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Toward a Practice-Based Theory of Mathematical Knowledge for Teaching[1]

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Mathematics professor: *The situation is terrible: Only one of the students in my mathematics content course for teachers can correctly divide .0045789 by 3.45.*

Fifth grader: *Ms. Ball, I can't remember how to divide decimals. There's something my stepfather showed me about getting rid of the decimal point, but I can't remember what he said and, besides, I don't think that would work.*

With all the talk of teachers' weak mathematical knowledge, we begin with a reminder that the problem on the table is the quality of mathematics teaching and learning, not—in itself—the quality of teachers' knowledge. We seek in the end to improve *students'* learning of mathematics, not just produce *teachers* who know more mathematics.

Why, then, talk about teacher knowledge here? We focus on teacher knowledge based on the working assumption that how well teachers know their subjects affects how well they can teach. In other words, the goal of improving students' learning depends on improving teachers' knowledge. This premise—widely shared as it may be, however—is not well supported empirically. We begin with a brief glimpse of the territory in which the problem on which we are working fits. Our purpose is to set the context for our proposal for reframing the problem.

The Problem: What Mathematics Do Teachers Need to Know to Teach Effectively?

The earliest attempts to investigate the relationship between teachers' mathematics knowledge and their students' achievement met with results that surprised many people. Perhaps the best known among these is Begle's (1979) analysis of the relationship between the number of courses teachers had taken past calculus and student performance. He found that taking advanced mathematics courses[3] produced positive main effects on students' achievement in only 10% of the cases, and, perhaps more unsettling, negative main effects in 8%. That taking courses could be negatively associated with teacher effects is interesting because the negative main effects are not easily explained by the criticism that advanced mathematics courses are not relevant to teaching, or that course-taking is a poor proxy for teachers' actual mathematical knowledge. Such claims support finding *no* effects, but not *negative* effects.

So why might these variables be associated with negative effects? One explanation might rest with the compression of knowledge that accompanies increasingly advanced mathematical work, a compression that may interfere with the unpacking of content that teachers need to do (Ball & Bass, 2000a). Another explanation might be that more coursework in mathematics is accompanied by more experience with conventional approaches to teaching mathematics. Such experience may impress teachers with pedagogical images and habits that do not contribute to their effectiveness with young students (Ball, 1988).

Observational studies of beginning and experienced teachers reveal that teachers' understanding of and agility with the mathematical content does affect the quality of their

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teaching. For example, Eisenhart, Borko, Underhill, Brown, Jones, and Agard (1993) describe the case of a middle school student teacher, Ms. Daniels, who was asked by a child to explain why the invert-and-multiply algorithm for dividing fractions works. Ms. Daniels tried to create a word problem for three-quarters divided by one-half by saying that three quarters of a wall was unpainted. However, there was only enough paint to cover half of the unpainted area. As she drew a rectangle to represent the wall and began to illustrate the problem, she realized that something was not right. She aborted the problem and her explanation in favor of telling the children to "just use our rule for right now" (p. 198).

Despite having taken two years of calculus, a course in proof, a course in modern algebra, and four computer science courses, Ms. Daniels was unable to provide a correct representation for division of fractions or to explain why the invert-and-multiply algorithm works. In fact, she represented multiplication, rather than division, of fractions.

Many other studies reveal the difficulties teachers face when they are uncertain or unfamiliar with the content. In 1996, the National Commission on Teaching and America's Future (NCTAF) released its report which proposed a series of strong recommendations for improving the nation's schools that consisted of "a blueprint for recruiting, preparing, and supporting excellent teachers in all of America's schools" (p. vi). Asserting that what teachers know and can do is the most important influence on what students learn, the report argues that teachers' knowledge affects students' opportunities to learn and learning. Teachers must know the content "thoroughly" in order to be able to present it clearly, to make the ideas accessible to a wide variety of students, and to engage students in challenging work.

The report's authors cite studies that show that teacher knowledge makes a substantial contribution to student achievement. They argue that "differences in teacher qualifications accounted for more than 90% of the variation in student achievement in reading and mathematics" (Armour-Thomas, Clay, et al., 1989, cited in National Commission on Teaching and America's Future, 1996, p. 8). Still, what constitutes necessary knowledge for teaching remains elusive.

