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CHAPTER 3

Making Mathematics Reasonable in School

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Benny, a sixth grader whose teacher regards him as one of her best pupils, is interviewed by a classroom visitor (Erlwanger, 1973). The interviewer probes his knowledge of decimals:

Interviewer: What would you get if you add .3 + .4?

Benny: That would be... oh, seven [07]... point oh seven [.07].

Interviewer: How do you decide where to put the point?

Benny: Because there's two points: at the front of the 4 and the front of the 3. So you have to have two numbers after the decimal, because... you know... two decimals. Now like if I had .44, .44 (i.e., .44 + .44), I have to have four numbers after the decimal (i.e., .0088). (p. 4)

He does the same thing when he multiplies decimals.

Interviewer: What about .7 × .5?

Benny: That would be .35.

Interviewer: And how do you decide on the point?

Benny: Because there's two points, one in front of each number; so you have to add both of the numbers left... 1 and 1 is 2; so there has to be two numbers left for the decimal. (p. 5)

Using these methods, Benny produces such answers as 4 + 1.6 = 2.0 and 7.48 – 7 = 7.41 and yet is unaware that his answers are wrong. Because Benny knows that numbers can be represented in different ways, if his answer is different from the teacher's answer or from the answer in the answer key, he simply assumes that his answer is just another way to write the correct one. "It's like a wild goose chase," he explains emotionally . Mathematics is anything but reasonable for Benny. His experience and the conclusions he draws about mathematics are also not uncommon. Contrast his experience in school with that of the children in the following excerpt from Deborah Ball's third-grade class (transcript of class, 19 January 1990). In this example, all names are pseudonyms, standardized across published analyses of these data and selected to be culturally similar to the children's real names. Near the end of a class, the children are concluding a discussion with their teacher:

Riba: (to the class) So Sean is saying that some even numbers, in a pattern, can be even and odd, and some can't. Four can't, because it's two groups. Six can. Eight can't. Ten can. *(Pointing at the number line above the chalkboard, she uses a pointer to mark off consecutive even numbers.)* Can't. Can. Can't. Can....

Ofala: Well, I just think that just because twenty-two is eleven groups, that doesn't mean it's an odd number. My conjecture, I think it's always true, is that if all twos are circled in a number, then it's an even number.

Sean: What conjecture?

Ball: Ofala, tell him what you're talking about when you talk about your conjecture. He's not sure what you're referring to.

Ofala: That conjecture I already...

Sean: That's not a conjecture. That's a *definition.*

These children and their classmates are struggling with concepts of evenness and oddness as a consequence of one child's claim that six could be even, and it could also be odd because you have an odd number of groups of two. Not unlike Benny, they are confronting puzzling mathematics. They had thought that they understood even and odd numbers. How could the number six be both even and odd? But rather than simply accept the student's notion, appeal to the teacher, or dismiss what might be seen as nonsense, the student's classmates are actively reasoning about elaborations of the claim and about the student's reasons for it. They have developed

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resources for inspecting and judging mathematical claims and arguments and for revising and developing mathematical ideas. In the process, they are solidifying their understanding of the definitions of even and odd. For them, unlike Benny, mathematics is reasonable, that is, something about which one can reason.

Mathematical Reasoning and Proof: Essential as Both End and Means

NCTM's *Principles and Standards for School Mathematics* (2000) makes a strong statement about the centrality of mathematical reasoning by including a major standard on reasoning and proof for all grades (p. 56):

> **Reasoning and Proof**
>
> Instructional programs pre-kindergarten through grade 12 should enable all students to—
>
> - recognize reasoning and proof as fundamental aspects of mathematics;
> - make and investigate mathematical conjectures;
> - develop and evaluate mathematical arguments and proofs;
> - select and use various types of reasoning and methods of proof.

Some may regard a standard on reasoning and proof as a nice, but esoteric, embellishment to the main curricular goals in mathematics. Some might even consider the standard expendable. Quite the contrary. Mathematical reasoning is no less than a basic skill. Why do we make this claim?

First, the notion of *mathematical understanding* is meaningless without a serious emphasis on reasoning. What, after all, would mathematical "understanding" mean if it were not founded on mathematical reasoning? Take, for example, understanding multiplication of decimals. Benny, like many adults, counted decimal places to determine the number of places in an answer, but he had no idea what it meant. Not understanding the reasons underlying the procedure meant that he made senseless mistakes. Why does multiplying .7 × .5 produce an answer with two decimal places—.35—whereas adding the same numbers, .7 + .5, yields an answer with just one decimal place—1.2? Unjustified knowledge is unreasoned and, hence, easily becomes unreasonable.

A second reason for claiming that mathematical reasoning is a basic skill is that such reasoning is fundamental to using mathematics. Knowing particular mathematical ideas and procedures as mere fact or routine is insufficient for using those ideas flexibly in diverse cases. First graders who have learned to use the equals sign to signal the result of an operation on two numbers are baffled when presented with 8 = __ + 5. They think that the 8 does not "tell" them to "do" anything, and so they often say there is no number to write in the blank. Their thinking results from using the equals sign in number sentences without reasoning about the concept of equality (Carpenter & Franke, 2001; Falkner, Levi, & Carpenter, 1999). Or consider people who know that the probability of two independent events can be calculated by multiplying the probability of the first event by the probability of the second. For flips of two fair coins, they thus may correctly calculate the probability of two heads, or of two tails, as 1/4 (= 1/2 × 1/2) and yet say that the probability of one head and one tail is 1/3, seeing this case as one of three possibilities (two heads, two tails, or mixed), failing to take account of the fact that the mixed case has two ways of occurring.

Third, mathematical reasoning is fundamental to reconstructing faded knowledge when a demand for it arises. A person who once knew how to divide fractions but has forgotten the algorithm can rebuild a reasonable procedure if he can use the meaning of division and of fractions to reason about dividing one fraction by another. Analyzing the basic meaning of division allows him to see dividing fractions as not essentially different from dividing any whole number by another: Dividing 4/5 by 2/3 is conceptually like dividing 6 by 3, even though the numbers involved in the former example have more complex descriptions. Or, consider a person who learned—but has since forgotten—the formulas to calculate probabilities of independent and of mutually exclusive events. When asked what the probability is of tossing two coins and getting two the same (i.e., two heads or two tails), she may be unsure what operations to perform. Should she multiply the probabilities of each outcome? Add them? What are the outcomes? If she can reason, however, about the logic of a probabilistic situation to analyze whether the outcomes are independent or mutually exclusive, she will likely be able to figure out that she must first multiply the probability of flipping one head times the probability of flipping a second and must do the same for two tails; the successive tosses are independent. Each of these probabilities is 1/2 ×

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1/2, or 1/4. But to calculate the probability of getting two coins the same, she will realize that she then must add the chance of tossing two heads to that of tossing two tails; the two-heads and two-tails outcomes are mutually exclusive. Being able to reason mathematically allows her to recapture a way to work successfully with the problem.

Our point is that mathematical reasoning is as fundamental to knowing and using mathematics as comprehension of text is to reading. Readers who can only decode words can hardly be said to know how to read. Reading competently depends on being able to understand the structures of texts and nuances of language; to interpret authors' ideas; and to visualize, evaluate, and infer meanings. Likewise, merely being able to operate mathematically does not assure being able to do and use mathematics in useful ways. Procedural operations are fundamental to reasonable mathematical activity but are by themselves little more than the analog of reciting text based on the phonetic and structural analysis of words. Making mathematics reasonable means making it reasoned and, therefore, known in useful and usable ways.

