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Crossing Boundaries To Examine the Mathematics Entailed in Elementary Teaching

Deborah Loewenberg Ball

Beginning in 1996, Hyman Bass, a professional mathematician, and I, an educational researcher and elementary school teacher, launched into a line of original work in educational research. The problem on which we have been working is one that is central to both professional education and instructional improvement: What mathematical knowledge is needed to teach elementary school mathematics well? This paper describes the foundations, focus, and methods of our work on this problem, and provides some examples of its results.

Three features of this work are unusual. The first is that, in our research, we turn the usual approaches to this problem on their head. Instead of producing lists of what teachers should know, we begin with an examination of practice itself. We identify and analyze core activities of teaching - figuring out what students know, choosing and managing representations of mathematical ideas, appraising, selecting, and modifying textbooks, deciding among alternative courses of action - and we seek to identify the mathematical resources entailed in these teacher activities. This approach, a kind of "job analysis" of classroom teaching focused on the actual work that teachers do, offers a view of mathematics as it is used in practice. The aim of this research is to analyze and articulate ways in which mathematical insight, sensibilities, and knowledge are entailed in the practice of teaching. Such a mathematical perspective on the work of teaching is needed to extend what we currently know about the mathematical resources needed for teaching, the role of such resources in practice, and, by implication, what opportunities teachers and prospective teachers need to develop in order to teach mathematics well.

A second feature of this work is the data with which we work: extensive multimedia records of one year of third grade mathematics teaching, including videos of

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daily lessons, audiotapes, transcripts, teacher journals, students' daily work, homework, quizzes, and standardized tests.[1] This database represents a primary record of teaching and learning, allowing a form of unusually direct access to practice.

A third feature of the research is the collaboration itself, unprecedented in research on teaching. Examining what it takes for two scholars from such different fields to work together is itself an interesting question worthy of analysis. The research advances a set of new questions on teacher knowledge, develops new methods of research, and is yielding new results, as a function of crossing these disciplinary boundaries.

Neither of us could have anticipated doing this sort of work together. Hyman Bass' forays into education began slowly and with no particular design, but escalated rapidly as his interests were piqued and he showed himself to be an intellectually formidable and yet respectful participant in discussions about mathematics education. Beginning in the late 1980s, he found himself increasingly invited to participate on education panels or attend meetings where issues of mathematics education were discussed. As a National Academy of Sciences member, he became involved in the Mathematical Sciences Education Board (MSEB) at a critical moment in its history, first as a member and later, to his astonishment, as its chair.[2] Around the same time, the American Mathematical Society formed the Committee on Education, a major step for this professional organization focused principally on the discipline and its own considerable particular challenges and interests. In 1995, Bass also became the chair of this Committee on Education. Rapidly, other opportunities arose - serving on advisory panels, reading and critiquing manuscripts, attending meetings about mathematics curriculum, giving a keynote talk about reform calculus, speaking about the conflicts over the reform of K-12 mathematics instruction. Although he continued to do his own mathematical research, publishing several important papers and two books during this same period, Bass devoted substantial time to the requests that came his way. Later, Bass became president of the International Commission on Mathematics Instruction (ICMI), a role centered on building connections between mathematics education and mathematics (through its parent organization, the International Mathematics Union [IMU]).

As he participated and increasingly provided leadership to these various groups and organizations, Bass was finding many of the issues more intellectually interesting that he had anticipated. Although he had launched into this as professional service, his curiosity and interest heightened. The problems in mathematics education were complex, their framing, pursuit, and solution elusive. The culture of education seemed foreign, permeated with unfamiliar assumptions and language. He sought out and learned from the teachers on the MSEB. He listened closely as they described the problems they faced in their work, and tried to understand the frustrations they expressed and he began to glimpse a fascinating set of issues that

[1]The database on which we draw was collected with funding from the National Science Foundation in 1989-90, through a grant to Deborah L. Ball and Magdalene Lampert. The funding supported the documentation (videotaping, audiotaping, photcopying, transcribing) of a year's worth of mathematics lessons, student work and tests, and teacher notes in two classrooms: Ball's third grade and Lampert's fifth grade. The original aims of the research and how it evolved are reported in Lampert, M. & Ball, D. L. (1998). *Teaching, multimedia, and mathematics: Investigations of real practice.* New York: Teachers College Press.

[2]Since 1993, Bass has chaired the National Research Council's Mathematical Sciences Education Board, an interdisciplinary panel of professionals concerned with mathematics education. The MSEB's work focuses on policy issues related to K-16 mathematics education.

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were entailed in the teaching of mathematics. The diverse membership of MSEB, and the complexity of the issues that the group tried to address, made this a site of considerable intellectual ferment in his thinking, even as he tried to cross some of the tough communication boundaries. In 1996, Bass gave a paper at a meeting at the Mathematical Sciences Research Institute (MSRI) in which he argued that mathematicians must take their roles as educators more seriously. Leaving mathematicians' preparation for teaching to chance, such that some become good teachers and others never do, was wasteful in a profession so centrally dependent on teaching, he claimed. When a revised version of this paper appeared in the *Notices of the American Mathematical Society* in January 1997, it generated considerable discussion for its bold claim that mathematicians and professional educators would need to collaborate if teaching at the college level and below was to be significantly improved.[3]

As Bass became better acquainted with educational issues and problems, questions sharpened in his mind. How were teachers actually prepared and why? Why were certain changes being made in the school curriculum? Who was determining these changes? What really went on in math classrooms? The precollege mathematics curriculum appeared as one undifferentiated landscape that seemed to be puzzlingly missing some critical ideas: Did students ever encounter the notion of a real number? What was "algebra" in the K-12 curriculum? What was going on with calculators? Despite his increasingly extensive involvement in education groups, Bass did not find it easy to pursue his questions, to probe issues of mathematics teaching and learning. He was convinced that there were some serious problems with the curriculum and with teacher preparation which could be substantially improved if he and other mathematicians devoted their attention to them and he was growing interested enough to want to do so. He thought, for example, that if he decided to write a text for a mathematics course for elementary teachers, he could design something much better than the texts currently in use. (Several years later, after working as he has been, he was heard to say that not only was he not qualified to write such a text, but that he would never be so - alone.)