An important contribution to the question of what it means to know content for teaching has been the concept of "pedagogical content knowledge" (Grossman, 1990; Shulman, 1986, 1987; Wilson, Shulman, & Richert, 1987). Pedagogical content knowledge, as Shulman and his colleagues conceived it, identifies the special kind of teacher knowledge that links content and pedagogy. In addition to general pedagogical knowledge and knowledge of the content, teachers need to know things like what topics children find interesting or difficult and the representations most useful for teaching a specific content idea. Pedagogical content knowledge is a unique kind of knowledge that intertwines content with aspects of teaching and learning.

The introduction of the notion of pedagogical content knowledge has brought to the fore questions about the content and nature of teachers' subject matter understanding in ways that the previous focus on teachers' course-taking did not. It suggests that even expert personal knowledge of mathematics often may be inadequate for teaching. Knowing mathematics for teaching requires a transcendence of the tacit understanding that characterizes much personal knowledge (Polanyi, 1958). It also requires a unique understanding that intertwines aspects of teaching and learning with content.

In 1999, Liping Ma's book, *Knowing and Teaching Elementary Mathematics* attracted still broader interest in this issue. In her study, Ma compared Chinese and U.S. elementary teachers' mathematical knowledge. Producing a portrait of dramatic differences between the two groups, Ma used her data to develop a notion of "profound understanding of fundamental mathematics", an argument for a kind of connected, curricularly-structured, and longitudinally coherent knowledge of core mathematical ideas.

What is revealed by the work in the years since Begle's (1979) famous analysis? Although his work failed to show expected connections between teachers' level of mathematics and their students' learning, it seems clearer now that mathematical knowledge for teaching has features that are rooted in the mathematical demands of teaching itself. These are not easily detected by how much mathematics someone has studied. We are poised to make new

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gains on an old and continuing question: What do teachers need to know to teach mathematics well? But we are poised to make those gains by approaching the question in new ways.

Reframing the Problem: What Mathematical Work Do Teachers Have to *Do* to Teach Effectively?

The substantial efforts to trace the effects of teacher knowledge on student learning, and the problem of what constitutes important knowledge for teaching, led our research group[4] to the idea of working bottom up, beginning with practice. We were struck with the fact that the nature of the knowledge required for teaching is underspecified. On one hand, what teachers need to know seems obvious: They need to know *mathematics*. Who can imagine teachers being able to explain how to find equivalent fractions, answer student questions about primes or factors, or represent place value, without understanding the mathematical content? On the other hand, less obvious is what "understanding mathematical content" *for teaching* entails: *How* do teachers need to know such mathematics? What else do teachers need to know of and about mathematics? And how and where might teachers use such mathematical knowledge in practice?

Hence, instead of investigating what teachers need to know by looking at what they need to teach, or by examining the curricula they use, we decided to focus on their work. What do teachers *do*, and how does what they do demand mathematical reasoning, insight, understanding, and skill? We began to try to unearth the ways in which mathematics is entailed by its regular day-to-day, moment-to-moment demands. These analyses help to support the development of a *practice-based theory of mathematical knowledge for teaching*. We see this approach as a kind of "job analysis", similar to analyses done of other mathematically intensive occupations, from nursing to engineering and physics (Hoyles, Noss, & Pozzi, 2001; Noss, Healy, & Hoyles, 1997), to carpentry and waiting tables. In this case, we ask:

- *What* mathematical knowledge is entailed by the work of teaching mathematics?
- *Where* and *how* is mathematical knowledge used in teaching mathematics? How is mathematical knowledge intertwined with other knowledge and sensibilities in the course of that work?

How We Do Our Work

Central to our work is a large longitudinal NSF-funded database, documenting an entire year of the mathematics teaching in a third grade public school classroom during 1989–90.[5] The records collected across that year include videotapes and audiotapes of the classroom lessons, transcripts, copies of students' written class work, homework, and quizzes, as well as the teacher's plans, notes, and reflections. By analyzing these detailed records of practice, we seek to develop a theory of mathematical knowledge as it is entailed by and used in teaching. We look not only at specific episodes but also consider instruction over time, examining the work of developing both mathematics and students across the school year. What sort of larger picture of a mathematical topic and its associated practices is needed for teaching over time? How do students' ideas and practices develop and what does this imply about the mathematical work of teachers?

