seven, and then three more, eight, nine, ten.

Perhaps it is through this counting that Riba first proved *to herself* that Tembe's formulation was valid. The teacher was requiring that this reasoning be made public, to persuade the class as well. The teacher then publicly validates Riba's work, underlining early in the year a standard for explanation and justification that is more than simple restating of the assertion:

> Do you see the difference in Riba's second explanation? Did you see how she really showed us how it equals 10? The first time you just read it. And the second time you explained it. That was really nice.

A process of reasoning typically consists of a sequence of steps, each of which has the form of justifying one claim by invocation of another, to which the first claim is logically reduced. This process, which merely transforms one claim into another, is not a vicious circle because the reduced claim is typically of a more elementary or accessible nature, and, in a finite number of such steps, one arrives at *a claim which requires no further warrant*, because it is part of the base of publicly shared knowledge, and is therefore universally persuasive within a

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particular community of reasoners. Thus, the base of public knowledge defines the primordial steps, requiring no further warrant, which form the stepping stones of an argument. In this sense, what is publicly shared knowledge defines the *granularity of acceptable mathematical reasoning* within a given context. In the example above, the addition represented in the equation 1 + 1 + 1 + 1 + 1 + 1 + 1 + 3 = 10 was not presumed to be part of the base of established knowledge of this class, and so Riba was pressed to reduce the claim to an iterated counting, keeping track of the total as she counted. At this point, Tembe's assertion was sufficiently reduced to a level that relied on knowledge common to the class, and required no further justification.

The crucial issue is: *How can one justify a mathematical claim?* One way is to state a claim, and to undergird its truth by the sheer force of authority. Students often receive mathematical knowledge in school that is justified by little else than the textbook's or the teacher's assertion. By default the book has epistemic authority: Teachers explain assignments to pupils by saying, "This is what *they* want you to do here," and the right answers are found in the answer key. Learning to "play by the rules" often entails a "suspension of sense-making" in school mathematics.[21] But that route is the antithesis of warranting claims through a process of mathematical reasoning.

The base of publicly shared knowledge consists of knowledge of certain facts and concepts, of the meanings of mathematical terms and expressions, of procedures and resources for calculation and other problem solving. It is always present, in both latent and active forms, though it may be tacit, and only implicit in the discourse of the community, whether they be mathematicians or third graders.

Whether a particular piece of knowledge is in fact commonly shared is an empirical question, one that a teacher must often appraise. Did everyone in the class understand and agree that Riba's enumerated explanation satisfactorily proved Tembe's claim? And, further, how many children were not already adequately convinced by his initial statement? These are difficult questions to resolve fully; they are also not completely within a teacher's or students' view as a class discussion proceeds. Still, reasoning within a community depends on the *presumption* of common knowledge and shared established methods. This is most often represented by the use of knowledge already used and established publicly. Arguments that do not build on publicly shared knowledge are unlikely to be effective in convincing others. At the same time, the process of reasoning can in fact help to build and extend a group's common knowledge. As claims are proved, and ideas

developed, they may become part of the legacy of public knowledge, on which subsequent claims may depend and build.

In our analysis, *mathematical language* is the foundation of mathematical reasoning complementary to the base of publicly shared knowledge. "Language" is used here expansively, comprising all of the linguistic infrastructure that supports mathematical communication, with its requirements for precision, clarity, and economy of expression. Language is crucial for mathematical reasoning and communication with others about mathematical ideas, claims, and explanations, and proofs. Mathematical language is not simply an inert canon, inherited and learned from a distant past. It is also a medium in which mathematics is enacted, used, and created.

Language in our framework includes the nature and role of definitions in mathematics, the nature and rules for manipulation of symbolic notation, and the conceptual compression afforded by timely use of such notation. Definitions and terms play a crucial role: Not simply delivered names to be memorized, they originate in and emerge from new ideas and concepts, and develop through active investigation and reflection. They facilitate reasoning about those new ideas by naming and specification. Decisions about what to name, when to name it, and how to specify that which is being named is an important component of mathematical sensibility and discrimination central to the construction of mathematical knowledge. Using symbolic and other representations to encode ideas, as well as decoding ideas represented in symbolic or other form are critical communicative tools for the construction of mathematical knowledge. Notation can be used to compress ideas into forms that, when done skillfully, reveal their structure. Wise use of notation can reduce computation and manipulation to manageable proportions; how and when to do this is an important skill of mathematical representation useful in reasoning.

Mathematical language is crucial to reasoning—to the construction of mathematical knowledge—for it provides the medium in which claims are developed, made, and justified. Magdalene Lampert writes:

> Mathematical discourse is about figuring out what is true, once the members of the discourse community agree on their definitions and assumptions. These definitions and assumptions are not given, but are negotiated in the process of figuring out what is true.[22]

Some disagreements stem from divergent or unreconciled uses of terminology, while others are rooted in substantive and conflicting mathematical claims. The ability to distinguish these requires a sensitivity

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to the nature and role of language in mathematics. We return to this element of language in the examples analyzed in the next section of this chapter.