This chapter examines what is entailed by mathematical reasoning, and what this looks like as it develops in students in classrooms. Drawing from work on elementary school teaching and learning, and on the practices of mathematics, we discuss a framework for what we call the reasoning of justification—or proof. Next, we turn to what it might take to make mathematics reasonable—that is, what can support the reasoning about mathematics in school.

What Do We Mean by Mathematical Reasoning?

Making mathematics reasonable is more than individual sense making. Making sense refers to making mathematical ideas sensible, or perceptible, and allows for understanding based only on personal conviction. Reasoning, as we use it, comprises a set of practices and norms that are collective, not merely individual or idiosyncratic, and rooted in the discipline. Making mathematics reasonable entails making it subject to, and the result of, such reasoning. That an idea makes sense to me is not the same as reasoning toward understandings that are shared by others with whom I discuss and critically examine that idea toward a shared conviction.

The desire to know and to understand has led people to develop disciplined means of reasoning, of exploring and verifying, of hypothesizing and justifying, in many arenas of human activity. Historians reason about evidence from the past, physicians reason about patients' symptoms, chefs reason about composing ingredients under particular conditions, and pilots reason about instrument readings. In none of these examples do individuals make sense in whatever ways they choose. Instead, in each of these arenas, people have developed methods of reliable thinking that afford inspection, analysis, judgment, and conclusions. These methods of reasoning are the particular means of constructing and evaluating knowledge in a domain.

Much has been written in recent years about constructivist theories of learning and their implications for instruction. Indeed, *constructivism* has arguably been one of the most influential—and most multiply interpreted—ideas in mathematics education. Our research analyzes classroom mathematics learning and teaching in light of ideas about constructing knowledge that are rooted in mathematics as a discipline. Lampert (1990, 1992) has been exploring similar resonances between the practices of knowing mathematics in school and those of knowing mathematics in the discipline. When students are at work in a mathematics class, for example, we see them as constructing mathematical knowledge. Looking at the development of students' knowledge in this way highlights the fundamentally mathematical nature of their—and hence, their teachers'—work. The ways in which students seek to justify claims, convince their classmates and teacher, and participate in the collective development of publicly accepted mathematical knowledge have powerful resonances with mathematicians' work. As students explore problems, make and inspect claims, and seek to prove their validity, even as young children, they engage in substantial forms of mathematical reasoning and make use of mathematical resources. Smith (1999) provides a vivid portrait of this through his close analyses of four nine-year-olds' individual mathematical reasoning. His account focuses on their use of language and representations as they draw together and use mathematical resources to solve problems. This mathematical perspective makes visible some fundamental aspects of mathematics teaching and learning that are hidden when instruction is viewed from a purely cognitive or sociocultural perspective. In particular, this analysis allows for and explores a subject-specific view of learning.

This work finds company in recent advances in other fields (e.g., Wilson, 2001; Wineburg, 1996). Shari Levine Rose (1999), in a study of fourth graders' learning of history in her own classroom, distinguishes between what she was able to see in her students' work when she viewed it from the perspective of generic theories of learning and when she later began to view their work using a lens of historical reasoning. Initially, she explains, she was "influenced by constructivist theories of learning,... [believing] that children drew upon knowledge, values, and beliefs in actively *making sense* of new information." She argues,

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however, that the generic perspective did not help her see how the children were constructing meaning of historical events. But with the historical lens, she was struck by the "historical nature" of children's sense making. They repeatedly sought understanding through constructing stories, much as historians fashion narratives, embedding meaning and interpretation in context. Rose writes about how the historical perspectives that she brought to bear in hearing and interpreting her students made visible how the children came to know the past and constructed meaning of historical events in ways that were much more rooted in the nature of historical reasoning.

Viewed from the perspective of the practicing mathematician, reasoning is one of the principal instruments for developing mathematical understanding and for constructing new mathematical knowledge. Mathematical reasoning can serve as an instrument of inquiry in discovering and exploring new ideas, a process that we call the *reasoning of inquiry*. Mathematical reasoning also functions centrally in justifying or proving mathematical claims, a process that we call the *reasoning of justification*, the focus of this chapter.

Historically, this sort of mathematical reasoning has primarily been found in the high school geometry curriculum in the context of constructing two-column proofs, sometimes treated more as ritual than as an instrument of sense making. What might be entailed by a broader conception and practice of mathematical reasoning in school, as called for by the NCTM's standard on reasoning and proof? In this chapter, we offer and illustrate a conceptual framework for learning and teaching mathematical reasoning.

Teaching Commitments to Mathematics, Students, and Community

Our study of mathematical reasoning is framed by a conception of teaching founded on three specific commitments—to the integrity of the discipline, to taking individual students' thinking seriously, and to the collective as an intellectual community. These commitments orient, but do not determine, practice. First is a commitment to draw from mathematics as a discipline in intellectually sound and honest ways (Ball, 1993; Ball & Bass, 2000a, 2000b; Bruner 1960; Lampert, 1990, 1992, 2001). So in the case of this analysis, we ask, What mathematical habits and dispositions are crucial to doing and learning mathematics? What is the basis of mathematical reasoning? How is mathematical knowledge constructed in the discipline? How do mathematicians interact as a community over knowledge claims? These and other questions have guided our study of making mathematics reasonable, by and for children, in school.

In counterpoint, we assume that teaching demands a sensitivity and responsiveness to students' ideas, interests, lives, and trajectories. Teachers strive to hear their students, to work with them as they investigate and interpret their worlds. Respecting students means attending to who they are and what they bring as well as helping them grow beyond where they are now or where they think they can go. But attending to individual students' interests and proclivities is not enough. In school, teachers must be concerned with "covering" the mandated curriculum so that each student is prepared for the next grade and for the standardized tests used to chart his or her progress. And while many seek to redefine what "*covering the curriculum*" might mean (Lampert 1992, 2001), caring for students means also being responsible to current definitions of progress and learning (Delpit, 1985). Understanding teaching as centrally guided by students' ideas and thinking has led our work on mathematical reasoning to close examination of how students come to hold and believe in mathematical knowledge (Ball & Bass, 2000b).

Finally, the teaching in which we are interested aims to create a classroom community in which differences are valued; in which students learn to care about and respect one another; and in which commitments to a just, democratic, and rational society are embodied and learned (Dewey, 1916; Schwab, 1976). Care and respect for others includes listening to, hearing, and being able to represent others' ideas, even those with which one disagrees. Respect also means taking others' ideas seriously, appraising them critically, and evaluating their validity. In this work, we consider mathematical reasoning as producing more than individual conviction: as generating public knowledge that is usable by the collective.

A Framework for Mathematical Reasoning

The reasoning of justification in mathematics rests on two foundations. One foundation is a body of public knowledge on which to stand as a point of departure and that defines the granularity of acceptable mathematical reasoning within a given context or community. The second foundation of mathematical reasoning is language—symbols, terms, and other representations and their definitions—and rules of logic and syntax for their meaningful use in formulating claims and the networks of relationships used to justify them.