A database of the scale and completeness of this archive affords a kind of surrogate for the replicable experiment. More precisely, the close study of small segments of the data supports the making of provisional hypotheses (about teacher actions, about student thinking, about the pedagogical dynamics), and even theoretical constructs. These hypotheses or constructs can then be "tested" and, in principle, refuted, using other data with this archive itself. We can inspect what happened days (or weeks) later, or earlier, or look at a student's notebook, or at the teacher's journal for evidence that confirms or challenges an idea. Further, when theoretical ideas emerge from observations of patterns across the data, we can use them as a lens for viewing other records, of other teachers' practices, and either reinforce or modify or reject our theoretical ideas in line with their adaptability to the new data.

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Structured data like those collected in this archive can constitute a kind of public "text" for the study of teaching and learning by a community of researchers. This would permit the discussion of theoretical ideas to be grounded in a publicly shared body of data, inherently connected to actual practice. As norms for such discourse are developed, so also would the expansion of such data sets to support such scholarly communication be encouraged. In our experience, disciplined inquiry focused on such a practice-based "text" tends to dissipate ideologically based disputes, and to assure that theoretical constructs remain connected to practice.

Even with such records of practice in which much is available to be seen, casual observation will no more produce insight about teaching and learning than unsophisticated reading of a good mathematics text will produce mathematical insight. Teaching and learning are complex and dynamic phenomena in which, even with the best of records, much remains hidden and needing interpretation and analysis. Our approach to this work has been to mobilize an interdisciplinary group representing expertise in teaching practice, in disciplinary mathematics, in cognitive and social psychology, and in educational research. Over time we have collectively crafted well-honed skills for sensitive observation of records (particularly video) of teaching practice. One of our research aims is to articulate some of the demands, skills, and norms that this entails; in short, a kind of *methodology of interdisciplinary observation of teaching*.

Our work uses methods of mathematical and pedagogical analysis developed in previous research (see, for example, Ball & Bass, 2000a, b; 2003). Using a framework for examining practice, we focus on mathematics as it emerges within the core task domains of teachers' work. Examples of this work include representing and making mathematical ideas available to students; attending to, interpreting, and handling students' oral and written productions; giving and evaluating mathematical explanations and justifications; and establishing and managing the discourse and collectivity of the class for mathematics learning. As we analyze particular segments of teaching, we seek to identify the mathematical resources used and needed by the teacher. For example, when a student offers an unfamiliar solution, we will look for signs of whether and how the teacher understands the solution, and what he or she did, and what the mathematical moves and decisions are. Our coding scheme includes both mathematical content (topics, procedures, and the like) and practices (mathematical processes and skills, such as investigating equivalence, reconciling discrepancies, verifying solutions, proving claims). The goal of the analysis is twofold: First, to examine how and where mathematical issues arise in teaching, and how that impacts the course of the students' and teacher's work together; and second, to understand in more detail, and in new ways, what elements of mathematical content and practice are used—or might be used—and in what ways in teaching.

What Mathematical Problems Do Teachers Have to Solve?

This approach has led us to a new perspective on the work of mathematics teaching. We see many things teachers do when teaching mathematics that teachers of any subject must do—keep the classroom orderly, keep track of students' progress, communicate with parents, and build relationships with students. Teachers select and modify instructional tasks, make up quizzes, manage discussions, interpret and use curriculum materials, pose questions, evaluate student answers, and decide what to take up and what to leave. At first, these may sound like generic pedagogical tasks. Closer examination, however, reveals that doing them requires substantial mathematical knowledge and reasoning. In some cases, the work requires teachers to think carefully about a particular mathematical idea together with something about learners or learning. In other cases, the work involves teachers in a kind of mathematical reasoning, unencumbered by considerations of students, but applied in a pedagogical context. Our analyses have helped us to see that teaching is a form of mathematical work. Teaching involves a steady stream of mathematical problems that teachers must solve.