Constructing Knowledge in a Classroom: Mathematical Reasoning at Work

We turn next to a closer look at three examples from Ball's third grade classroom. Whereas other analyses of mathematics learning view the construction of knowledge as either an individual process of interpretation and sense-making based on what learners bring, or as a social process—the product of participation in a set of shared cultural practices—our examination looks through the lens of the discipline of mathematics. How are mathematical ideas developed and justified in a class of eight-year-olds? How is public mathematical knowledge used to validate claims, and how does an idea or a method become part of the shared knowledge to be used by the class? How do issues of formal and informal language play out in the students' reasoning about emergent and established mathematical ideas? Magdalene Lampert's studies of fifth graders engaged in mathematical discourse provide portraits of them seeking to validate what is true "not by the teacher, or another student, saying that an answer is right or wrong, but by mustering the evidence to prove or disprove an assertion."[23] We attempt in this chapter to probe closely the anatomy of children's mathematical reasoning, with the nature of reasoning and justification in the discipline of mathematics as our lens for examining the children's efforts to produce conviction.

We selected the three examples to portray different cases of mathematical reasoning in a group of young children's construction of mathematical knowledge. In the first example, two children attempt to reason about whether zero is even or odd. Their exchange provides a simple example of children's efforts at basic mathematical reasoning that relies both on the presumption of publicly shared knowledge and on mathematical language. The second example provides a case of failed collective mathematical reasoning: a child proposes a division sentence as a solution to a problem, and because the children lack the shared knowledge required to verify the validity of her claim, the solution is ultimately tabled. And the third, most complex, example reveals a portrait of the children's joint construction of mathematical knowledge. A child makes a mathematical observation that conflicts with the shared knowledge of the group and working definitions that they hold. As the children argue with his observation, their collective work

that leads to the construction of a new mathematical idea reveals sophisticated use of a base of publicly shared knowledge and the development of mathematical language.

MUCH ADDUCED ABOUT NOTHING: SEEKING COMMON GROUNDS FOR BELIEF ABOUT ZERO

In mid-January, the third graders' fascination with zero is in full bloom. They often bring zero up as a special or boundary case when exploring conjectures and they like using it in number sentences or in solutions to problems. They also wonder about its nature: Is it even or odd, or perhaps neither? Is it really a number? It is, after all, "nothing." Some ask their parents and bring to school their parents' views about zero.

Near the beginning of class one day, Sheena is musing about something she had heard the day before:

> Well, I didn't think that zero was—zero, um—even or odd until yesterday they said that it could be even because of the ones on each side is odd, so that couldn't be odd. So that helped me understand it.

A closer look at Sheena's statement reveals that it is a structured mathematical argument. She makes an implicit claim: Zero is even.

She provides a justification for her claim, relying on something she heard another student say. The justification has two elements, each part of the publicly shared knowledge of the group:

> The ones on each side is odd . . .
> . . . so that couldn't be odd.

"The ones on each side is odd" refers to the status of one and negative one, each of which Sheena believes to be odd. This is an interesting case: Although the number one has been publicly established to be odd, and although the children have worked with negative numbers, no explicit consideration has been given to the expansion of their definition of odd numbers used for whole numbers to the enlarged system of the integers. This is an aspect of the collective work with mathematical language that we refer to analytically as "expanding definitions," but in any case, it has not yet happened in the class. Still, Sheena seems confident that negative one is odd and uses it, presumably assuming that others share this idea by extension.

"So that couldn't be odd," the second step in Sheena's argument, refers to the zero, positioned on the integer number line which is mounted prominently above the chalkboard, between negative one

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and one. Here she is using one of the publicly shared definitions of even and odd numbers—that they alternate on the whole number line.

Sheena's argument, then, is that zero is even, and she provides an argument for her claim built on shared public knowledge and shared mathematical language.

Sean sits up, however, and takes issue with Sheena's claim:

> Um, I—I—I just want to say something to Sheena, when she—what she said about um that, that one, um—zero has to be an odd, an even number because—I disagree because, um, because what, what two things can you put together to make it?

Like Sheena's argument, Sean's has a clear structure of claim and reasons. His implicit claim is: Zero cannot be even.

His justification relies on just one piece of mathematical knowledge, one that he likely thinks is shared among his classmates:

> What two things can you put together to make it?

This one requires some translation. By asking Sheena, "What two things can you put together to make it?" he seems to be invoking another piece of public knowledge about even numbers—that is, that they are the sum of two other equal whole numbers. Eight is the sum of four plus four. Twenty is the sum of ten plus ten. Sean's test for whether or not a number is even is different from the one used by his classmates, although it relies on the class's current working definition:

> If you have a number that you can split up evenly without having to split one in half, then it's an even number.

Whereas other children determine whether a number is even or not by splitting it in half, Sean seeks a number that can be doubled to yield the given number: He has transformed the definition into a parallel form in which even numbers are those numbers that are "made up of" two equal whole numbers. But what is he saying here? "What two things can you put together to make [zero]?" implies that he does not think that zero can be written as the sum of two equal numbers. It is not clear what he is thinking, but he might be looking to negative numbers to find one that can be doubled to equal zero. With a positive even number, after all, the number that is doubled to produce it is to its left on the number line. For example, eight is "made up of" four plus four, and four is to the left of eight. He thus might be seeking, and failing to find, a number to the left of zero whose double is zero. He may be thinking something else; it is not clear. However, in any case, he does not seem to think of zero as "made up of" zero plus zero. This is an interesting instance of mathematical reasoning, in part, because he does invoke publicly shared knowledge (even numbers as double another number) but cannot apply it in this case, since he cannot think of a number that is half of zero. By claiming that negative one was odd, something that had not in fact been publicly determined, Sheena, too, was facing ambiguities in using the publicly shared definition of odd numbers.

Sheena moves to respond to Sean's challenge:

> Well, I could show you it. (*Moves toward the chalkboard and points to the number line above the chalkboard.*) This one (*points to the 1 on the number line*)—this one is odd and this one (*points to the -1 on the number line*) is odd, so this one (*points to the zero*) has to be even.