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We first discuss the *base of public knowledge*, the term we use to refer to the knowledge on which claims and arguments are based within some context. Yackel and Cobb (1996) use the label *taken-as-shared* to refer to the meanings, norms, and ideas that are negotiated and used as common within a classroom. Edwards and Mercer (1987; 1989) also write about the development of common knowledge in teaching and learning and focus particularly on the discourse patterns whereby teachers establish such common knowledge. With Yackel and Cobb, we are interested in normative aspects of mathematics discussions specific to students' mathematical activity, such as agreements about what counts as mathematically different solutions or what counts as an acceptable mathematical explanation. And like Edwards and Mercer, we are interested in the development of common knowledge. In our framework, we focus further on the specific mathematical knowledge that is available for public use by a particular community in constructing mathematical claims and in seeking to justify those claims to others. This knowledge is of particular ideas, accepted procedures, defined terms, and methods of mathematical investigation and verification. This knowledge is already assumed or developed—part of the record of the children's prior experience or the class's past work. By identifying it as public, we seek to avoid implying that each member of the community knows it individually in the same way; ascertaining how mathematical knowledge is known individually is an empirical question beyond the scope of our analysis. We mean, rather, to call attention to the knowledge that can comfortably be assumed and used publicly without additional explanation. We contrast such knowledge with ideas or procedures that are not shared and must therefore be established before they can be used to justify claims in the collective discourse of a community.

This base of public knowledge is defined relative to a particular community of reasoners. For professional mathematicians, the base of public knowledge might consist of an axiom system for some mathematical structure (e.g., Euclidean geometry or group theory), simply admitted as given, plus a body of previously developed and publicly accepted mathematical knowledge derived from those axioms. We argue that this idea of a base of public knowledge is useful in understanding the work of a class of elementary school students as well, where this base of public knowledge comprises the expanding set of publicly established ideas and shared knowledge that can be used by the class in explanation or justification.

Take a rudimentary example. Early in September one day, in Ball's third-grade class, the children were working on this problem: "Write number sentences for 10." In the early stages of their work, most students were writing simple equations: $4 + 6 = 10$, $3 + 7 = 10$, and $8 + 2 = 10$. Pressing them gently into more complex solutions, Ball gave the children the following challenge (transcript of class, 18 September 1989):

> I wonder if someone can think of a number sentence that uses more than two numbers here. Just so we have a bunch of ideas of how we could do this. Who can make a number sentence that equals 10 but has more than two numbers adding up to 10?

Quickly, Tembe began, "One plus one plus one plus one—..."

Ball wrote on the board as Tembe rattled off numbers: "One plus one plus one plus one plus one plus one plus one. Plus three."

"Why does that equal ten?" asked Ball, who then called on Haroun to justify Tembe's claim. Haroun, repeating, said, "That's just one plus one plus one plus one...." Ball asked the students, "How do we know that that equals ten?" Riba, eager to respond, explained, "Because one plus one plus one plus one plus one plus one plus one and plus three equals ten." Ball, still not satisfied, said, "You're just sort of reading it. How could you prove it to somebody who wasn't sure?"

Riba: Because I counted it.

Ball: What did you count? What did you find out?

Riba: There's one and the next one is two and the next one is three, next one is four, next one is five, next one is six, next one is seven, next one is—seven, and then three more, eight, nine, ten."

In this simple segment, Tembe's offered solution, the first that included more than two terms, was not something automatically presumed to be within the common knowledge of the class, and Ball asked for justification. If this exchange had taken place in a fifth-grade class, one would likely presume differently. A young child's initial sense of addition comes from counting, which is adding one at a time. Adding many terms at once or adding two numbers larger than one in a single step are higher-order operations, not only for young children but also mathematically. When Riba is called on to explain why the string of terms equals 10, she first just recites the equation. When Ball presses her, "But how could you prove it to somebody who wasn't sure?" Riba

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replies that she "counted," and Ball then encourages her further to make this counting public. In response, Riba expands her explanation:

> There's one and the next one is two and the next one is three, next one is four, next one is five, next one is six, next one is seven, next one is—seven, and then three more, eight, nine, ten.

Perhaps it is through this counting that Riba first proves to herself that Tembe's formulation was valid. Her teacher is requiring her to make this reasoning public to persuade the class as well. Her teacher then publicly validates Riba's work, underlining early in the year a standard for explanation and justification that is more than simple restating of the assertion:

> Do you see the difference in Riba's second explanation? Did you see how she really showed us how it equals ten? The first time you just read it. And the second time you explained it. That was really nice.

Here the teacher does more than praise Riba. She points explicitly to Riba's work—the mathematical explanation she has constructed—and comments on the difference between repeating a statement and explaining it.

A process of reasoning typically consists of a sequence of steps, each of which has the form of justifying one claim by invocation of another, to which the first claim is logically reduced. This process, which merely transforms one claim into another, is not a vicious circle, because the reduced claim is typically of a more elementary or accessible nature and, in a finite number of such steps, one arrives at a claim that requires no further warrant because of being part of the base of publicly shared knowledge and therefore universally persuasive within a particular community of reasoners. Thus, the base of public knowledge both constrains and enables the stepping-stones of an argument. In that sense, publicly shared knowledge defines the granularity of acceptable mathematical reasoning within a given context. In the example above, the addition represented in the equation $1 + 1 + 1 + 1 + 1 + 1 + 1 + 3 = 10$ was not, at that moment, presumed to be part of the base of established knowledge of that class, and so Riba was pressed to reduce the claim to an iterated counting, keeping track of the total as she counted. At that point, Tembe's assertion was sufficiently reduced to a level that relied on knowledge common to the class—counting by ones—and required no further justification.

The crucial issue is how to justify a mathematical claim. One way to justify a claim is to state the claim and to undergird its truth by the sheer force of authority. Students often receive mathematical knowledge in school that is justified by little else than the textbook's or the teacher's assertion. By default, the book has epistemic authority: Teachers explain assignments to pupils by saying, "This is what they want you to do here," and the right answers are found in the answer key. According to Davis (1967), learning to "play by the rules" often involves a "suspension of sense-making" in school mathematics. But that route is the antithesis of warranting claims through a process of mathematical reasoning.

The base of public knowledge consists of knowledge of certain facts and concepts; of the meanings of mathematical terms and expressions; and of procedures and resources for calculation, for problem solving, and even for reasoning. The base of public knowledge is always present, in both latent and active forms, although it may be tacit and only implicit in the discourse of the community, whether mathematicians or third graders.

Whether a particular piece of knowledge is in fact commonly shared is an empirical question, one that a teacher must often assess. Did everyone in the class understand and agree that Riba's elaborated explanation satisfactorily proved Tembe's claim? And, further, how many children were not already adequately convinced by his initial statement? These questions are difficult to resolve fully; they are also not completely within the teacher's or the students' view as a class discussion proceeds. Still, reasoning within a community depends on the presumption of common knowledge and shared established methods. This presumption is most often represented by the use of knowledge already used and established publicly. Arguments that do not build on publicly shared knowledge are unlikely to produce grounded conviction in others. At the same time, the process of reasoning can in fact help build and extend a group's common knowledge. As claims are proved, and ideas developed, the claims may become part of the legacy of public knowledge on which subsequent claims may depend and build.

In our analysis, mathematical language is the foundation of mathematical reasoning that is complementary to the base of publicly shared knowledge. Language is used here expansively, comprising the entire linguistic infrastructure that supports mathematical communication with its requirements for precision, clarity, and economy of expression. Language is essential for mathematical reasoning and for communicating about mathematical ideas, claims, explanations, and proofs. Language is a medium in which mathematics is enacted, used, and created.

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In our framework, language includes the nature and role of definitions in mathematics; the nature of, and rules for, manipulating symbolic notation; and the conceptual compression afforded by timely use of such notation. Definitions and terms play a crucial role: Not simply delivered names to be memorized, definitions and terms originate in, and emerge from, new ideas and concepts and develop through active investigation and reflection. Definitions and terms facilitate reasoning about those new ideas by naming and specification. Decisions about what to name, when to name it, and how to specify that which is being named are important components of mathematical sensibility and discrimination central to the construction of mathematical knowledge. Using symbolic and other representations to encode ideas, as well as decoding ideas represented in symbolic or other forms, are essential communicative tools for the construction of mathematical knowledge. Precise language is also needed to articulate the correspondences between equivalent representations of the same mathematical entity or concept. Notation can be used to compress ideas into forms that, when done skillfully, can reduce computation and manipulation to manageable proportions; how and when to do this is an important skill of mathematical representation useful in reasoning.