Let us consider an example. Teachers often encounter students using methods and solutions different from the ones with which they are familiar. This can arise for a variety of reasons, but when teachers see methods they have not seen before, they must be able to ask and answer—for themselves—a crucial *mathematical* question: What, if any, is the method, and

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will it work for all cases? No pedagogical decision can be made prior to asking and answering this question. Consider, for example, three alternative methods for multiplying 35 × 25:

(A)
35
x 25
125
+ 75
875

(B)
35
x 25
175
+ 700
875

(C)
35
x 25
25
150
100
+ 600
875

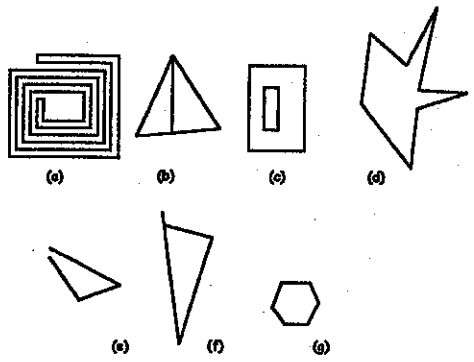
A teacher must be able to ask what is going on in each of these approaches, and to know which of these is a method that works for multiplying any two whole numbers. These are quintessential *mathematical*—not *pedagogical*—questions. Knowing to ask and how to answer such mathematical questions is essential to being able to make wise judgments in teaching. For instance, a decision about whether or not to examine such alternative methods with the students depends on first sizing up the mathematical issues involved in the particular approach, and whether they afford possibilities for worthwhile mathematics learning for these students at this point in time.

Being able to sort out the three examples above requires more of teachers than simply being able to multiply 35 × 25 themselves. Suppose, for example, a teacher knew the method used in (B). If a student produced this solution, the teacher would have little difficulty recognizing it, and could feel confident that the student was using a reliable and generalizable method. This knowledge would not, however, help that same teacher uncover what is going on in (A) or (C).

Take solution (A) for instance. Where do the numbers 125 and 75 come from? And how does 125 + 75 = 875? Sorting this out requires insight into place value (that 75 represents 750, for example) and commutativity (that 25 × 35 is equivalent to 35 × 25), just as solution (C) makes use of distributivity (that 35 × 25 = (30 × 20) + (5 × 20) + (30 × 5) + (5 × 5). Even once the solution methods are clarified, establishing whether or not each of these generalizes still requires justification.

Significant to this example is that a teacher's own ability to solve a mathematical problem of multiplication (35 × 25) is not sufficient to solve the mathematical problem of teaching—to inspect alternative methods, examine their mathematical structure and principles, and to judge whether or not they can be generalized.

Let us consider a second example. This example again helps to make visible the mathematical demands of simple, everyday tasks of teaching. Different from the first, however, it reveals that the mathematical demands are not always so closely aligned to the content outlines of the curriculum (in the example above, multiplication). Suppose that, in studying polygons, students produce or encounter some unusual figures and ask whether any of them is a polygon.



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This is an natural mathematical question. Knowing how to answer it involves mathematical knowledge, skill, and appreciation. An essential mathematical move at this point is to consider the definition: What makes a figure a polygon? A teacher should know to consult the textbook's definition, but may well find an inadequate definition, such as this one, found in a current textbook:

> *A closed flat two-dimensional shape whose sides are formed by line segments.*

Knowing that it is inadequate requires appreciating what a mathematical definition needs to do. This one, for example, does not rule out (b) or (c) or (f), none of which is a polygon. But if the textbook definition is unusable, then teachers must know more than a formally correct mathematical definition, such as:

> *A simple closed plane curve formed by straight line segments.*

Teaching involves selecting definitions that are mathematically appropriate and also usable by students at a particular level. For example, fifth graders studying polygons would not know definitions for "simple" or "curve", and therefore would not be able to use this definition to sort out the aberrant figures from those we would call polygons.

To determine a mathematically appropriate and usable definition for "polygon", a teacher might try to develop a suitable definition, better than those found in the available textbooks. Consider this effort:

> *A sequence of three or more line segments in the plane, each one ending where the next one begins, and the last one ending where the first one begins. Except for these endpoints, shared only by two neighboring segments, the line segments have no other points in common.*

This definition, unlike the previous one in the textbook, is mathematically acceptable, as it does properly eliminate (b), (c), and (f), as well as (e). But a teacher would still need to consider whether or not her students can use it. Definitions must be based on elements that are themselves already defined and understood. Do these students already have defined knowledge of terms such as "line segments", "endpoints", and "plane", and do they know what "neighboring" and "in common" mean? In place of "neighboring", would either "adjacent" or "consecutive" be preferable? Knowing definitions for teaching, therefore, requires being able to understand and work with them sensibly, treating them in a way that is consistent with the centrality of definitions in doing and knowing mathematics. Knowing how definitions function, and what they are supposed to do, together with also knowing a well-accepted definition in the discipline, would equip a teacher for the task of developing a definition that has mathematical integrity and is also comprehensible to students. A definition of a mathematical object is useless, no matter how mathematically refined or elegant, if it includes terms that are beyond the prospective user's knowledge.