In her response, Sheena does not answer Sean's direct question, which is based on a different definition of even numbers than the one she is invoking.

Viewed from the perspective of the discipline, this is a striking mathematical exchange. Each child's argument is structured with a claim, and supported by smaller stepping stones of publicly shared mathematical knowledge. However, because they are using two shared, but different and unreconciled, definitions of even numbers, they are unable to find common ground for reaching agreement on the parity of zero. Neither seems convinced by the other's reasoning and the question is left unresolved. What is needed in particular is a process for comparing and reconciling these different definitions of even numbers so that Sheena and Sean can reason together about zero.

In the next example, the class considers a claim for which there is no supporting publicly shared knowledge or established mathematical language: a hundred divided by ten equals ten. After some efforts to verify this claim, it becomes clear that, lacking the resources for building the chain of reasoning, this valid claim must be set aside, unverified and, for now, unverifiable within this community.

UNCOMMON—AND UNCONVINCING—NUMBER SENTENCES FOR TEN

Early in September, Ball gives her third graders the problem referred to earlier, "Write number sentences for 10." After the early discussion described above, in which Ball presses the children to generate solutions that use more than two terms, they work alone and in pairs

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for about half an hour, generating many solutions to the problem. Later in the class period, Ball pulls them back together to begin sharing and inspecting solutions. She opens by inviting students to consider what they might want to bring up:

> Let's see if we can share some of the things you can come up with. Look over your list, and pick a couple that you feel especially—are especially interesting, that you think other people might not have thought of or that you were pleased to get or something. And I would like you to listen to each other's. If you liked one that somebody else brings up you can add it to your list. (*pause*) Lucy, do you have one that you like?

Lucy says she does, and announces "a hundred divided by ten equals ten." Ball tells the class to listen closely to Lucy, and repeats her solution.

> Ball: She said a hundred divided by ten equals ten. Lucy, can you explain that?
> Lucy: What do you mean by "explain"?
> Ball: How do we know that that's right?

Lucy begins to explain it and then pauses, "I don't know really." Ball asks who else can explain. "How do we know that what Lucy said is right?" Lisa volunteers. She says that she agrees with what Lucy had said about the number sentence. When Ball asks her why she thinks it is right, Lisa says she doesn't know, but she just thinks it's right "for some reason." Mei agrees: "I had it on my paper." Ball presses. "But how do you know it's right? Just because a couple of people have it on their paper doesn't mean it's right." Lucy ventures, "Mei said her mom taught her about dividing." She adds, "and so I believe Mei."

> Ball: Okay, so Mei told you, and her mother taught her, and so you believe her. Mei—
> Lucy: Um, well um, I have a lot of other ones, and like, if I had twenty divided by two, Mei said it was right, so then—when I said that, she said it was right, so if I—if I have ten divided—ten divided by one equals ten, and then I have it—and then I have like fifty divided by five equals ten, and Mei said that it was right, and so, like if you have a hundred divided by ten it would be right because—because if you went from like five then it would be five more, and then—'cause fifty is five less than a hundred so—so it would be ten cause if it's five divided by—fifty divided by five it would equal ten.

Ball wonders what others are understanding of Lucy's tumble of divisions. Lucy's enthusiasm notwithstanding, Ball is concerned whether others are following and whether the children will be able to justify her claims. The class have not yet worked on division, although they will later in the year. Ball asks others what they think. Betsy says she thinks she knows what Lucy is saying. "Like the fifty she just put—she just plussed fifty plus fifty and made it into a hundred, five plus five, made it into a ten, and then she divided it and put it into ten, made it ten." This doesn't exactly seem clear. "*Is* that what you are saying, Lucy?"

"Yeah, sort of," she replies doubtfully.

Ball turns out to the class: "Does anyone have anything they could say that would help us know whether we should believe this or not? Devin and Tembe? (*pause*) Mei, you think you have something?"

> Mei: Those kinds of problems are really the opposite of times. And like if ten times five, it would be fifty, so fifty divided in five would be ten.

Multiplication has not yet been a topic in the class, although Ball suspects that some children may know some multiplication. In order to follow Lucy's and Mei's claims, and to justify them, they will have to be able to use knowledge of multiplication. Ball probes the class, "What does she mean, ten times five is fifty?" Jeannie says something vague about Mei writing up two numbers on the board, and no one else comments. The children do not seem to be rising into the discussion, and Lucy and Mei themselves do not seem able to construct a line of reasoning accessible to the others. The discussion appears to be grounded for now, lacking the base of publicly shared knowledge necessary for its development.

> Ball: It seems like maybe we should go on right now. Other people seem to be working on other things and I'm not sure that everyone is thinking about whether or not we should believe this. I'm not sure we should—we should include this on our list until we have some way of showing that we know that it's right. Like, remember before when Riba explained one plus one plus one plus one plus one plus one plus one plus three. She proved to us that it made sense. But right now we don't have any way of really knowing if these are right or not. And I'm not sure we should have them on our list unless we have a way to show that they make sense.

Ball places brackets around the division sentence, $100 \div 10 = 10$, commenting, "This doesn't mean that it's wrong, but until we have some way of deciding if it's right, we're not sure about it yet."