Hyman Bass and I first met in 1994 at a small meeting focused on research on collegiate mathematics education. I had been doing research on teachers' mathematical knowledge, and on their mathematical preparation, and was at the time deeply engaged in working with the videotapes of mathematics teaching that had been filmed across a year in my third grade classroom.[4] Both in my teaching and in my research, I had had a long interest in the nature and role of mathematics knowledge in teaching. I had worked at teaching mathematics in ways that had mathematical integrity, and I had been trying to learn math and learn to make use of it in my teaching. It was not always easy to do this, for helpful opportunities to learn mathematics were few. Deeply interested in how teachers think about, understand, and use mathematics, I had also worked to develop methods for studying teachers' mathematical knowledge and reasoning. Magdalene Lampert and I were at this point investigating ways that artifacts from real classrooms might be used in the preparation of teachers by designing opportunities for them to study teaching and learning, as well as mathematical content. We had worked with

[3]Bass, H. (1997). Mathematicians as educators. *Notices of the American Mathematical Society*, 18–23.

[4]Lampert & Ball (1998), *op cit.*

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other mathematicians, showing them videotape and asking them to comment on the mathematical issues that struck them as they watched young children and their teachers at work.[5] This had been fascinating for us and we had imagined ourselves eventually creating a library of mathematical annotations created to accompany the materials, to help foreground and explicate mathematical issues entailed in the lessons and children's work. I asked Bass whether he might be interested in seeing one of these tapes and commenting, and he said that he would. Over the next couple of years, we had one or two occasions to view tape, and to talk about some issues of elementary mathematics content, and teachers' mathematical preparation. We corresponded a bit and, at his request, I sent him some of my and others' papers to read. Bass often responded critically, suggesting ways teachers in the articles should or could have handled the content. The opportunity to read and occasionally view examples of actual mathematics teaching seemed to draw his interest. This was quite different from sitting at meetings discussing education policy and improvement. He became engrossed in the mathematical details of a particular task or in child's idea or language. Our correspondence was engaged and focused, but often quite contentious. We disagreed on interpretations of what students meant,or of what was possible for children to do. We differed over what constituted an issue of mathematical terminology and what was, for eight-year-olds, a matter of ideas. The differences in our perspectives, while often challenging to discuss and understand, also seemed increasingly a valuable resource. The concrete materials of teaching were a critical ingredient of our exchanges. We were not disagreeing over large matters of curricular reform or learning theory or teacher education. We were looking at actual teaching, at mathematics in play, and our correspondence was grounded in the common material to which we could both refer.

An example from our early work illustrates the nature of the undertaking, with the difficulties it posed, as well as the potential it held. The following example comes from a videotape of a lesson that Bass and I were examining within the first months of our efforts.

In my third grade class, near the beginning of the lesson one day, a boy named Benny said:[6]

> Um, first I said that um, zero was even but then I guess I revised so that zero, I think, is special because um, I– um, even numbers, like they – they make even numbers; like two, um, two makes four, and four is an even number; and four makes eight; eight is an even number; and um, like that. And, and go on like that and like one plus one and go on adding the same numbers with the same numbers. And so I, I think zero's special.

What was he saying? Was he saying that two even numbers when added together yield even sums? Was he talking about powers of two? What did he mean that zero is "special"?

Seeking further insight into his thinking, I asked Benny:

> Can I ask you a question about what you just said? Were you saying that when you put even numbers together, you get another

[5]Among the mathematicians who worked with us over time were: Herbert Clemens (University of Utah), Peter Hilton (SUNY-Binghamton), and Phil Kutzko (University of Iowa).

[6]Data from transcript of January 19, 1990, Deborah Ball's third grade class.

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> even number – or were you saying that all even numbers are made up of even numbers?

He assented, but it was not clear why. Was that what he was saying, that all even numbers are "made up of even numbers"? I was still unsure what he was thinking.

A moment later, a girl named Bernadette disagreed with what she thought Benny was agreeing to, pointing out the number 6. Shea, another classmate, joined in, saying that, "Six is two odd numbers to make an even, to make an even number." Bernadette agreed. Benny, pondering this, said:

> I know that, but um, um I'm talking about like two plus two is four, and four plus four is eight and I just skipped the six so I just added the ones that, that add. Like the two plus two is four, and four is an even number and I'm just talking about the things that um, like what I just said – the, un, two plus two is four and four plus four is eight . . .

Bernadette interrupted, asking:

> So what you're doing is you're going by twos and then what two equals from then you go from – all the way up?

Benny replied, "Yeah, I'm not going by every single number like two, four, six, eight."