Teaching requires, then, a special sort of sensitivity to the need for precision in mathematics. Precision requires that language and ideas be meticulously specified so that mathematical problem solving is not unnecessarily impeded by ambiguities of meaning and interpretation. But the need for precision is relative to context and use. For example, a rigorous and precise definition for odd numbers as those numbers of the form $(2k + 1)$, or of even numbers as multiples of two, would not be precise for first graders first encountering the notion of "even number". Because they cannot decode the meaning of $(2k + 1)$ and do not have a definition of "multiple", the elements used to create a precise definition remain obscure and unusable to six-year-olds. Needed for teaching are definitions that are both correct and useful. Knowing what definitions are supposed to do, and how to make or select definitions that are appropriately and usefully precise for students at a certain point, demands a flexible and serious understanding of mathematical language and what it means for something to be *precise*.

Taken together, these two examples show that knowing mathematics in and for teaching includes both elements of mathematics as found *in* the student curriculum—that is, standard computational algorithms, multiplication, and polygons—as well as aspects of knowing and doing mathematics that are less visible in the textbook's table of contents—

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sensitivity to definitions or inspecting the generality of a method, for example. These examples also provide a glimpse of how centrally mathematical reasoning and problem solving figure in the work of teaching.

Examples of Mathematical Problems of Teaching

To illustrate ways in which solving mathematical problems is a recurrent part of the work of teaching, we turn next to some examples. Each of our examples was chosen to show different aspects of the mathematical work of teaching, and to develop the portrait of the mathematics that teaching entails, and the ways in which mathematics is used to solve problems of teaching mathematics.

1. *Choosing a task to assess student understanding: Decimals*

   One thing that teachers do is monitor whether or not students are learning. To do that, on an informal basis, they pose questions and tasks that can provide indicators of whether or not students are "getting it".

   Suppose you wanted to find out if your students could put decimal numbers in order. Which of the following lists of numbers would give you best evidence of students' understanding?

a)	.5	7	.01	11.4
b)	.60	2.53	3.14	.45
c)	.6	4.25	.565	2.5

   Obviously, any of these lists of numbers can be ordered. One possible decision, then, is that the string makes no difference—that a correct ordering of any of the lists is as good as any other.

   However, a closer look reveals differences among the lists. It is possible to order (a) and (b) correctly without paying any attention to the decimal point at all. Students who merely looked at the numbers, with no concern for decimal notation, would still put the numbers into the correct order. List (c), however, requires attention to the decimal places: If a student ignored the decimal point, and interpreted the list as a set of whole numbers, he would order the numbers as follows:

   .6 2.5 4.25 .565

   instead of:

   .565 .6 2.5 4.25

   So what sort of mathematical reasoning by the teacher is involved? More than being able to put the numbers in the correct order, required here is an analysis of what there is to understand about order, a central mathematical notion, when it is applied to decimals. And it also requires thinking about how ordering decimals is different from ordering whole numbers. For example, when ordering whole numbers, the number of digits is always associated with the size of the number: Numbers with more digits are larger than numbers with fewer. Not so with decimals. 135 is larger than 9, but .135 is not larger than 9. This mathematical perspective is one that matters for teaching, for, as students learn, their number universe expands, from whole numbers to rationals and integers. Hence, teaching requires considering how students' understanding must correspondingly expand and change.

2. *Interpreting and evaluating students' non-standard mathematical ideas: Subtraction algorithms*

   Teachers regularly encounter approaches and methods with which they are not familiar. Sometimes students invent alternative methods and bring them to their teachers. In other cases, students have been taught different methods.

   Suppose you had students who showed you these methods for multi-digit subtraction. First, you would need to figure out what is going on, and whether it makes sense mathematically. Second, you would want to know whether either of these methods works in general.