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Ball's decision not to take up multiplication and division in the context of this problem, "Write number sentences that equal ten," may seem a puzzling pedagogical move, missing an opportunity to open a territory for the children's work. But considering what it takes to justify a mathematical claim reveals that the children, confronted with division sentences, did not share a base of public knowledge with which to reason about and validate them. Lucy herself, who introduced this, could not explain it. She believed that a hundred divided by ten was ten because Mei's mother taught Mei, and Lucy believed Mei. Although she did exhibit substantial signs of emergent understanding, her conviction seemed based more on her trust in Mei than her ability to reason mathematically about division herself. Her "explanation" did not seem to work to engage or persuade her classmates. Moreover, the group lacked the publicly shared knowledge to inspect and verify the division sentences. They did not share definitions for multiplication and division, and even the terms and notation were not shared. Without publicly shared knowledge or language with which claims can be discussed and validated, the discussion cannot be founded on mathematical reason, but is reduced to the authority of whoever puts forth an idea. The children were not yet equipped mathematically to reason about multiplication and division, at least not without some development, which Ball decided not to do on the fly. To develop knowledge of multiplication and division is not something to do by way of mention or quick illustration.

We turn next to an example, where drawing on a base of publicly shared knowledge and language, the children were successfully able to handle and develop a novel idea proposed by one child. In contrast with the example of division sentences above, in the following case, children are able to reason constructively about the mathematics in play.

SEAN NUMBERS: THE COLLECTIVE CONSTRUCTION OF A NEW MATHEMATICAL IDEA

One Friday in January, in the midst of a unit on even and odd numbers, Sean proposes to the class that the number six "can be even and it can be odd." After a few minutes, he clarifies: "Three twos to make that, and two threes make six." When he says this, many of the children appeared dubious and most seemed to think he was wrong. Six could not be considered odd. Several tried to persuade him of this.

Cassandra, using a pointer on the number line hung above the chalkboard, invokes the even-odd-even-odd pattern:

> I disagree with Sean when he says that six can be an odd number. I think six can't be an odd number because . . . look—(*she gets up and comes up to the board*) . . . Six can't be an odd number because this is (*she points to the number line, starting with zero*) even, odd, even, odd, even, odd, even, how can it be an odd number because (*starting with zero again*) that's odd, even, odd, even, odd, even, odd. Because zero's not an odd *number*.

Sean continues his point:

> There can be three of something to make six, and three of something is like *odd*, like see, um, you can make two, four, six. Three twos to make that and two threes make six.

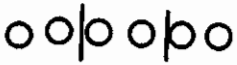
Keith, unconvinced, looks at him:

> That doesn't mean that six is *odd*.

Tembe requests that Sean "prove it to us," and Ball, sensing the need to clarify terms, asks what the "working definition of an even number is." Sean doesn't remember, but several other children do, and Jeannie articulates the current public working definition:

> It is um, if you have a number that you can split up evenly without um, having, having to um, split one in half, then um, it's an even number.

Sean agrees that 6 is even. But, he says, "It can fit the definition of odd, too." "Prove it to us," demands Tembe. "*Prove* it to us." Sean does not appear to hear this as anything other than a reasonable request for justification. He walks slowly up to the board, where he draws six circles and places lines between each pair of circles:



He explains:

> Well, see, there's two, (*he draws*) number two over here, put that there. Put this here. There's two, two, and two. And that would make six.

It gradually dawns on Mei what Sean is saying. "I think I know what he is saying," she says. "I think what he is saying is that it's almost, see, I think what he's saying is that you have three groups of two. And three is an odd number so six can be an odd number *and* an even number." Ball looks at Sean and asks whether that is what he is claiming.

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He nods. Mei becomes excited and says she disagrees. She asks Ball whether she can go to the board.

> It's not *according* to how many groups it is. Let's say that I have (*pauses*) Let's see. If you call six an odd number, why don't (*pause*) let's see (*pause*) let's see—ten. One, two . . . (*draws circles on board*) and here are ten circles. And then you would split them, let's say I wanted to split, spit them, split them by twos . . . (*she draws*)



> one, two, three, four, five—then why do you not call *ten* a, like—a, an odd number and an even number, or why don't you call *other* numbers an odd number and an even number?

Sean looks carefully at what Mei has drawn and replies:

> I didn't think of it that way. Thank you for bringing it up, *so*—I say it's—ten can be an odd and an even.

Mei becomes agitated. If he goes "on like that," she says, maybe "it will turn out that all numbers will be odd and even" and that "wouldn't make sense. She continues excitedly:

> Then it won't make sense that all numbers should be odd and even, because if all numbers were odd *and* even, we wouldn't be even having this *discussion!*

In this segment, Sean has initially noticed something interesting about the number six. He sees suddenly, based on some discussion among his classmates about structure of even numbers, that there is something noteworthy about six—namely, that when it is divided into twos, there are three groups of two and three is an odd number . . . This seems to strike him as interesting and he proposes that it could therefore be considered "odd," too.

Sean is making a mathematical observation about the number six. Calling six "odd," however, conflicts with the current public definitions of even and odd, which are implicitly considered to be mutually exclusive. As Mei points out, the definition does not depend on the "number of groups it is." Sean's nonstandard use of terminology agitates his classmates. Moreover, six is already known to be even: This is public knowledge. Sean's efforts to make this new claim about the

number six falters both because he is using the term "odd" in ways that do not correspond to the collective working definition, and because his argument is built on a way of understanding the concept of a number's parity that departs from where the class has been. In other words, the reasoning does not convince his classmates because it is not built on, and in fact conflicts with, publicly shared knowledge and common definitions of terms. Mei's sense is that if his argument were to erode all distinctions between evenness and oddness, then that would not make sense, and the discussion they were having would not even occur.

However, as the discussion unfolds, gradually his idea is liberated from its connection to the terms "even" and "odd." Mei's proposal to Sean, "Why don't you call *ten* an even and an odd number?" launches the class into a *generalization* of Sean's idea to all numbers which are an odd number of groups of two.