Bass, on viewing this segment, was quickly firmly convinced that Benny had discovered powers of two. He wrote:

> Benny has effectively, and explicitly given the recursive algorithm that constitutes the very mathematical definition of powers of 2, and all that is lacking in his description is the mathematical name for it. Moreover, his possession of the idea can be confirmed operationally by giving him any power of 2, for example 128, and then asking him for the next term. I [am] confident that, allowing him time to make the computation, he would produce the next power of two. . . . this constitutes what I consider to be virtually a definition of what is meant by powers of 2.

I countered with cautions born of experience in the challenges of probing students' thinking. Certainly, at times adults fail to recognize the sophistication underlying a child's awkward or unusual phrasing of an idea. But sometimes adults overinterpret what students said, imbuing their talk with mature understandings and reading more into the child's words that was intended.[7] I thought this was an instance where, Bass' mathematical perspective was leading him to read more into Benny's comment than was warranted. I disagreed, saying that I did not think there was enough evidence to support his claim with such definite certainty. We argued and each sought to provide data from the transcript and the videotape to convince the other. Although there was no way to establish with certainty either

[7] I have written about the problems of interpreting students' talk in Ball, D. (1997) What do students know? Facing challenges of distance, context, and desire in trying to hear children. In B. Biddle, T. Good, & I. Goodson (Eds.), *International handbook on teachers and teaching* (Vol. II), (pp. 679–718). Dordrecht, Netherlands: Kluwer Press.

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claim about what Benny might have meant, our capacity to ground our work in the data developed.

With high degrees of precision and attention to detail, Bass' work with these data grew increasingly sharp and focused. I had rarely seen anyone, mathematician or other, watch videotape with as much care. At the same time, his mathematical eye sometimes led him to make interpretations that flew far from the evidence. This was all so new for a scientist who had never done empirical work. It was also new for me to have this level of mathematical scrutiny trained on my practice. We both learned. We were both challenged by the intellectual complexity of communicating about mathematics situated in the work of this third grade class. Bass brought his fluent mathematical sensibilities and expansive perspective on the domain; I brought my fluency with listening to students and my experience as a teacher who had been trying to engage young students in mathematical work. The gulf between us was immense, and yet our use of these concrete records of teaching made it possible to find a common ground into which our perspectives could be sown and our exchanges built. We began to make headway in probing the mathematics between and on the lines of the class' actual work. We explored mathematical possibilities unrealized in the particular lessons we studied.

I asked Bass whether he might want to try writing a mathematical commentary on a videotape or two. He seemed interested on the basis of our exchanges to this point. What would constitute the genre of "mathematical commentary" was not specified, and we set out to work on a particular couple of tapes. Bass watched them over and over again, making notes on the written transcript. He sent me his comments on email. Sometimes he found errors in the transcript. His mathematical eye was caught by some specific issues - the emergence and use of definitions, the structure of the children's mathematical reasoning, particular children's thinking. He wrote notes and we discussed them. Quite often he criticized things he found on the tape. Sometimes he offered praise for what he saw. Often we disagreed over our differing interpretations - of what a child meant, of what mathematics was in reach of the discussion in the class One day, Bass reported that he had found a particular videotape fascinating. Eager to hear what he had found so intriguing, I asked for his comments. But he had none. I asked why. "Because you did everything right," he replied. In a flash, I realized that our conception of analysis and commentary had to this point differed. Whereas I had been asking him to make visible the mathematical issues that stood out to him as he studied the materials, he had been looking for mathematical errors, flaws, and missed opportunities. This was an important moment in our work, for in it we had bumped into a fundamental difference in our orientations to analysis and commentary.

The more we worked on these commentaries, the more it began to seem that our discussions were yielding mathematical analyses that could themselves be mined for improved understanding of the mathematical issues and insights critical to elementary teaching and learning. This marked the beginning of our more formal work on the question of what mathematics is entailed in elementary teaching. This paper provides both background for this work, and a description of the aims and structure of the research, as well as some glimpses of what we are learning.

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How Has Teacher Knowledge of Mathematics Been Considered and Studied in the Past?

That teachers' own knowledge of the subject affects what they teach and how they teach it seems so obvious as to be trivial. However, the empirical support for this "obvious" fact has been surprisingly elusive. Teachers' mathematical knowledge has shown weak and, more often, no effects on student performance. In a well-known meta-analysis of studies of teacher effects, for example, the number of credits in mathematics beginning at calculus was shown to have positive main effects in only 10% of the cases, and, perhaps more jolting, had negative main effects in 8%.[8] Perhaps higher level mathematics makes little difference in practice, but more likely this is a problem of measurement: Course-taking is not a good proxy for knowledge. Still, there remain unexamined questions regarding how content knowledge matters to good teaching, and the field's lack of understanding plagues policy about professional preparation and certification, teacher education, and weakens research on teaching and teacher learning.

What are the weaknesses in current widely shared ideas about teacher content knowledge? First, subject matter knowledge for teaching is usually defined simply by the subject matter knowledge that students are to learn – that is, by the curricular goals for students. Put simply, most people assume that what teachers need to know is what they teach. Many would also add to the list, arguing that teachers must know more in order to have a broad perspective on where their students are heading. Nothing is inherently wrong with this perspective. However, the lists it produces are both arbitrary and unsubstantiated. What evidence is there that these lists comprise the mathematics that teachers must use in teaching? On what basis do we believe that this mathematical knowledge is what it takes to do the work of mathematics teaching? What might be missing from these curriculum-based lists? Without evidence for their role or significance, all items on the list are of equal importance, and there is no way to determine the lists' completeness or adequacy.