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$$\begin{array}{r} 307 \\ -\ 168 \\ \hline 2\ -6\ -1 \\ \hline 1\ 3\ 9 \end{array} \qquad \begin{array}{r} {}^{29}\cancel{3}\cancel{0}7 \\ -\ 168 \\ \hline 139 \end{array}$$

The first method uses integers to avoid the standard, error-prone, method of regrouping. It surely works, for it reduces the algorithm to a simple procedure that relies on the composition of numbers, and does not require "borrowing". The second regroups 307 by regarding it, cleverly, as 30 tens plus 7 ones, to 29 tens and 17 ones. Asking mathematical questions, a teacher might ask himself: Even if the methods work, what would either one look like with a 10-digit number? Do both work as "nicely" with any numbers? Skills and habits for analyzing and evaluating the mathematical features and validity of alternative methods play an important role in this example. Note, once again, that this is different from merely being able to subtract 307 – 168 oneself.

3. *Making and evaluating explanations: Multiplication*

Independent of any particular pedagogical approach, teachers are frequently engaged in the work of mathematical explanation. Teachers explain mathematics; they also judge the adequacy of explanations—in textbooks, from their students, or in mathematics resource books for teachers.

Take a very basic example. In multiplying decimals, say 1.3 × 2.7, one algorithm involves carrying out the multiplication much as if the problem were to multiply the whole numbers 13 and 27. One multiplies, ignoring the decimal points.

$$\begin{array}{r} 1.3 \\ \times\ 2.7 \\ \hline 91 \\ 26\ \ \\ \hline 351 \end{array}$$

Then, because the numbers are decimals, the algorithm counts over two places from the right, yielding a product of 3.51.

But suppose one wants to explain why this execution of the algorithm is wrong:

$$\begin{array}{r} 1.3 \\ \times\ 2.7 \\ \hline 91 \\ 26\ \ \\ \hline 11.7 \end{array}$$

and to explain why the standard algorithm works? In this typical instance, a student has not "moved over" the 26 on the second line, and has, in addition, simply placed the decimal point in the position consistent with the original problem.

Is it sufficient to explain by saying that the 26 has to be moved over to line up with the 6 under the 9? And to count the decimal places and insert the decimal point two places from the right?

These are not adequate mathematical explanations. Teaching involves explaining why the 26 should be slid over so that the 6 is under the 9: this involves knowing what the 26 actually represents. In whole number multiplication, if this were 13 × 27, then the 26 on the second line would represent the product of 13 and 20, or 260. In this case, the 26 represents the product of 1.3 and 2—260 tenths, or 2.6. Developing sound explanations that justify the steps of the algorithm, and explain their meaning, involves knowing much more about the algorithm than simply being able to perform it. It also involves sensitivity to what constitutes an explanation in mathematics.

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What Does Examining the Work of Teaching Imply About Knowing Mathematics for Teaching?

Standing back from our investigation thus far, we offer three observations. First, our examination of mathematics teaching shows that teaching can be seen as involving substantial mathematical work. Looking in this way can illuminate the mathematics that teachers have to do in the course of their work. Each of these involves mathematical problem solving:

- Design mathematically accurate explanations that are comprehensible and useful for students
- Use mathematically appropriate and comprehensible definitions;
- Represent ideas carefully, mapping between a physical or graphical model, the symbolic notation, and the operation or process;
- Interpret and make mathematical and pedagogical judgments about students' questions, solutions, problems, and insights (both predictable and unusual);
- Be able to respond productively to students' mathematical questions and curiosities;
- Make judgments about the mathematical quality of instructional materials and modify as necessary;
- Be able to pose good mathematical questions and problems that are productive for students' learning;
- Assess students' mathematics learning and take next steps.