First, the class revisits the definition of an odd number. Matching Sean's drawings of numbers divided into groups of two, Ofala points out that "even numbers have two in them, and also odd numbers have two in them, except they have one left":



Prompted for a definition of odd numbers, she explains:

> Ofala: Well, an odd number is something that has one number left over.
> Ball: After you do what?
> Ofala: After you circle the twos.

Ball guides the students through some experimentation with other numbers they know to be odd, testing Ofala's definition, when suddenly Riba calls out, "Twenty doesn't work. There's *ten* groups." Sean looks at her, "I'm not *saying* that twenty can be an odd number. I'm saying *twenty-two* can be an odd number." This fires up the class, as they race to locate other even numbers like six, ten, and twenty-two. When Ball asks whether there is a pattern to the ones that Sean is calling odd, several children chorus yes, and Sean says he knows what it is:

> Every four numbers, like um, there's one starting out like that, and it can be an odd number, and then four more . . . The fourth one would be

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another odd number. Because you can split it in odd groups. Odd number of groups.

Riba, at the board, uses a pointer to underscore Sean's pattern: Four can't, six can, eight can't, ten can, can, can't, can, can't (*marking off alternating even numbers*). The number two surprises everyone, including Sean.

Sean: Two isn't, two isn't, because you can't make it an odd number. There's only one group there.
Riba: (*and several other students*) One's an odd!!
Riba recapitulates Sean's idea that some even numbers can be even and odd and some can't. "That's what I'm trying to say," agrees Sean.

In the unfolding of the class's work with Sean's original idea about the number six, his idea had been developed and become public. It was no longer an idea only about six, but about that subset of even numbers that are an odd multiple of two (e.g., 6, 10, 14, 18). An emergent new idea had been born from Sean's initial observation. Remaining was need of a name to detach it from its confounded interaction with evenness and oddness: "Sean numbers." Without a label, and without a developed sense of the importance of naming, Sean had quite reasonably used the names he knew—even and odd—and used them together to denote the properties he is noting. Sean numbers, as newly defined mathematical objects, then proved worthy of the same questions of interest that are asked of its cousins, even and odd numbers. What do you get, for instance, when you add a Sean number to another Sean number?

Many readers might puzzle over the significance of third graders developing a new idea that has to them no known place in mathematics. Sean numbers? What are they? What is the point of children constructing knowledge in school separate from the familiar body of knowledge we call "mathematics"? We take a closer look at the mathematical idea that the students have constructed. Just as even and odd numbers can be thought of as whole numbers leaving a remainder of zero or one, respectively, when divided by two, Sean numbers are characterized as those leaving a remainder of two when divided by four. Six, ten, and twenty-two are all Sean numbers—even numbers with an odd number of groups of two. Six divided by four is one with two left over. Ten divided by four is two with a remainder of two. Twenty-two divided by four is five with a remainder of two. And so on. The arithmetic study of the remainders upon division by a fixed

number m (m is the "modulus," the number being used as the divisor) is a domain of mathematics called "modular arithmetic." This is useful, for example, in studying periodic phenomena; a familiar example is "every other number" which is period two. This topic is sometimes referred to as "clock arithmetic," since the analog clock is period twelve. Thus, Sean's idea took the class from modular arithmetic mod two (the study of evenness and oddness) to an encounter with a phenomenon of period four, since Sean numbers occur in steps of four (2, 6, 10, 14, and so on). This led to the class construction of a piece of modular arithmetic mod four, a natural mathematical idea.

Labeled and defined, Sean's idea had traveled from an initial isolated observation (or confusion) to a maturely developed mathematical concept as a result of the class's collective work on his claim. While his original argument faltered because it violated publicly shared knowledge and did not fit the collective working definition of the term "odd," the children together built up a core of publicly shared knowledge about this newly discovered set of "Sean numbers." They relied on, and in the process improved, their publicly shared knowledge of even and odd numbers. Making it clearer that the number of groups of two into which a number may be divided is not a relevant part of the definition of evenness or oddness helped to build the concept of "even" and "odd." The children also struggled with the imprecision of their working definitions, and through their work, improved those, too. When Sean argued, "Six can be split fairly and six can be shared not fairly," he was still able to use the currently shared working definitions to permit six to be considered odd. This helped to point out not that six was odd, but that the definition needed to be sharpened. That the children were working with three different definitions of even numbers also complicated the class's reasoning. When she went to the board to use the number line as a tool to reason about Sean's claim, Cassandra used the definition of even numbers as alternating whole numbers ("every other number" on the number line when you start with 0). Several other children were using as their definition for even those numbers that can be divided into two equal whole-number groups with no remainder ($2k$). Sean, and later Ofala, were working from the definition that specifies even numbers as those numbers that can be divided into groups of two with none left ($k2$). These three definitions, while all viable, had not been compared and each called attention to a different aspect of evenness. The children's work with Sean's claim and his reasoning helped to reveal the need to compare and reconcile the three definitions.

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In this example, the children constructed a new idea together. Initiated by the private musings of one child, his observation gradually evolved into a public object of first, dispute, and later, exploration.[24] His claim, built on the use of ideas that at first seemed to conflict with the provisional publicly shared knowledge, became more clearly a site where names and definitions emerged as crucial to the group's work. Calling what he was noticing about the number six "odd and even" complicated the children's explorations of the idea. As they focused increasingly on the structure and pattern of these numbers that they together identified, they were able to move from an argument about terms to the development of a mathematical idea, which was eventually given its own name, unconfounded with the working definitions of even and odd numbers. On the unit quiz given two weeks later, all the children were correctly able to place even and odd numbers into a Venn diagram. Despite the fact that two of the even numbers given were Sean numbers, no one placed those in the intersection (as even and odd). The base of public knowledge was a critical resource in their work, as was their sensitivity to definitions and use of terms.