Second, the subject matter knowledge required for teaching is most often measured in terms of credentials or particular courses (e.g., teachers should have a minor in math, should take algebra, geometry, analysis, and discrete mathematics). It is convenient to represent what teachers need to know in terms of what they take, presuming that these circumscribe the topics over which they should have command.[9]

Finally, whatever the content and extent of the lists, the use of mathematical knowledge in teaching is often taken for granted. The mathematical problems teachers confront in their daily work are left unexplored, the occasions that require mathematical sensitivity and insight unprobed. Hence, the content and nature of the mathematical knowledge needed in practice is insufficiently understood. Moreover, the role played by such knowledge is also left unexamined. A recent result provides a glimpse of the importance of this distinction in understanding how mathematics knowledge matters in good teaching. In their 1997 Sociology of Education

[8]Begle, E. (1979). *Critical variables in mathematics education: Findings from a survey of the empirical literature.* Washington, D.C.: Mathematical Association of America and the National Council of Teachers of Mathematics.

[9]Kennedy, M.; Ball, D.; McDiarmid, G. (1993). *A study package for examining and tracking changes in teachers' knowledge.* (Technical Series 93-1). East Lansing, Michigan: National Center for Research on Teacher Learning, Michigan State University.

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article describing their analysis of data from the National Education Longitudinal Study of 1988, Rowan and his colleagues report strong positive correlations between teachers' responses to items designed to measure not just mathematics knowledge, but its use in teaching, and student performance.[10] This analysis provides some confirmation that understanding the use of mathematics in the work of teaching is a critical area worth further examination. It is not just what mathematics teachers know, but what they are able to mobilize mathematically in the course of teaching.

Because the association of teachers' knowledge of mathematics, teaching practice, and student learning has been so poorly mapped, our work seeks to investigate those connections. The work builds on two recent lines of research - one on teachers' subject matter knowledge and its role in teaching; and the second on the interplay of mathematics and pedagogy in teaching and teachers' learning. Both lines of research have been active for over the past decade, spurred by Lee Shulman's 1985 presidential address to the American Educational Research Association,[11] and both provide crucial foundations for the research in which we are engaged. In his address, Shulman argued for the centrality of subject matter in teaching and teacher education. Researchers had in recent years been focusing on teachers' thinking, their planning and decision-making, their reflections. Less attention had been paid to how teachers understood what they were teaching, or how teachers' conceptions of their content affected their practice. One exception to this in mathematics was Alba Thompson's study of three junior high school mathematics teachers which revealed how strongly and consistently their conceptions of mathematics influenced their teaching.[12] A few other studies probed connections between teachers' own ideas about mathematics and their teaching of the subject, but focused on secondary teachers.[13] But, as Shulman claimed, subject matter was not the dominant paradigm for the study of teachers or teaching in 1985.

In the years that followed, interest in teachers' subject matter knowledge was renewed, both as an object of inquiry and a matter of concern in teaching and teacher education. A growing number of studies began to examine teachers and prospective teachers' subject matter understanding - in history, science, English, as well as in mathematics.[14] Perhaps nowhere were these results as dramatic as they were in mathematics. Using new methods subtler than earlier analyses of teachers' grades and years of study[15] - for example, interviews designed around tasks of teaching that involve mathematics, observations and discussion of classroom teaching[16]

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[10]These items were developed at the National Center for Research on Teacher Learning, Michigan State University. See Kennedy, Ball, and McDiarmid (1993).

[11]Shulman, L. S. (1985). Those who understand, teach: Knowledge growth in teaching. *Educational Researcher*, 15(2), 4–14.

[12]Thompson, A. (1984). The relationship of teachers' conceptions of mathematics and mathematics teaching to instructional practice. *Educational Studies in Mathematics*, 15, 105-127.

[13]See, for example, Bush, W. (1983). *Preservice secondary mathematics teachers' knowledge about teaching mathematics and decision-making during teacher training.* Unpublished doctoral dissertation, Athens, GA: University of Georgia; Steinberg, R., Haymore, J., & Marks, R. (1985, April). *Teachers' knowledge and content structuring in mathematics.* Paper presented at the annual meeting of the American Educational Research Association, Chicago.

[14]Wilson, Wineburg, Grossman, others.

[15]Begle, E. (1979). *Op cit.*

[16]See Ball (1988) and Kennedy, Ball, and McDiarmid (1993) for theoretical discussions of methods for the study of teacher knowledge.

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- researchers probed teachers' mathematical knowledge. Teachers' deficits were reported repeatedly. Once researchers began to look closely, their analyses revealed the thinness of most U.S. elementary teachers' mathematical understanding.

A quick overview of studies that flocked around elementary teachers' mathematical knowledge shows the pervasiveness of the lacks that researchers uncovered. For example, Simon's[17] study of 33 prospective elementary teachers found that their conceptual knowledge of division was weak in the way they understood the common algorithms, the different interpretations of division, and the connections between symbolic division and real-world problems. This study reinforced results from some of Ball's earlier analysis of prospective teachers' knowledge of division[18] - specifically, that while they could get "right answers," they lacked understanding of the meanings of the calculation procedures or of the solutions. Their understandings were often fragmented, and ideas that might have been connected (e.g., whole number division, fractions, decimals, division in algebraic expressions) were held separately. Other researchers probed other specific areas of teachers' knowledge, each revealing the thin and inexplicit nature of elementary teachers' mathematical understanding.[19] Borko and Eisenhart, et al.[20], in a seminal case study of one beginning elementary teacher, Ms. Daniels, traced her opportunities to learn mathematics as part of her teacher preparation program, and into student teaching. Even when the researchers were able to follow the development of the same mathematical idea (division of fractions) across contexts (e.g., her opportunity to learn it in teacher education, her first occasion to teach it as a student teacher), they discovered that Ms. Daniels was not able to pull together an adequate conceptual understanding of the topic in the face of teaching it.