Mathematical language is central to constructing mathematical knowledge; it provides resources with which claims are developed, made, and justified. Lampert writes,

> Mathematical discourse is about figuring out what is true, once the members of the discourse community agree on their definitions and assumptions. These definitions and assumptions are not given, but are negotiated in the process of figuring out what is true. (1990, p. 42)

Some disagreements stem from divergent or unreconciled uses of terminology, whereas others are rooted in substantive and conflicting mathematical claims (Crumbaugh, 1998; Lampert, 1998). The ability to distinguish between issues of terminology and issues of mathematical claims requires sensitivity to the nature and role of language in mathematics. We return to this aspect of language in the examples analyzed in the next section of this chapter.

Developing Mathematical Reasoning in a Third-Grade Class

In this section, we take a close look at two classroom episodes in which elementary school students are learning to engage in mathematical reasoning. These classroom episodes are based on Ball's third-grade class records for the school year from 1989 to 1990 (see Lampert & Ball, 1998). We have chosen two excerpts of instruction from the same classroom, one less than 5 months after the first. In fact, the first episode is from the first day of mathematics class in September, and the second, from a day late in January. Our purpose in choosing these two segments is to examine the evolution of the reasoning of justification. What were the students and teacher doing as they sought to reason about mathematics in September, and in what ways were those approaches the same or different in January, less than 5 months later? What do the students seem to be learning? Comparing these two points in time helps establish that mathematical reasoning is something that students can learn to do and, hence, that teachers can teach. Our analysis leads to the final section of the chapter, in which we turn to considering what approaches teachers might adopt to help make mathematics reasonable in school.

In the first episode, early in September, the students are working on an arithmetic problem that has multiple, but finitely many—six—solutions. The task is to find all the solutions and then to show that all solutions have been found. In the later episode, in January, students are working on a conjecture about addition of odd numbers: that an odd number plus an odd number equals an even number. The task is to determine whether this conjecture is true. In both instances, the mathematical work for students calls for justification: How do you know that you have found all the answers? Can you prove that this statement is always true? What the students do in January entails more complex reasoning in that it concerns a claim about all the infinitely many pairs of odd numbers; this work shows substantial development of reasoning skills and sensibility since September.

September: How Do You Know That You Have Them All?

In the first regular mathematics period of the school year, the students are working on the following problem:

> I have pennies, nickels, and dimes in my pocket. Suppose I pull out two coins. How much money might I have?

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In setting up the task, Ball reads the problem with the students and they try an example together. She asks the class, "I'm going to pull out two coins. How much money could I pull out? Like this, like I'm not even looking and I'm going to reach in and pull out one coin and reach in and pull out another. How much money might I have?" "Ten cents," proposes Lucy.

Ball asks for more explanation: "How could I pull out 10 cents?"

Lucy: Two 5s.

Ball: What do other people think about that? If I pulled out two nickels, would I have 10 cents?

Students: Yeah.

Ball: How do you know that? How do you know that that would be 10 cents? Ofala?

Ofala: Because five plus five is ten.

The students work on the task for about 20 minutes while Ball circulates, noting what different students are doing and asking and answering an occasional question. A few times during class, Ball calls the group together to share bits of work or to discuss how the work is proceeding. We zoom in on the class discussion of solutions to the problem because it affords a close look at the group's early efforts to reason mathematically, as individuals and as a group.

About 20 minutes before the end of class, Ball brings them back together. She elicits solutions to the problem from different students. The class discusses and verifies each proposed solution. Ball records the answers on the board in a list:

15¢
20¢
6¢
11¢
2¢
10¢

Ball then reads the solutions off the board:

Ball: We have 15 cents, 20 cents, 6 cents, 11 cents, 2 cents, and 10 cents. Any more? Look at your lists, and see if you have anything that we didn't put on the board. (pause) No, Ofala? You don't have anything else in your notebook? (pause) Jeannie, do you have anything else in your notebook? Riba, do you? Does anybody? How many different possible answers did we find for this problem? How many answers did we come up with here?

Latifa: Six.

Latifa is right. They have come up with six solutions for the problem. However, Ball does not affirm Latifa's answer. Instead, she presses a bit, pushing for justification.

Ball: Six answers. How do we know that we have them all, though? How do we know there isn't a seventh one? Or an eighth one that we didn't find yet?

Ofala, noting a constraint in the given statement of the problem, says she thinks that six is all they can make because they cannot use quarters. Mei comments, "I think we have them all 'cause we had lots of them already and all the people had six."

Lisa speaks next, but her voice is almost inaudible. Ball interrupts to tell her to "talk so that other people can hear you," and directs the rest of the class to listen to what Lisa is saying. Lisa declares, more audibly, "We can only pick up two coins, and if we pick up seven, then we would be picking up three or four." Ball, seeking to understand Lisa's argument, asks for clarification, "So to get a seventh answer, we'd have to pick up three coins?" Lisa assents, and Ball asks her how she knows. Lisa says something about a nickel and a penny, "and if we add another penny, it will be seven and three coins." She seems to talking about a solution for seven cents, but, because it requires three coins, she may be trying to show that the production of a seventh solution—Ball's original question—would require more than two coins.

At this juncture, Sheena raises her hand. She says that she has been working on the question, and she keeps trying to find more solutions but keeps "getting the same answers." Ball repeats her idea to the class: "If you keep picking them up, you'll get the same answers? How many people solved this problem by picking up coins until they got the same ones again?... How many people reached into the box or onto their pile and kept picking them up until they started to repeat?" Many students raise their hands. "What's another way to do the problem?" she asks, looking around.

Mei offers another method: "You first think of what you can make from, you can make out of nickels, dimes, and pennies; and then you take, and then you write them down and you think about it some more until you're—(pause) then you'll get them all."

The children seem reasonably satisfied that they have found all the solutions. They believe they have found them all because they cannot find any more. This argument is empirical or inductive, not deductive. It is the kind of empirical reasoning that can increase confidence in a scientific hypothesis, but it is not a mathematical proof. Still, the children find it persuasive, and they do

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not seem to conceive any method by which one could confirm more definitively that they have all solutions. Ball ends class by urging them to "think very hard before math tomorrow" to see whether they can find any more answers to this problem. By not affirming the completeness of the solution—with six answers—she seeks to maintain some need to show more firmly that they found all the answers, something more than trying many solutions or being personally convinced.

At this early stage, the children have not witnessed or been formally introduced to the notion of mathematical proof. But we see here that the imperative to mathematically justify is being seeded in their work, before they even know what a proof is or looks like. The challenges to justify their own conjectures serve to motivate them to construct, through reflection and analysis, some of the intellectual architecture of mathematical reasoning.

January: Can We Prove That Betsy's Conjecture Is Always True?

We revisit the class almost five months later. The students have been working on problems that involve patterns with sums. For example, they have worked on such problems as the following:

> Erasers cost 2¢, and pencils cost 7¢. How many different combinations of erasers and pencils can you buy if you want to spend exactly 30¢?