Making Believe: Mathematics and Elementary Teaching and Learning

A recurrent dispute about school mathematics learning has been whether children should function as "little mathematicians." Our analysis does not take a position on whether or not they *should* function as such, but suggests that when their interactions and work are viewed from a disciplinary perspective, particular features of the construction of mathematical knowledge often unseen become visible. We do not advocate that students "make believe" they are mathematicians. Instead, we argue that "making believe," or producing conviction, is central to constructing mathematics and that it has special subject-specific characteristics with which even young children contend. Sheena's sudden insight that they had never really proved something—since numbers go on forever—presented her and her classmates with a special challenge of knowing that is unique to mathematics. Sheena's and Sean's struggles to talk with each other about whether zero was an even number similarly put them in a quintessentially mathematical situation—seeking common ground on which to reconcile disagreement, but finding it impossible to do when the linguistic components of their arguments lacked commonality. Similarly, the tabling of Lucy's effort to introduce advanced mathematics (division) into the third grade reveals what is

required for a community to consider and verify a new idea, and the role played by publicly shared and established knowledge in constructing mathematics in a community. The example of Sean's observation about the number six and the ways in which his classmates picked up and challenged, argued against, and eventually developed the idea highlights the critical role played by mathematical definitions and other terms in constructive mathematical work.

What does this disciplinary perspective on students' learning of mathematics mean for the practice of mathematics teaching? Too often, the focus on "constructivism" has been transformed into imperatives about teaching: teachers are to listen more, to talk and tell less, and to create and orchestrate situations in which students will "construct understanding." Giving knowledge to students has been portrayed as being in conflict with constructivist theories, for students are supposed to construct their "own" ideas. Telling students things that they have not constructed themselves is sometimes seen as pointless, as if to assume that students could not develop understanding by listening or watching.[25] Helping students construct valid and usable knowledge of mathematics is a complex challenge of practice. Global imperatives and slogans tend to obscure the subtleties entailed in the active work of helping students construct knowledge. Our analysis of children's classroom mathematics learning seen as the construction of mathematical knowledge suggests to us a different path for the development of practice in support of students' learning. We highlight three elements that connect ideas about the construction of mathematical knowledge to the work of teaching. These elements are bound together by their role in the establishment of conviction, of *making believe*, in mathematics teaching and learning.

First is the centrality of the *reasoning of justification* in the construction of mathematical knowledge. Recognizing the role of such reasoning to investigate claims provides a medium for inspecting and verifying representations and solutions produced individually and in class. It suggests the need for the development of classroom participation structures and intellectual and social norms that would enable students to reach sound mathematical conclusions that are neither idiosyncratic ("this makes sense to me") nor rooted in assumptions of others' authority ("the book says so").

Placing an increased emphasis on the reasoning of justification does not suggest who should produce these explanations. However, whether teachers explain and justify or whether students do, reasoning should provide a chain of supporting knowledge, linked logically, in

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ways that establish the validity of a claim and that are at the level of granularity appropriate for the current understandings of a particular group of children. Justification should seek to establish warranted conviction, making oneself and others believe for good mathematical reasons. This is core to the construction of mathematical *knowledge*.

Second is attention to what it means to support the identification, use, and ongoing development of a base of publicly shared knowledge within a classroom. Calling attention to conclusions reached, to shared assumptions, to common understanding is one component. Another may be to develop practices that create public records and naming ideas on which subsequent mathematical work can draw. The third graders in Ball's class had accumulated a small set of conjectures that had been proposed and agreed upon across the school year, each labeled with a name and printed on construction paper and posted publicly in the room. Much as mathematicians use theorems in constructing proofs, the children used these "conjectures" by name as they reasoned about ideas, representation, and solutions. Betsy's conjecture, for example, was, "An even number plus an even number equals an even number," and was used to build other arguments. Several conjectures about fractions were proposed and validated; these formed some of the resources for the children's mathematical reasoning on other problems. With encouragement to do so, the children also perused and drew from their notebooks ideas that had been publicly determined and were hence usable for reasoning. If publicly shared knowledge constitutes a crucial foundation for sound mathematical reasoning in a community, then it is an essential resource for individuals' and groups' efforts to construct mathematical knowledge. Therefore, developing ways to develop and make it both visible and usable is crucial.

Closely related is the third component—the significance of mathematical language in reasoning. Definitions, notation, and terms, as well as the syntax of their use in representing mathematical ideas, are important resources in mathematical reasoning. Children will invent terms and notation; respecting, reinforcing, and using those at times, and linking them to formal mathematical language at others, is a third area in need of focused attention. How and when might what forms of precision be a goal? When are students' informal ways of representing ideas important? If language is a medium for the construction of mathematical knowledge, then concern for its development and use matters. Language must be comprehensible to oneself, and also to others. The construction of mathematical knowledge depends on the comprehensibility and validity of the forms in which new ideas are

represented, and old ones used. Taking more seriously the development of children's usable mathematical language implies a kind of substantive attention to definitions, symbolic and other notations, and issues of naming that is not typical in most classrooms. As the third element that emerges from our analysis of the construction of mathematical knowledge, mathematical language is a crucial cornerstone of moving toward the construction of publicly shared and validated mathematical knowledge.