Second, looking at teaching as mathematical work highlights some essential features of knowing mathematics *for teaching*. One such feature is that mathematical knowledge needs to be *unpacked*. This may be a distinctive feature of knowledge for teaching. Consider, in contrast, that a powerful characteristic of mathematics is its capacity to *compress* information into abstract and highly usable forms. When ideas are represented in compressed symbolic form, their structure becomes evident, and new ideas and actions are possible because of the simplification afforded by the compression and abstraction. Mathematicians rely on this compression in their work. However, teachers work with mathematics as it is being learned, which requires a kind of decompression, or "unpacking", of ideas. Consider the learning of fractions. When children learn about fractions they do not begin with the notion of a real number, nor even a rational number. They begin by encountering quantities that are parts of wholes, and by seeking to represent and then operate with those quantities.[6] They also encounter other situations that call for fractional notation: distances or points on the number line between the familiar whole numbers, the result of dividing quantities that do not come out "evenly" (e.g., 13 ÷ 4, and later 3 ÷ 4). Across different mathematical and everyday contexts, children work with the elements that come together to compose quantities represented conveniently with fraction notation. Meanwhile, their experiences with the expansion of place value notation to decimals develops another territory that they will later join with fractions to constitute an emergent concept of rational numbers. Teachers would not be able to manage the development of children's understanding with only a compressed conception of real numbers, or formal definition of a rational number. So, although such a conception has high utility for the work of mathematics, it is inadequate for the work of teaching mathematics.

Another important aspect of knowledge for teaching is its connectedness, both across mathematical domains at a given level, and across time as mathematical ideas develop and extend. Teaching requires teachers to help students connect ideas they are learning—geometry to arithmetic, for example. In learning to multiply, students often use grouping: 35×25 could be represented with 35 groups of 25 objects. But, for example, to show that $35 \times 25 = 25 \times 35$, or that multiplication is commutative, grouping is not illuminating. More useful is being able to represent 35×25 as a rectangular area, with lengths of 25 and 35 and an area of 875 square units. This representation makes it possible to prove commutativity, simply by rotating the rectangle, showing $a \times b = b \times a$. Or, later, helping students understand the meaning of $x^2 + y^2$ and how it is different from $(x + y)^2$, it is useful to be able to connect the algebraic notions to a geometric representation:

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Using these two diagrams helps to show that $x^2 + y^2$ is not the same as the $x^2 + 2xy + y^2$ produced by multiplying $(x + y)^2$.

Teaching involves making connections across mathematical domains, helping students build links and coherence in their knowledge. This can also involve seeing themes. For example, the regrouping of numbers that is part of the standard multi-digit subtraction algorithm is not unlike the renaming of fractions into equivalent forms. In each case, numbers are written in equivalent forms useful to the mathematical procedure at hand. To add two fractions with unlike denominators, it is useful to be able to rewrite them so that they have the same denominator. In subtraction, to subtract 82 – 38, it is useful to be able to rewrite 82 as "7 12" (7 tens and 12 ones)—also an equivalent form. Seeing this connection is useful in helping students appreciate that, to be strategic and clever in mathematics, quantities can be written in equivalent, useful forms.

Teaching also requires teachers to anticipate how mathematical ideas change and grow. Teachers need to have their eye on students' "mathematical horizons" even as they unpack the details of the ideas in focus at the moment (Ball, 1993). For example, second grade teachers may need to be aware of the fact that saying, "You can't subtract a larger number from a smaller one", is to say something that, although pragmatic when teaching whole number subtraction, is soon to be false. Are there mathematically honest things to say instead that more properly anticipate the expansion to integers, and the accompanying changes in what is true or permissible?[7]

One final observation about what we are finding by examining teaching as mathematical work: In our analyses, we discover that the critical mathematical issues at play in the lesson are not merely those of the curricular topic at hand. For example, in a lesson on subtraction with regrouping, we saw the students grappling with three different representations of subtraction and struggling with whether these were all valid, and, if so, whether and how they represented the *same* mathematical operation. They were examining correspondences among representations, investigating whether or not they were equivalent. Although the content was subtraction, the mathematical entailments of the lesson included notions of equivalence and mapping. In other instances, we have seen students struggling over language, where terms were incompletely or inconsistently defined, and we have seen discussions which run aground because mathematical reasoning is limited by a lack of established knowledge foundational to the point at hand. These lessons brought to the surface important aspects of mathematical reasoning, notation, use of terms and representation. Entailed for the teacher would be both the particular mathematical ideas under discussion as well as these other elements of knowing, learning, and doing mathematics. We have seen many moments where the teachers' attentions to one of these aspects of *mathematical practice* is crucial to the navigation of the lesson, and we have also seen opportunities missed because of teachers' lack of mathematical sensibilities and knowledge of fundamental mathematical practices.