Two other lines of work add to the power of these empirical findings. One has been the examination of secondary teachers' knowledge of mathematics. Studies of secondary teachers revealed, to some people's surprise, that their knowledge was much thinner and less robust that most assumed, despite the fact that they - unlike their elementary counterparts - usually major in mathematics. Indeed in studies at several major universities, mathematics majors stumbled over similar concepts and had difficulty justifying similar procedures and solutions as did elementary teachers.[21]

A second line of work, a cross-cultural analysis of teachers' mathematical knowledge, has uncovered dramatic differences in Chinese and U.S. teachers' understanding. Liping Ma compared interviews with 29 U.S. teachers with those she conducted

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[17]Simon, M. (1993). Prospective elementary teachers' knowledge of division. *Journal for Research in Mathematics Education*, **24**, 233–254.

[18]For example, see: Ball, D. L. (1990a). Prospective elementary and secondary teachers' understandings of division. *Journal for Research in Mathematics Education*, **21**, 132–144; (1990b). The mathematical understandings that prospective teachers bring to teacher education. *Elementary School Journal*, **90**, 449–466.

[19]See, e.g., Wheeler, M .M . & Feghali, I. (1993). Much ado about nothing: Preservice elementary teachers' conceptions of zero. *Journal for Research in Mathematics Education*, **24**, 147–155; and Graeber, A., & Tirosh, D. (1991). The effect of problem type and common misconceptions on preservice elementary teachers' thinking about division. *School Science and Mathematics*, **91**, 157–164.

[20]Borko, H., Eisenhart, M., Brown, C., Underhill, R., Jones, D., Agard, P. (1993). Learning to teach hard mathematics: Do novice teachers and their instructors give up too easily? *Journal for Research in Mathematics Education*, **23**, 194–222.

[21]See, for example, Steinberg, Haymore, and Marks (1985); Ball (1988, 1990); Ball & Wilson (1990).

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with 72 Chinese teachers.[22] Her analyses showed dramatic differences in teachers' abilities to unpack basic ideas and procedures, to use representations wisely, and to connect significant ideas. Ma's work develops what she calls "profound understanding of fundamental mathematics" as a standard for the mathematical understanding critical for elementary teaching. This research provides the first glimpse of what robust elementary mathematical knowledge, well-suited for teaching, might look like.

Despite this flurry of interest in teachers' knowledge of mathematics, much remained unexamined and unsolved. Studies of teacher knowledge have only mined a few key elementary mathematics topics.[23] Division has garnered enormous attention, followed by fractions, rational numbers, and multiplication.[24] Obviously, many other key topics warrant attention - discrete mathematics, number systems, infinity, integers, geometry, to name but a few. And substantive knowledge has dominated researchers' attention: Little research has probed "knowledge about mathematics".[25] Simon and Blume have examined prospective teachers' ideas about proof[26], an element of mathematical knowledge that Ball also probed with both elementary and secondary teachers.[27] While this distinction between "knowledge of" and "knowledge about" mathematics was intended to call attention to epistemological aspects of mathematics, it has been more often cited and mentioned than investigated empirically. We know quite little about the elements of mathematical ways of knowing that count in teaching elementary school. How does "proof" matter in teaching eight-year-olds? How do issues of symbolization, abstraction, and generalization arise?[28] What might teachers need to know - and what do they know - about mathematical epistemology, about the role and nature of definitions, about the structure and elements of effective mathematical justification or argument?

Moreover, overall, most studies have focused on preservice teachers' knowledge. Fewer studies have explored practicing teachers' knowledge, or the role of teachers'

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[22]Ma, L. (1999). *Knowing and teaching elementary mathematics.* Hillsdale, NJ: Erlbaum.

[23]Research on secondary teachers' knowledge has focused overwhelmingly on teachers' knowledge of functions.

[24]e.g., Simon, M. & Blume, G. (1994). Building and understanding multiplicative relationships: A study of prospective elementary teachers. *Journal for Research in Mathematics Education*, **25**, 472–494; Post, T., Harel, G., Behr, M., & Lesh, R. (1988). Intermediate teachers' knowledge of rational number concepts. In E. Fennema, T. Carpenter, & S. Lamon (Eds.), *Integrating research on teaching and learning mathematics* (pp. 194–219). Madison: Wisconsin Center for Educational Research, University of Wisconsin; Simon, M. & Blume, G. (1994). Mathematical modeling as a component of understanding ratio-as-measure: A study of prospective elementary teachers. *Journal of Mathematical Behavior*, **13**, 183–197.

[25]See Ball, D. L. (1991). Research on teaching mathematics: Making subject matter part of the equation. In J. Brophy (Ed.), *Advances in research on teaching* (Vol. 2), (pp. 1–48). Greenwich, CT: JAI Press.

[26]Simon, M. & Blume, G. (1994). *Mathematical justification: A classroom teaching experiment with prospective teachers.* In B. Pence (Ed.), Proceedings of the Fifteenth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, (Vol. II, 152–158.). Asilomar, CA.

[27]Ball, D. L. (1988). *Op cit.*

[28]See Harel, G. & Tall, D. (1991). The general, the abstract, and the generic in advanced mathematics. *For the Learning of Mathematics*, 11(1), 38–42. The authors examine the processes of generalization, abstraction, and generic abstraction, offering a specific conceptual framing of each process in ways that could be very generative for the frames we are developing.