We close with a final glimpse at children engaged in constructing mathematical knowledge, a look back that entails the ideas of mathematical reasoning on which this chapter has focused. Betsy and three other children, late in the work on the problem, "Write number sentences for ten," decided that they had a way to show that the answers to this problem were infinite, and they presented it to the whole class. Here we see them seeking to convince others of their idea, drawing on publicly shared knowledge and language, and assisted by the teacher. The segment begins with Betsy's explanation of their reasoning: they could take any number, subtract that number from itself, and add ten, and that number sentences of this form would always produce sums of ten. They link this to the shared knowledge that "numbers go on forever," to show that solutions to this problem "go on and on and on":

> See, what we did is we would take any number, it wouldn't matter what number—say two hundred. And then we would minus two hundred, then we would plus, ten, and it would always equal ten. So you could go on for, oh, a long, long time, just keep on doing that. . . . So, since numbers they never stop, you could go on and on and on and on and on and on and on.

Betsy's explanation draws on several things that she presumes to be shared by her classmates. For example, she leaves unstated the fact that two hundred minus two hundred equals zero; she assumes that everyone knows this. The fact that numbers "never stop" was similarly assumed. She apparently also expected others to share the knowledge that "any number" could replace her example of two hundred in the expression.

Ball tries to call attention to these children's claim by writing it down for collective consideration, and by giving a name for the first time this year to the kind of significant mathematical thinking that it represents—a *conjecture*.

> Ball: Could you say what your idea was again? I want to write it down for us to think about.

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(*Later.*) Okay. Now, I put a title on this piece of paper, it's important. I wrote, "Betsy, Riba, Mark and Nathan's Conjecture."

Sean: What does that mean?

Ball: Conjecture is an important word that we're going to use this year. A conjecture is when you come up with an idea, something you think is true, that you're trying to prove to other people. How did they prove to us that they think this is true? Who understands what they were doing, that tried to prove to us that the answers would go on forever? What did they show us? What did Betsy do on the board to try to make you believe that this is true? Jeannie, what did she do?

Jeannie: She wrote it up on the board until we thought that they were right.

At this response, Ball tries to focus Jeannie and the others more closely on the specifics of Betsy's reasoning so that they can examine the claim and its supporting argument.

Ball: Well, what kind of example did she use? Do you remember what she showed?

Jeannie: She took away two hundred, and then that's zero, and then she added ten. She said she had a number—Betsy had a number, then she took it away and that was zero, and then she added ten.

Ball: Why does that—why does that help us to think that this is true? Why does that show that the answers would go on forever? I thought maybe people weren't sure about that. Can you—this group say anything more about why you think—oh, Lisa, you think you know why that proves it?

Lisa: Because numbers go on forever, and you can go like infinity take away infinity plus ten equals ten. Because numbers go on and on.

What had started as a relatively ordinary mathematics problem was evolving toward a sophisticated moment of mathematical reasoning, and the construction of a solution to the problem well beyond two-addend sums. The children, engaged in mathematical reasoning, were at the mathematical work of making believe. Taking a disciplinary perspective on classroom teaching and learning can offer ideas for supporting the construction of mathematical knowledge in school by honoring the central role played by the reasoning of justification and seeing it in incipient and sometimes astonishingly developed form even among young children. How we can learn to see and make use of this in practice, linking theories of learning with ideas about the construction of mathematical knowledge in the discipline, is the challenge.

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Notes

1. Transcript of Ball's third grade class, January 19, 1990.

2. See, for example, Ernst von Glasersfeld, "Learning as a Constructive Activity," in *Problems of Representation in the Teaching and Learning of Mathematics*, ed. Claude Janvier (Hillsdale, NJ: Erlbaum, 1987): 3-18.

3. Paul Cobb explores possible tensions and complementarities between individual and sociocultural views of learning in "Where Is the Mind? Constructivist and Sociocultural Perspectives on Mathematical Development," *Educational Researcher* 25 (October 1994): 13-20.

4. D. C. Phillips, "The Good, the Bad, and the Ugly: The Many Faces of Constructivism," *Educational Researcher* 24 (October 1995): 5-12.

5. D. C. Phillips, "How, Why, What, When, and Where: Perspectives on Constructivism in Psychology and Education," in *Issues in Education: Contributions from Educational Psychology* 3(2), ed. Jerry S. Carlson and Robert Calfee (Stamford, CT: JAI Press, 1997), p. 190.

6. Magdalene Lampert has been exploring similar resonances between the practices of knowing mathematics in school and in the discipline. See Magdalene Lampert, "When the Problem Is Not the Question and the Answer Is Not the Solution: Mathematical Knowing and Teaching," *American Educational Research Journal* 27 (Spring 1990): 29-63; also, Magdalene Lampert, "Practices and Problems in Teaching Authentic Mathematics," in *Effective and Responsible Teaching: The New Synthesis*, ed. Fritz Oser, D. Andreas, J. Patry (San Francisco: Jossey-Bass, 1992): pp. 295-314.

7. See Stephen P. Smith, *Children, Learning Theory, and Mathematics: An Analysis of the Role of Language and Representations in Children's Mathematical Reasoning* (PhD dissertation, Michigan State University, 1999). Smith analyzes four nine-year-olds' individual mathematical reasoning, focusing on their use of language and representations as they draw together and use mathematical resources to solve problems.

8. Samuel S. Wineburg, "The Psychology of Teaching and Learning History," in *Handbook of Educational Psychology*, ed. David Berliner and Robert Calfee (New York: Simon & Schuster, 1996): 423-437.