Attending to *mathematical practices as a component of mathematical knowledge* makes sense. As children—or mathematicians, for that matter—do and learn mathematics, they are engaged in using and doing mathematics, as are their teachers. They are representing ideas, developing and using definitions, interpreting and introducing notation, figuring out whether a solution is valid, and noticing patterns. They are engaged in mathematical *practices* as they engage in learning mathematics. For example we often see students whose limited ability to interpret and use symbolic notation, or other forms of representation impedes their work and their learning. Similarly, being able to inspect, investigate, and determine whether two solutions, two representations, or two definitions are similar, or equivalent is fundamental to many arenas of school mathematics. Students and teachers are constantly engaged in

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situations in which mathematical practices are salient. Yet, to date, studies of mathematical knowledge for teaching have barely probed the surface of what of mathematical practices teachers would need to know and how they would use such knowledge.

Conclusion: Learning Mathematics for Teaching

What we know about teachers' mathematical knowledge, learning mathematics for teaching, and the demands of teaching mathematics suggests the need to reframe the problem of preparing teachers to know mathematics for teaching. First, although many U.S. teachers lack adequate mathematical knowledge, most know some mathematics—especially some basic mathematics. Identifying what teachers know well and what they know less well is an important question for leveraging resources wisely toward the improvement of teachers' opportunities to learn mathematics. What many teachers lack is mathematical knowledge that is useful to and usable for teaching. Of course, some teachers do learn some mathematics in this way from their teaching, from using curriculum materials thoughtfully and by analyzing student work. However, many do not. Inadequate opportunities exist for teachers to learn mathematics in ways that prepare them for the work, and few curriculum materials effectively realize their potential to provide mathematical guidance and learning opportunities for teachers. Also important to realize is that professional mathematicians may often not know mathematics in these ways, either. This is not surprising, for the mathematics they use and the uses to which they put it are different from the mathematical work of teaching children mathematics. They, too, in helping teachers, will have mathematics to learn, and new problems to learn to solve, even as they also contribute resources. This summary suggests that reframing the problem and working on it productively is both promising and challenging.

Our analysis suggests that teachers' opportunities to learn mathematics should include experiences in unpacking familiar mathematical ideas, procedures, and principles. But, as the polygon example shows, learning mathematics for teaching must also afford opportunities to consider other aspects of proficiency with mathematics—such as understanding the role of definitions and choosing and using them skillfully, knowing what constitutes an adequate explanation or justification, and using representations with care. Knowing mathematics for teaching often entails making sense of methods and solutions different from one's own, and so learning to size up other methods, determine their adequacy, and compare them, is an essential mathematical skill for teaching, and opportunities to engage in such analytic and comparative work is likely to be useful for teachers. As we examine the work of teaching, we are struck repeatedly with how much mathematical problem solving is involved. It is mathematical problem solving both like and unlike the problem solving done by mathematicians or others who use mathematics in their work. Practice in solving the mathematical problems they will face in their work would help teachers learn to use mathematics in the ways they will do so in practice, and is likely also to strengthen and deepen their understanding of the ideas. For example, a group of teachers could analyze the three multiplication solutions presented here, determine their validity and generality, map them carefully onto one another. They could also represent them in a common representational context, such as a grid diagram or an area representation of the multiplication of 35 × 25 (see Ball, 2003).

Seeing teaching as mathematically-intensive work, involving significant and challenging mathematical reasoning and problem solving, can offer a perspective on the mathematical education of teachers, both preservice and across their careers. It opens the door to making professional education of teachers of mathematics both more intellectually and mathematically challenging, and, at the same time, more deeply useful and practical.

Notes

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2. The authors acknowledge Heather Hill for her contributions to the ideas discussed in this paper.
3. An "advanced course" was defined as a course past the calculus sequence.

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4. Members of the Mathematics Teaching and Learning to Teach Project include Mark Hoover, Jennifer Lewis, Ed Wall, Rhonda Cohen, Laurie Sleep, and Andreas Stylianides.
5. These data were collected under a 1989 National Science Foundation grant to Ball and Magdalene Lampert, then at Michigan State University.
6. Understanding the development of ideas was implied by Dewey in his distinction between the psychological and the logical aspects of subject matter in *The Child and the Curriculum* (1902). By "psychological", he did not mean the way in which a particular idea might be learned, but the epistemological composition of its growth.
7. A group of prospective teachers suggested saying, "We can't subtract larger numbers from smaller ones using the numbers we have right now".

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