The numbers in this problem have been chosen deliberately so that the children might notice that an even number of pencils must be bought if the total has to be 30—an even number—because 7 is an odd number. Indeed, solving such problems has generated a series of conjectures about even and odd numbers:

Even + even = even
Even + odd = odd
Odd + odd = even

Ball has helped the children formulate these conjectures and understand them and has challenged the children to see whether they could prove that these statements are always true. The students have generated long lists of examples in their notebooks, seeking to confirm the conjectures or to find examples that do not work and therefore would show that a conjecture is not (always) true. They are working in small groups on a particular conjecture.

In the course of this work, something new happens. For the first time, the thought occurs to some of the children that to prove such a conjecture is something they have not done before and that doing so presents challenges they have not previously appreciated. Jeannie and Sheena, who are working on the conjecture that an odd number plus an odd number equals an even number, report on their work:

Jeannie: Me and Sheena were working together, but we didn't find one that didn't work. We were trying to prove that... you can't prove that Betsy's conjecture (*odd + odd = even*) always works. Because, um, there's, um, like numbers go on and on forever, and that means odd numbers and even numbers go on forever, so you couldn't prove that *all* of them work.

Ofala protests. She declares, looking closely into her notebook, that she has tried "almost eighteen" of them and even some special cases, and they have all "worked," so she thinks that "it can always work."

Mei then offers a very different kind of objection. "I think it could always work because with those conjectures [motioning to several previously discussed and widely agreed-on conjectures posted above the chalkboard], we haven't even tried them with all the numbers there is, so why do you say that those work? We haven't tried those with all the numbers that there ever could be."

Ball asks whether this statement means that she is disagreeing with Jeannie or agreeing with her. "I disagree," replies Mei, emphatically. Ball asks her to clarify what she is saying, and she repeats and amplifies her point that the class has already agreed to accept other conjectures even though they were not able to check them with every number. But Jeannie and Sheena are not to be deterred. Jeannie says that she never said that those other conjectures were true all the time. "Then why didn't you disagree when everyone agreed with those conjectures?" presses Mei. Sheena explains that they had never thought about any of this before, and now they see that a problem exists because numbers go on forever. Class ends with this issue unresolved, and Ball tells the students to think some more about what Jeannie and Sheena are claiming: "What do other people think? Do you think we can't prove that it's always true, or do you think that we can prove that it's always true?"

A few days later, the class takes a big step. Betsy, together with a few classmates, presents a proof of the conjecture, which they first illustrate with 7 + 7:

> What we figured out how it's always true is that we would have seven dots, or lines, plus seven lines (draws fourteen hash marks on the board, seven at a time)



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> . . . and then (counts the lines), and then we said that we had to circle them by twos (she starts circling twos from right to left)...



> and also we said that... an even number is... just a second (finishes circling groups of two lines) that if you added another even one to an odd number, or another 1 to an odd number, then it would equal an even number 'cause all odd numbers if you circle them, what we found out, all odd numbers if you circle them by twos, there's 1 left over; so if you... plus 1, um, or *if you plus another odd number, then the two 1's left over will group together, and it will make an even number* (italics added).

And then after a couple of questions by other children, Betsy elaborates:

> If an odd number plus an odd number, if you add another number with an odd number, then it equals an even number because an even number plus 1 equals an odd number, so *if you added two odd numbers together, you can add the 1's left over and it would always equal an even number* (italics added).

Ball turns to the class:

> Does anybody want to comment either on the example or on Betsy's proof? Do people think that does prove that an odd plus an odd would always be even?

Mei, seeming to focus mainly on the illustrated example, 7 + 7, questions whether this proves that the conjecture would always be true. She wonders about checking all numbers. She says, "I don't think so, because you don't know about, like, in the thousands, and you don't know the numbers, like, you don't even know how you pronounce it, or how you say it." Jeannie disagrees, arguing that Betsy is showing that this would work for all numbers: "It could be any two odd numbers because, um, there's always one left."

Mei still does not accept this idea: "I think what I am trying to say... I'm trying to say that that's [the diagram that Betsy has drawn on the board] only one example. You can't really say that it will work for every odd number, even... I don't think it would work for numbers that we can't say or figure out what they are."

Riba wonders, too: "How does Betsy know that it will always work? She never tried all the numbers." Betsy herself concedes the last point:

Betsy: Mathematicians can't even do that. You would die before you counted every number.

Riba: I know, that's why I mean, that's what I mean... you don't know if it always works.

Ball: Anyone else have a comment? Sheena?

Sheena: I agree sort of with Riba, but you don't have to try all the numbers, because you would die before you tried all the numbers, because there are some numbers you can't even pronounce and some numbers you don't even know that they're there. But still, I still think it's true because, um, we've tried like enough of, of examples already. It proves it's true.

Riba: I don't think so. Like if... even though we don't know how to pronounce it, we could write it down and say, like, "this number."

The discussion continues for a few more minutes. The children wrestle with whether Betsy's argument shows that any two odd numbers will always add to an even number. Unlike the discussion in September, here the lack of agreement on a proof prevents consensus in the group that the conjecture is true. Struggling with whether they can prove the conjecture, they are still not willing to believe something is true without being able to prove it. Some children claim that they have tried enough examples to be convinced; others remind them that "there are numbers we have not checked" that might show the conjecture to be false. But the idea of a general proof is beginning to capture many of the students. And Riba, struggling to articulate some way of referring to the idea of "some number that we don't know what it is," unwittingly highlights the need for new mathematical language.

Ball sends the children back into their groups again, seeking to prove or disprove the conjectures. A bit later in that same class, she asks groups to report on their work and progress.

Mark, who has been quiet throughout the preceding discussions, reports on his work with Nathan: "Well, first, me and Nathan, we were just getting answers and we weren't thinking about proof. And we were still getting answers and [then] we were trying to prove, and Betsy came and she had proved it, and then we all agreed that it would work." Mark and Nathan's work has been transformed by what has happened. No longer content to produce lists of examples, they glimpse what it might mean to prove a statement, and they see the possibility of

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doing so. Moreover, they are convinced by Betsy's argument that her conjecture is always true.

The work of this class is a long way from what they were doing just less than 5 months earlier. We turn in the next section to consider the nature of the children's mathematical reasoning and explore the teaching and learning of reasoning in class.

September: What Is It Going to Take to "Do Mathematics" in Third Grade?

Our first episode takes place at the beginning of the year. Many students do not know one another, and the teacher is unacquainted with what they know. Norms for mathematical work have yet to be developed, and the language and ideas on which the class's work can be based are just emerging. If students are to engage collectively in substantial mathematical work, they need to learn to listen attentively, and critically, to the views of others. They need to learn to offer, justify, and critically evaluate mathematical claims, both their own and those of others. These skills are likely not ones that they already possess. Neither are these skills ones they are likely to develop simply by being expected to behave or think in such ways. How will they develop these practices of mathematical reasoning and acquire the knowledge and skills that support them?

One site for such learning is situated in the nature of the mathematical tasks on which students are asked to work. Consider the problem on this first day of class:

> I have pennies, nickels, and dimes in my pocket. Suppose I pull out two coins. How much money might I have?

As stated, this problem could be solved by providing a single answer—for example, 6¢. In this class, however, the enacted task involves several levels of mathematical work beyond this outcome. First, students are expected to justify their answers. How is six cents an answer to this problem? Why is it an answer to this problem? Second, students are directed to find multiple answers to the problem and, together, to find all the possible solutions. Finally, in the whole-group discussion, the teacher asks the class to prove that they have found all such solutions. These layers of additional mathematical work, part of the task in its enacted version, create contexts for mathematical reasoning implied but not automatically entailed by the original problem statement.