9. Suzanne M. Wilson, "Research on History Teaching," in *Handbook of Research on Teaching*, ed. Virginia Richardson (New York: Macmillan, in press).

10. Shari Levine Rose, *Understanding Children's Historical Sense-making: A View from the Classroom* (PhD dissertation, Michigan State University, 1999).

11. Ibid., p. 215.

12. We thank Raven Wallace for identifying the salience and significance of this third commitment. For a more complete explanation of this teaching and some of its attendant dilemmas, see Deborah Loewenberg Ball, "With an Eye on the Mathematical Horizon: Dilemmas of Teaching Elementary School Mathematics," *Elementary School Journal* 93 (1993): 373-397.

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13. Erna Yackel and Paul Cobb describe inquiry classrooms in which producing and sharing *mathematically different* solutions was encouraged. See Erna Yackel and Paul Cobb, "Sociomathematical Norms, Argumentation, and Intellectual Autonomy in Mathematics," *Journal for Research in Mathematics Education* 27 (1996): 458-477.

14. Data from January 1990 lessons in Ball's third grade class.

15. These records were collected in Ball's third grade mathematics class and Magdalene Lampert's fifth grade, with support from the National Science Foundation, for a project in which we set out to investigate the potential of using new technologies together with extensive records of practice to design new approaches to the pedagogy and curriculum of teacher education. For a discussion of this project and its results, see Magdalene Lampert and Deborah Loewenberg Ball, *Teaching, Multimedia, and Mathematics: Investigations of Real Teaching* (New York: Teachers College Press, 1998).

16. For discussion of what it means to study teaching "from the perspective of practice," see Magdalene Lampert, "Studying Teaching as a Thinking Practice," in *Thinking Practices in Mathematics and Science*, ed. James Greeno and Shelly V. Goldman (Hillsdale, NJ: Lawrence Erlbaum and Associates, 1998): pp. 53-78. Also, see Deborah Loewenberg Ball and Magdalene Lampert, "Multiples of Evidence, Time, and Perspective: Revising the Study of Teaching and Learning," in *Issues in Education Research: Problems and Possibilities*, ed. Ellen Lagemann and Lee S. Shulman (San Francisco: Jossey-Bass, 1999): pp. 371-398.

17. Yackel and Cobb, "Sociomathematical Norms, Argumentation, and Intellectual Autonomy in Mathematics."

18. See Derek Edwards and Neil Mercer, *Common Knowledge* (London: Routledge, 1989); Derek Edwards and Neil Mercer, "Reconstructing Context: The Conventionalization of Classroom Knowledge," *Discourse Processes* 12 (1997): 91-104.

19. Yackel and Cobb, "Sociomathematical Norms, Argumentation, and Intellectual Autonomy in Mathematics."

20. All names are pseudonyms, standardized across published analyses of these data, and selected to be culturally similar to the children's real names. For example, Tembe was from Kenya, and his pseudonym was selected from among similar Kenyan boys' names.

21. Robert Davis, "Mathematics Teaching—With Special Reference to Epistemological Problems," *Journal of Research and Development in Education* (Monograph No. 1, 1967).

22. Lampert, "When the Problem Is Not the Question," p. 42.

23. Ibid., p. 55.

24. Rogers Hall and Andee Rubin propose three participation structures used in the course of the construction of mathematical knowledge—private, local, and public. In the story of Sean numbers, we see an interesting movement across those three. See Rogers Hall and Andee Rubin, "... There's Five Little Notches in Here: Dilemmas in Teaching and Learning the Conventional Structure of Rate," in *Thinking Practices in Mathematics and Science*, ed. James Greeno and Shelly V. Goldman (Hillsdale, NJ: Lawrence Erlbaum and Associates, 1998): pp. 189-235.

25. See Daniel Chazan and Deborah Loewenberg Ball, "Beyond Being Told What to Tell," *For the Learning of Mathematics* 9 (July 1999): 2-10, for an analysis of the teacher's role in the context of discussion-intensive approaches to teaching mathematics.

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Section Four

The Impact of Constructivism on Researchers

Editor's Introduction

There are various ways in which important new ideas, or the ideas of an exciting theorist, can impact the work of researchers. They can suddenly gain attention like a searing flash of light, one that brings unexpected illumination to a field; Charles Darwin's impact on biologists (and some philosophers) in the years immediately following the publication of his book in 1859 seems to have been of this kind. Sometimes they can win slow acceptance and have impact that grows over time. Sometimes the new ideas get co-opted into a movement or research program that is already in existence, in order to give it additional support. Herbert Spencer, the enormously popular late-nineteenth century essayist, treated Darwin's theory this way, regarding it as added proof that his own prior theories were valid. And no doubt there are other ways in which new ideas come to have an effect.

How have the major constructivist ideas impacted researchers? Probably in all the ways depicted above; but, like true love, the course of exciting new ideas does not always run smooth, as shall be seen in the pair of chapters by Kenneth Tobin and Richard Gunstone. Both are internationally known science educators. Tobin has as his main focus the preparation of teachers, and Gunstone is chiefly concerned with studying classroom learning (not that there is a clearcut division between these interests). Both also would identify themselves as constructivists although, as will become clear, they mean different things by this label.

Both Tobin and Gunstone were asked to reflect on the impact that constructivism has had upon them, and how it (whatever they interpreted "it" to be) has influenced their own research and the schools or traditions of research with which they were most familiar. What results are two interesting—and remarkably different—sets of personal reflections by individuals who have very detailed knowledge of the work of important groups of science education researchers around the globe.