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understandings on what and how they teach. One reason may have to do with researchers' sensitivity to "testing" or "exposing" teachers. Still, of those studies that have studied experienced teachers, most have uncovered profiles of understanding similar to those of prospective teachers, suggesting that there is nothing automatic about teaching experience that leads to teachers learning more content in and from their practice.[29] One exception to this was a study conducted by Leinhardt and Smith. in which they reported that experienced teachers had more elaborated and structured knowledge of fractions than did the novice teachers.[30] However, in our research group's case studies of teachers dealing with mathematics reforms, we repeatedly found teachers struggling with the new content, or with old content approached more conceptually.[31] We did not see evidence in our cases of experienced teachers with deep or developed knowledge of mathematics. A consequence of the paucity of examples is that there has been little opportunity to examine how mathematical knowledge plays out in experienced and highly developed teaching practice.

Because the investigation of teachers' subject matter knowledge has been unevenly intense, our frameworks for the mathematics that teachers need to know - or better. the mathematical entailments of elementary teaching - are underdeveloped. The scheme of "knowledge of" and "knowledge about" mathematics, drawn from the notion of disciplinary structures,[32] calls attention to both substantive knowledge(i.e., the topics, definitions, theorems, and major ideas of mathematics, and their interconnections) and syntactical knowledge (the ways of constructing mathematical knowledge - e.g., approaches to exploring, asking questions, conjecturing, experimenting, generalizing, abstracting, proving; what it means for something to be "true"). Although this helps to highlight the critical role of epistemology in mathematical knowledge, it is nonetheless a primitive framework. Aspects of mathematical knowing, appreciation, and practice go unnoticed and unarticulated, despite the fact that they may be critical resources in teaching. The conceptual scheme for what it means to think mathematically is limited, often falling by sheer habit into all-too-familiar lists of topics. Moreover, there is a great deal more to

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[29]Schifter, D. (1996). (Ed.). *What's happening in math class? Volume 1: Envisioning new practices through teacher narratives, Volume 2: Reconstructing professional identities.* New York: Teachers College Press.

[30]Leinhardt, G., & Smith, D. (1985). Expertise in mathematics instruction: Subject matter knowledge. *Journal of Educational Psychology,* **77**, 247–271.

[31]I refer here to the Educational Policy and Practice Study (EPPS), co-directed by David K. Cohen, Penelope Peterson, Suzanne Wilson, and myself, in which we studied teachers' responses over a period of several years, in varying local contexts, to state-level math reforms in California, Michigan, and South Carolina. See, for example, Heaton, R.M. (1992). Who is minding the mathematics content? A case study of a fifth-grade teacher. *Elementary School Journal,* **93**, 153–162; Prawat, R.S., Remillard J., Putnam, R.T., & Heaton, R.M. (1992). Teaching mathematics for understanding: Case studies of four fifth-grade teachers. *Elementary School Journal,* **93**, 145–152; Geist, P. (1997). Policy in practice: Re-forming mathematics teaching. Unpublished doctoral dissertation, East Lansing: Michigan State University; Wiemers, N.J. (1990). Transformation and accommodation: A case study of Joe Scott. *Educational Evaluation and Policy Analysis,* 12(3), 281–292; Spillane, J. (in press). Reconstructing teaching: The subject-matter divide in one fifth grade teachers' opportunities to learn about reconstructing practice. *Elementary School Journal.*

[32]Schwab, J. J. (1978). Education and the structure of the disciplines. In I. Westbury & N. Wilkof (Eds.), *Science, curriculum, and liberal education: Selected essays* (pp. 229–272). Chicago: University of Chicago Press. (Original work published 1961.)

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understand about what any of these aspects of mathematical knowledge might look like or mean in teaching.

Considering questions of how mathematical knowledge matters in teaching takes us to a second major arena of prior research to which this project is linked. The renewed interest in teachers' subject matter knowledge was accompanied by Shulman's introduction of "pedagogical content knowledge" to the lexicon of research on teaching and teacher education.[33] The term called attention to a special kind of teacher knowledge that links content and pedagogy. In addition to general pedagogical knowledge and knowledge of the content, argued Shulman and his colleagues,[34] teachers needed to know things like what topics children find interesting or difficult or the representations most useful for teaching a specific content idea[35] This idea helped to underscore the importance of understanding subject matter in teaching. It also led to the crucial insight that even expert personal knowledge of mathematics could often be surprisingly inadequate for teaching. Knowing in and for teaching requires a transcendence of the tacit understanding that characterizes much personal knowledge.[36] It also requires a unique understanding that intertwines aspects of teaching and learning with content. Consider the following example. As an experienced classroom teacher, I knew that figuring out what my fifth graders knew about decimals depended in part on my knowledge of number systems and in part on my understanding of the kinds of errors that ten-year-olds typically make. For example, I knew that they would confuse .5 with .05 and that they would draw this confusion, in part, from their prior conviction that 5 and 05 were the same number. This meant that a fifth grade teacher needs to understand a lot about the base ten number system and about positional notation. When a fifth grader asks, "Where is the "oneths" place?" a teacher needs to be able to hear that this likely emanates from a ten year old's reasonable expectation that if there is a ones place to the left of the decimal point, and a tens place to the left of that, there should be a symmetry to the right of the decimal. In other words, why is the tenths place immediately to the right, and not a "oneths" place? But being able to hear this student is not enough. Why isn't there a "oneths" place? Answering this for oneself requires a certain explicit understanding of place value and of the multiplicative structure of the base ten system that goes beyond being able to name the places (ones, tens, hundreds, etc.) or read numbers. And then, beyond being clear about the mathematics, helping a fifth grader understand the missing "oneths" requires an intertwining of content and pedagogy, or pedagogical content knowledge.