Although this problem's opportunities for mathematical reasoning are not explicit in its basic formulation, the problem has a substantial curriculum potential. *Curriculum potential*, as Ben-Peretz (1990) develops it, is a characteristic of curriculum that describes the opportunities for learning and teaching offered by the structure and scope of the tasks. The problem's possibilities emerge and are developed as the students enact the problem in class. The opportunities that arise for learning to reason mathematically depend on how the problem is used. Enactment can either extend or degrade the mathematics learning potential of a given task (Stein, Grover, & Henningsen, 1996).

A second site for the development of mathematical reasoning is embedded in how the task is used in class and can be seen in the class discussion of solutions to the problem. Whether and how these solutions are discussed can hide or make visible the underlying reasoning. For example, in this discussion, we see the teacher ask, "How much money might I have?" When a child offers 10 cents as a solution, she does not simply affirm this answer or ask others whether they agree. Instead, she asks for justification: How could ten cents be made under the given conditions—two coins, using pennies, nickels, and dimes? Translating nickels to their numeric value, the child responds, "Two fives." Ball then asks the other children to verify this answer by asking a question: Do two nickels form ten cents? Several say yes, and she presses for an explanation: "How do you *know* that?" The unpacking of what underlies the solution "ten cents" becomes part of the work and opens up opportunities to develop practices of mathematical reasoning.

Later in the lesson, we see the teacher engage the class in another aspect of mathematical reasoning: whether the solutions identified and listed on the board cover all possible solutions. The process of recording on the board the various student solutions, which might otherwise remain unnoticed by some of the students, makes individuals' work available for public discussion and helps set up the resources for their collective work. First Ball asks them to look at their own work and see whether they have other solutions not listed on the board. That they can easily view all the solutions makes work on this question more accessible. When no one can produce other solutions, Ball asks whether they know that they have found them all. The children's responses reveal that they are relying exclusively on empirical information to determine the solution's completeness: They have tried over and over again but keep getting the same answers. For this problem, despite the teacher's repeated questions—Are there other answers? How do we know that there is not a seventh solution?—the children are not yet able to prove that exactly six solutions are possible. Merely questioning students is not sufficient to help them learn this particular aspect of mathematical

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reasoning. Still, although these students seem satisfied that they have found all possible solutions, their teacher does not permit the problem to be finally complete. She directs the students to think more about this question, leaving some doubt about the current solution.

Consider this: This problem does have exactly six solutions, and well before the end of class the children have successfully identified and justified them. This work has engaged the students in reasoning as a substantial component of the task. Some might say that the class has finished the work. But the mathematical reasoning possible—and necessary—is not complete. Obtaining six correct answers is not equivalent to showing that six is the total number of answers to this problem. The teacher is the one who moves this remaining mathematical issue into the public workspace, who presses it, and who maintains it. And more work—more opportunities to engage in this sort of reasoning, more teaching, and more learning—will be necessary to develop the students' ability to prove the completeness of a finite solution set. Students can, for example, be encouraged to imagine drawing first one coin and then a second, each of which can be a penny (*p*), nickel (*n*), or dime (*d*), leading to exactly nine (3×3) possibilities: *pp*, *pn*, *pd*, *np*, *nn*, *nd*, *dp*, *dn*, and *dd*. Of course, some of them, for example, *pn* and *np*, yield the same amount of money, 6 cents, so altogether only six different amounts of money are represented. Alternatively, students could be encouraged to use a more direct route that does not pay attention to the order in which the coins are drawn. Thus one might first consider those cases that draw at least one penny: *pp*, *pn*, and *pd*. With no more pennies, if at least one nickel is drawn, the possibilities are *nn* and *nd*. Finally, the only case involving neither pennies nor nickels is *dd*. Thus we have six possibilities, and each one gives a different amount of money. Even without this extra work, the teacher may have helped seed the children's awareness of an important mathematical question: Do I have all the solutions, and how do I know?"

That this class is observed from the very beginning of the school year is important. How does a foundation begin to be laid for the development of mathematical reasoning? First, the problem with its layers of possible mathematical engagement can likely be made accessible to every student. The arithmetic called for, that of adding combinations of 1, 5, and 10, is within reach of these third graders. And although some students are unfamiliar with the values of U.S. coins, the teacher talks about their worth and makes actual coins available for the students. This accessibility of the task helps create a common space for collective mathematical reasoning. Second, for a base of knowledge to be used in class, an approach that is acceptably usable by this group of children must be made visible and shared. To that end, the work that students do on this problem, individually and in group discussion, reveals what students know and can do, as well as aspects of their mathematical habits and dispositions. Finally, practices of mathematical work are begun during this class. Children are called on to communicate their solutions, not only to the teacher but also to the class. When Lisa offers a comment in class, Ball asks her to speak so that others can hear and directs the others to listen. Children are also asked to evaluate the productions and claims of the other students. The work of the class is recorded both in student notebooks for future reference and on the chalkboard for public discussion.

Late January: Examining the Foundations for Building Mathematical Knowledge in Third Grade

In the previous class example, we saw the foundations beginning to be laid for mathematical reasoning. We saw the initial stages of work that could serve to identify and make public the mathematical knowledge usable for making and justifying claims. And we saw practices of reasoning being first developed.

Four and a half months later, we glimpse the more established foundation for their work and the evolution of these practices. For example, in September the children were not used to providing justification, whereas in January they assumed this need. In the fall, we saw them unconcerned with one another's ideas, but at midyear we see them focused on, and attentive to, the collective work.

This episode provides a glimpse of the children using tools of mathematical reasoning to grapple with a problem. First, the task has a sophisticated quality: to prove (or disprove) "Betsy's conjecture," that "odd + odd = even." In other words, Betsy is suggesting that the sum of two odd numbers is even. Whereas the two-coin problem called for a finite number of solutions with the task of exhibiting them and showing that all solutions had been found, Betsy's conjecture, in contrast, asserts a property of any pair of odd numbers, and there are infinitely many odd numbers.

Here the task is not to seek or rule out some undiscovered example, and exhibiting all the examples is impossible because the claim is universally quantified over an infinite set. The challenge here, rather, is to confirm a property for all the examples. Because this confirmation requires reasoning about examples that cannot all be tried individually, two of the students, Jeanie and

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Sheena, make the revolutionary proposal that Betsy's conjecture cannot be proved. They have taken the class to a new precipice that challenges the very foundations of their work together: If you cannot try all the numbers, or even know what all of them are, then how can you ever say that something about all of them is really true? This sudden realization of the implications of the infinitude of numbers takes Jeannie and Sheena aback. They have encountered the force of making a claim about a property of an infinite set and the corresponding burden of trying to prove it.

The mathematical task—seeking to prove whether a conjecture is true—suggests that the students' base of mathematical knowledge and skills has developed since September. The students are familiar with and are able to use definitions of even and odd numbers. Their testing out of various pairs of odd numbers, including large cases, shows that they are now more able to add whole numbers quickly and accurately. They know to arrange and keep track of their empirical work as they experiment with these combinations, and they have acquired the skills to do so. They also are developing a language—*conjecture*, *definition*, and *proof*—about mathematical knowledge. Although their meanings for these terms are as yet informal, this vocabulary has emerged in the context of doing mathematical work and provides the students with terms of reference for knowing and doing mathematics. The students' work on the conjecture shows, too, that their mathematical reasoning skills have also developed. At the outset, the children verify many numerical cases of Betsy's conjecture. They seem to conclude that the conjecture must be true because they have verified a large number of diverse cases and not found a counterexample. Listing many examples suffices to convince many of the students that the conjecture is true. This stage of the work is, like the earlier work on the two-coin problem, empirical or inductive but not mathematically conclusive. Still, the children are more inclined to question and seek to prove the truth of a mathematical claim.

But for two children, Jeannie and Sheena, a new insight emerges. In working on this already challenging task, they begin to realize a new problem: They are trying to establish the truth of a claim for all numbers, but they cannot check all numbers. They observe, "You can't prove that Betsy's conjecture (*odd + odd = even*) always works. Because... , like, numbers go on and on forever and that means odd numbers and even numbers go on forever, so you couldn't prove that all of them work."

Jeannie and Sheena are making the dramatic claim that Betsy's conjecture cannot be proved, because, essentially, it represents an infinite number of claims—one for each pair of odd numbers—and verifying infinitely many cases is beyond human capacity. This insight into the nature of claims involving universal quantification—all—over an infinite set and the seeming paradox of having the capacity to prove such claims with a procedure of finite extent is profound.

These two students not only reach an important mathematical insight but are confident enough to share it with the class. Both were rather quiet and shy in September. Both were also then convinced that checking examples was an adequate means of justification.

That they reach this insight is not the only aspect to notice here, however. As the teacher turns their idea to the class, the other students take up on it immediately. They spontaneously consider the girls' claim and are both able and willing to respond. Lucy agrees, but Ofala objects, citing the long list of examples she has recorded in her notebook. And Mei, invoking a principle of logical consistency, points out all the other conjectures that have already been accepted as true without verifying every possible number.

The students have acquired habits of mathematical reasoning: They assume that a classmate's posited claim is theirs to respond to, and they draw on past work and knowledge as they respond. Even students who do not talk in the whole-group discussion on this day, such as Mark and Nathan, use this discussion to continue and amend their own work. Mark remarks later, "Well, first, me and Nathan, we were just getting answers and we weren't thinking about proof." He describes how they went back to work after this discussion and began trying to prove the conjecture, not just list more examples.

Over the next couple of classes the situation advances dramatically. The students manage to construct a convincing mathematical proof of Betsy's conjecture: "All odd numbers if you circle them by twos, there's one left over, so if you... plus one, um, or if you plus another odd number, then the two left over will group together, and it will make an even number."

This argument is a general one, about any two odd numbers, not a particular numerical example. It invokes a general property that derives from one of the definitions of odd numbers, namely, that a number is odd if, when it is grouped into twos, there is one left over. It also uses a definition of even numbers, namely, a number is even if nothing is left over when it is grouped into twos. Betsy's argument is that when combining two odd numbers, one combines the two ones left over to form a pair, and so the combination of the two odd numbers can be grouped into pairs with nothing left over and hence is

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even. Consequently, the proof is able to transcend the objection raised by Jeannie and Sheena—that if each pair of odd numbers cannot be inspected directly, determining the truth of the claim is impossible.

The proof relies critically on the definition of odd numbers, a definition that itself contains an assertion quantified over the infinite set of all odd numbers and so it has the capacity to support logical conclusions of similarly infinite purview.

We can distill here the compelling logic of this argument, which Betsy haltingly struggled to articulate but of which not all the students were immediately convinced. Sheena spoke for more than herself when she said, "I agree sort of with Riba, but you don't have to try all the numbers, because you would die before you tried all the numbers, because there are some numbers you can't even pronounce and some numbers you don't even know that they're there. But, still, I still think it's true because, um, we've tried like enough of, of examples already. It proves it's true." Riba responds, "I don't think so. Like if...even though we don't know how to pronounce it, we could write it down and say, like, 'this number.'" We can see in this proposal from Riba the conceptual germ of algebraic notation for a variable as representing any number in some class. Part of Betsy's and Riba's struggle is the lack of a developed mathematical language with which to designate an indeterminate quantity ranging over some infinite set of numbers—for example, the idea of "any odd number," which can be represented using algebraic notation in the form $2x + 1$, where x is any whole number. Riba's effort to name "this number" can be heard as a plea for this linguistic tool, which the work on this problem of proof naturally invites.

In this episode, we see among the children the first emergence of some fundamental mathematical constructs: the need and challenge, from Jeannie and Sheena; the possibility, from Betsy, of structural mathematical proof grounded in precise definitions; norms for logical consistency, offered by Mei; and quantified language for discussing properties of all elements of an infinite collection, stated by Betsy and Riba. And we see evidence of assimilation of these ideas by other students, here, Mark and Nathan. This episode provides a glimpse of how far the students have come in developing a base of public knowledge that they can use to reason about mathematical claims as well as a host of skills and habits, language, and practices for engaging in reasoning, individually and collectively. We see the migration of newly constructed student ideas and productions through the class as they become rejected or else assimilated and consensual, thereby becoming eventually a part of the base of public knowledge. That the sum of two odd numbers will always be an even number was to become one such item.

Teaching Reasonable Mathematics in School

One common reaction to such accounts as the ones in this chapter is "My students could never do this" or "These students must be gifted." We disagree. Students can reason in this way, but only when they have learned to do so. It would be peculiar to argue that students could never learn division or fractions or concepts of probability, but neither is it usually questioned that these things need to be taught and learned if they are to be known. Our exploration of mathematical reasoning as it began to develop in this third-grade class makes plain that such reasoning—just as anything else we teach in mathematics—is something that can be and is learned. The base of public knowledge that can be used to establish new ideas and the mathematical language for their expression and use lay the foundation for practices of constructing mathematics. We saw both the base and the language develop from September to January. Whereas in September, the students replied to questions when asked and focused on the teacher, by January, they engaged mathematical claims themselves and responded to and used one another's statements. We saw signs of the children's learning to ask such questions as "How can I prove that this solution is valid?" and "Do I have all the solutions to this problem?" We glimpsed them learning to expect explicitness from a definition and working with an agreed-on definition to prove a claim.

We argue in this chapter that mathematical reasoning is not auxiliary to basic goals of mathematics education but rather is fundamental to knowing and being proficient with mathematics—that mathematical reasoning is itself basic. If that argument is true, then the work of teaching includes helping students learn to engage in mathematical reasoning. What does this work entail for teachers? What are some of the things that teachers might do to enable the development of mathematical reasoning? We offer a preliminary set of ideas about practices of teaching mathematics in ways that support the learning of mathematics in ways that are reasonable, that is, subject to reasoning by students and their teachers.

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Designing and Using Mathematical Tasks

A first arena is to select and adapt tasks in order to create the need and opportunity for substantial mathematical reasoning. Many problems exist that, by design, create considerable space for mathematical reasoning and are worth seeking out and learning to use well. But good opportunities can also be created by enhancing ordinary mathematical work. The two-coin problem discussed above provides one example. Taken as originally worded and then enacted by letting students find one or more possible answers would have afforded much less space for exploring the problem's mathematical territory. Students might have been able to record 20¢ or 11¢ on their papers and handed them in. The task would have remained one that supplied application and some basic addition practice and little else. So one thing teachers can do is make justification a part of the tasks they use, adding such questions as "How do you know that you have all the solutions?" to problems, or "Suppose someone challenged your solution. How could you prove to them that your answer is right?" Teachers can ask students to evaluate solutions by giving them solved problems and asking them whether the solutions are valid. Problems can be adapted or designed to precipitate student conjectures. Carpenter and Franke (2001) are engaged in designing such sequences of tasks with teachers and are finding that even young students of elementary age can begin to notice patterns or generate conjectures from such work. For example, teachers might ask students first to calculate

$$\begin{array}{r} 68 \\ +57 \\ \hline \end{array}$$

and then the following:

$$\begin{array}{r} 67 \\ +58 \\ \hline \end{array} \qquad \begin{array}{r} 69 \\ +56 \\ \hline \end{array} \qquad \begin{array}{r} 70 \\ +55 \\ \hline \end{array}$$

Teachers might then ask students what should be added to 65 to yield the same sum as the original problem. Asking whether students notice a pattern might precipitate some useful general ideas about arithmetic expressions, such as that changing the numbers you are adding so that one goes up the same amount that the other goes down results in the same sum. These claims could also be examined and proved in their own right. This example illustrates how mathematical reasoning can be naturally integrated into basic work on addition computation, a site for important student practice. Teachers can also seek opportunities to augment such ordinary mathematical tasks. For example, the work above could be extended: What if the sum (125) is multiplied by 2? How might you change the terms you are adding to make the equation true? What if you multiplied 125 by 5?

$$\begin{array}{r} + \quad ? \\ \quad ? \\ \hline 2 \times 125 \end{array} \qquad \begin{array}{r} 68 \\ +\ 57 \\ \hline 125 \end{array} \qquad \begin{array}{r} + \quad ? \\ \quad ? \\ \hline 5 \times 125 \end{array}$$

Such explorations of operations could be further extended to subtraction, multiplication, and division.