This kind of knowledge is not something a mathematician would necessarily have, but neither would it be familiar to a high school social studies teacher. It is quite clearly mathematical, yet formulated around the need to make ideas accessible to others. "Pedagogical content knowledge" highlights the interplay of mathematics and pedagogy in teaching. Rooted in content knowledge, it comprises more than

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[33]Shulman, L. (1986) *Op cit.*

[34]Wilson, S. M., Shulman, L. S., & Richert, A. (1987). "150 different ways of knowing": Representations of knowledge in teaching. In J. Calderhead (Ed.), *Exploring teacher thinking* (pp. 104–124). Sussex: Holt, Rinehart & Winston.

[35]Shulman, ibid.; and (1987) Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review*, **57**, 1-22.

[36]Polanyi, M. (1958). *Personal knowledge: Towards a post-critical philosophy.* Chicago: University of Chicago Press.

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understanding the content oneself. For example, in their study of a middle school teacher's attempt to teach the concept of rate, Thompson and Thompson[37] highlight the crucial role played by language. They describe, vividly, that although the teacher understood the concept of rate well himself, he was restricted in his capacity to express or discuss the ideas in everyday language. Satisfied with computational language for his own purposes, when these did not help students understand, he was not able to find other means of expressing key ideas.

While I was fascinated with the notion of "pedagogical content knowledge", a repertoire of representations and content-specific ideas about children, of understandings flexible enough to accommodate many different forms of representation, I found myself wondering about the dynamic of content with pedagogy in teachers' reasoning. That is, I realized that teachers must bring to bear considerations of content, students, learning, and pedagogy as they meet novel situations in teaching. They must reason, not just reach for their repertoire of strategies and answers. When teachers look at student work, choose a text to read, design a task, or moderate a discussion, they must attend, interpret, decide, and make moves. Their thinking depends on their capacity to call into play different kinds of knowledge, from different domains. No repertoire of pedagogical content knowledge, no matter how extensive, can adequately anticipate what it is that students may think, how some topic may evolve in a class, the need for a new representation or explanation for a familiar topic.

An example from my own third-grade classroom illustrates this well.[38] In a well-known segment from one of my classes, a boy named Shea proposes that the number six "can be even and it can be odd". After a few minutes, he clarifies: "Three twos to make that, and two threes make six". What does it take to make sense of what he is thinking about? As he continues, in response to his classmates' objections and my puzzled questions, what are some of the mathematical possibilities? What are the pedagogical issues entailed in sizing up what is happening mathematically and in general in class? Navigating this event, and the myriad of others like it, is crucial to teaching mathematics. What are the mathematical entailments of making sense of what is happening, deciding what to do, and acting in the situation? I will return to this episode below to illustrate our current work.

How Are We Studying the Mathematical Knowledge Entailed in Teaching?

In our work, Bass and I are building on these two lines of research: one focused on teachers' mathematical knowledge and the second on the interplay of mathematics and pedagogy in teaching and in teachers' knowledge. We aim to probe this amalgam of mathematics and pedagogy, not just in teacher reasoning, but as evident in the interactive work of teaching.

One purpose, then, of our project is to re-examine the role played by particular aspects of mathematical knowledge in teaching. Our research aims to do this through close analyses of records of interactive teaching, examining the mathematical ideas and issues between and on the lines of this third grade classroom, supplemented by such records from other classrooms at other grade levels in other

[37]Thompson, P. & Thompson, A. (1994). Talking about rates conceptually, Part I: A teacher's struggle. *Journal for Research in Mathematics Education*, **25**, 279–303.

[38]Data from January 1990 lessons in Ball's third grade class.

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contexts. By studying records of practice closely, we seek to analyze how mathematical and pedagogical issues meet in teaching - at times intertwining, at times mutually supporting, and at times creating conflicts. Through analyses of mathematics in play in the context of teaching, the project extends and challenges existing assumptions of what it is about mathematics that elementary teachers need to know and appreciate, and where and how in teaching such understandings and appreciation are needed. For example, in addition to understanding division, or zero, or rational numbers, what other elements of mathematics emerge in planning lessons, listening to children, asking questions, posing examples or counterexamples, or helping develop, validate, and justify mathematical definitions, claims, and methods?

The mathematical resources that show up in our analyses are not always most comfortably labeled mathematics "knowledge", although each entails knowing something within or about mathematics. Some of the resources that we notice recurring as we analyze the tasks and challenges and what it might take to recognize, understand, and manage them, might be better seen as mathematical sensibilities or sensitivities, mathematical appreciation, or mathematical skills. We also notice that, despite the face validity of the mathematics involved, some of the things we see are not easily named. That is, they are recognizable to mathematically fluent people, who rely on and use these in their own mathematical thinking, but for whom the ideas in which we are interested are almost ineffable.