Making Records of Mathematical Knowledge

As we saw in this chapter, mathematical tasks themselves do not comprise the work that teachers and students do. How they are used is equally important. This section on making records to support teaching and learning draws on, and has benefited especially from, our research with Ben-Peretz and Cohen. If mathematical reasoning is to be central to the work that teachers and students do, then teachers play an important role in making mathematical knowledge and language public and in scaffolding their use. One element concerns making records of the mathematical work of the class: how the chalkboard is used; when the overhead projector, with its transitory record, is not used; what is recorded on pieces of paper and preserved on classroom walls; and what students are helped to record in notebooks. Doing this recording makes mathematical work public and available for collective development, scrutiny, and subsequent use. Definitions that have been worked out can be posted in the room. Students' conjectures may be provisionally recorded while they are being tested; established conclusions may later replace such conjectures. For example, once the students completed their work on the conjectures about adding odd and even numbers, the agreed-on conjectures were printed on colored paper and taped to the wall:

Odd + odd = even

Even + even = even

Odd + even = odd

Even during a class period, having available different solutions, diagrams, and ideas that teacher and students can use to reason about the problems at hand supports the work. Without such records, the "texts" of class work are remote from individual and collective attention and the history of an idea's or a problem's development is obscured. Being able to look at three distinct methods for

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solving a given problem permits such questions as "Are these all correct?" "Are these really different or the 'same'?" and "In what way are these the same?" Seeing arrayed the six solutions to the two-coin problem made it easier to consider whether the class had found all possible solutions. And having previous conjectures posted above the board gave Mei the grounds to remind the class that they had already agreed on these earlier claims. Obviously, what teachers decide to display on the board or in a permanent spot, who places it there, and how such public records are used are all matters worthy of careful attention.

Naming

Making records of mathematical work includes considering how to name or refer to ideas, methods, problems, solutions, and so on. *Betsy's conjecture* referred to the claim, suggested by Betsy, that an odd number plus an odd number equals an even number. Naming it as such provided a compact means of referring to this provisional idea. Naming it after Betsy reflected a common practice in mathematics of identifying mathematical ideas by their authors. As teachers choose to name ideas for students, sensitivity to whose ideas are named and to the possibility that several students may have come up with the same idea matters. Mathematical ideas, methods, and solutions can also be given other sorts of names. *Our working definitions of even and odd numbers* might be the label for a provisional articulation of these definitions. Worth considering, however, are ways to make knowledge easily retrievable and its referents collectively recognized.

Making Ideas Public

Making mathematical knowledge and language public also requires deftly moving individuals' ideas into the collective space. When individual students offer ideas or solutions, these often become bilateral exchanges between them and their teachers. For students to take note of and use one another's ideas requires deliberate work. First, students' contributions must be comprehensible to their peers. This step requires that students speak loudly enough to be heard and learn to listen closely to others' talk. When Lisa offered an idea in the September class, Ball asked her to speak up and told others to consider what she was saying. Teachers' moves may be as simple as that. But teachers may also have to help students articulate their ideas in ways that are both audible and understandable. They may have to ask individual students to repeat what they have said or ask them questions about what they are saying. These tactics help make the ideas that individual students are conveying more explicit, so their contributions do not remain private, vague, half-developed, and weakly articulated statements to which others cannot usefully respond. Making ideas public involves helping make them accessible for others' consideration. Once ideas are more clearly expressed, teachers may ask students to respond directly to another student's point, may ask students to explain what a classmate has said, or may ask students whether they can articulate how a classmate reached a conclusion. For example, in September, Ball asked Haroun to justify a claim that Tembe made. In January, Ball asked,

> What do other people think about this argument that Jillian and Lin are having? Lin's saying that she is still not convinced that this could apply for all numbers—big, big numbers, and Jillian is saying that it would always be true because there'd always be one left over from each odd number. Rania, what do you think about that?

She might also have asked Betsy's classmates whether they could see how she was using the definition of odd numbers to make her proof. As students' ideas become regular sources of the class's work, students will both speak more clearly and ask one another to speak more audibly.

Teachers also play an important role in modeling the use of others' ideas, using public mathematical knowledge, and using language carefully. Teachers can make references to "Lucy's method" or remark on uses of established ideas: "How is this idea related to Tembe's conjecture?" They can expect connections with public knowledge by asking such questions as "Are you using the definition that we agreed on for even numbers?" or "How does what we figured out about multiplying by 10 or 100 or 1000 help with this problem?"

Conclusion

We have argued that mathematical reasoning is inseparable from knowing mathematics with understanding, and that mathematical reasoning is itself a basic mathematical skill. Making mathematics reasonable in school is not a frill. Doing so, however, requires more than slogans and exhortation. Simply posing open-ended mathematical problems that require mathematical reasoning is not sufficient to help students learn to reason mathematically. Neither is merely asking students to explain their thinking.

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In this chapter, we characterized two foundations of the mathematical reasoning of justification: a base of public knowledge and mathematical language. The reasoning of justification requires pedagogy and design, just as the teaching of any other aspect of mathematics. Students must learn to use publicly established ideas, methods, and language to make, inspect, validate, improve, and extend the mathematical knowledge of the class. And teachers must provide the resources for, and create an environment that encourages and makes possible, complex student work.

In the last section of the chapter, we identified some initiatives and moves that teachers might make to support the development of students' mathematical reasoning. One is the design or enhancement of mathematical tasks to create a demand for mathematical reasoning. Another is to create lasting records of established as well as currently negotiated mathematical knowledge. Such records afford both reference to prior work and a shared sense of the class's base of public knowledge. This record making, and the classroom discourse that it enables, includes the development of mathematical language—such as working definitions of basic concepts, and names for important ideas. Putting students' mathematical productions into public space and helping make those productions comprehensible so that other students can critically examine them entail norms of communication, of both careful and audible articulation of mathematical ideas, and of attentive listening.

All these steps require learning. In the episodes analyzed above, we have tried not only to illustrate what substantial mathematical reasoning by third graders can look like but also to show that these skills and practices were learned across time, from the first day of class in September. We also have made explicit some of the pedagogical moves and designs that contributed to that learning, teacher practices that play an important role in making mathematics reasonable for students. But these practices are not ones that teachers can do simply because it is suggested that they do so. Just as students need to learn to reason mathematically, so, too, must teachers develop and learn practices to support such learning. If students are to acquire the basic skills to build mathematical knowledge by reasoning, then teachers must have opportunities to develop the knowledge and practice necessary to make mathematics reasonable in school.

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