An example of a mathematical resource that has these characteristics is something related to mathematical similarity we have provisionally labeled the "imperative to reconcile multiplicities" (of interpretation, definition, representation, solution paths and methods, solutions). What we mean is that, in mathematics, when more than one representation is offered for an idea or a problem, the imperative exists to investigate the extent to which these are mathematically isomorphic, equivalent, or similar. Doing so is part of the development as well as the verification of the work. The opportunity for this is ubiquitous in classrooms where students' ideas contribute substantially to the enacted curriculum. For example, in one discussion in the third grade class, several different definitions were offered for even numbers: as alternating whole numbers ("every other number" on the number line when you start with 0); as those numbers that can be divided into two equal whole-number groups with no remainder ($2k$); and as those numbers that can be divided into groups of two with none left ($k2$). The mathematical imperative to reconcile the differences among these definitions would involve asking whether each of these produces the same set of numbers, and if so, whether and how the three versions map onto one another. Pursuing this leads the teacher to other mathematical ideas. Comparing the second and the third, for example, involves understanding commutativity and understanding what it would take to establish that $2k = k2$. The issue of reconciling multiplicities in the course of classroom lessons has emerged repeatedly across our data, as well as in others' data. As children engage with interesting mathematical problems, they produce different solution strategies and solutions. Quite often, classes end with no comparative inspection of these. Instead, multiple representations, methods, solutions are left to an uneasy coexistence, implying either a mathematically unverified equivalence, or the message that multiple solutions simply exist, much like multiple interpretations of text. But mathematics and literary interpretation differ, and the imperative to inspect and compare multiple ideas is central to mathematics in ways that are often inadequately appreciated in

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the backlash against a "single right way" pedagogical approach. In mathematics multiplicity leads either to illuminating relations or equivalence of representations, or else it can conceal mathematical contradiction, on which ensuing mathematical reasoning may founder.

Other mathematical resources on which our analyses focus include: the nature of mathematical justification; the role and nature of definitions in mathematics; the construction and use of representations and modeling; and the use, comprehension, and construction of symbolic notation; isomorphism and other forms of mathematical similarity and equivalence.

Because this research is unusual, it might help to say what it is not. The study does not aim to revise the curriculum of third grade mathematics, asking how attention to mathematics as a discipline and concern for children's thinking can contribute to a better map of what we should try to teach children of this age. At the time these data were collected from my classroom, I was following the curriculum guidelines of the school district in which I taught. This is not a study of what ought to be taught at third grade; it is, instead, a study of the mathematics entailed in teaching a third grade class the usual topics of the contemporary mathematics curriculum: place value, addition and subtraction with regrouping, fractions, even and odd numbers, probability, multiplication, combinations and permutations. What are the mathematical issues that intertwine around the teaching of this third grade class across a year?

Ours is also not a study of the interplay of mathematics and pedagogy in any approach to mathematics teaching. The study focuses on a kind of teaching that aims to take seriously both mathematics as a discipline and children's mathematical ideas, and that sees mathematics as a collective intellectual endeavor situated within community.[39] What it means to take seriously either mathematics or children's ideas, or to treat mathematical work as collective, is of course highly contestable. "Mathematics as discipline" is hardly a monolith; and "taking children's ideas seriously" can imply wildly different actions. Neither aim provides clear guidance for practice, but together they function as twin imperatives for orienting the kind of substantive work children might do and how their learning might be fostered. "Collective intellectual work" has been too often distorted into a commitment to particular physical room layouts or small group work. As imperatives for practice, these three elements define a problem space for teaching and learning that shapes practical decisions, moves, interpretations, questions, and puzzles. This problem space also shapes the pedagogical and mathematical issues that arise. With different purposes for teaching mathematics, Shea's claim about the number six might be interpreted, quite simply, as a misunderstanding or a delay tactic. Here, the distinct possibility that he is edging up on factorization or on modular arithmetic mod 4, is a viable interpretation. Still, it is important to appreciate that these imperatives - to treat both mathematics and students with respect, and to consider mathematical work as a collective intellectual endeavor - do not prescribe a single approach, or a tightly-specified strategy of teaching.

[39] While the Standards ([1989]. *Curriculum and evaluation standards for school mathematics*, and [1991]. *Professional standards for teaching mathematics*. Reston, VA: Author) may not frame mathematics teaching in this way, these ideas are not inconsistent with the NCTM's emphasis on mathematical reasoning, problem-solving, conjecturing, and justifying, or on the attempt to broaden and deepen the mathematical scope of the curriculum.

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Collaboration With Mathematicians

One major strategy of this research is to develop modes of analyses that involve mathematicians working with a multidisciplinary research team. And a further aim of this project is to make even the collaborative inquiry itself also an object of inquiry and analysis. With educational problems as complex as they are, the time is right to explore new forms of border-crossing that expand the usable range of perspectives and intellectual resources. As Ellen Lagemann points out so eloquently in her recent essay, *Contested Terrain: A History of Educational Research in the United States, 1890-1990*, work in schools of education has been persistently plagued by its isolation from the disciplines and from practice in schools. This project is explicitly designed to bridge these worlds and to examine the course of the effort self-consciously.

The work so far suggests that this collaborative analysis is worthwhile, that ideas are generated and supported by the different intellectual resources we bring to the study of teaching. However, the work has also been challenging and difficult. We face many conflicts of interpretation as we navigate an unfamiliar borderland in which neither the norms of mathematics nor those of educational research quite work. What we are about, in part, is the development of method, of ways to study teaching that are grounded in practice and which deliberately and explicitly include mathematical lenses and structures.

Eisenhart and Borko[40] claim that "interdisciplinary collaboration holds special promise for research on teaching and teacher education because it addresses the multifaceted nature of teaching and learning to teach". They propose two different views of how different disciplines might be combined: one more additive or parallel, the other deliberately dialogic and interactive. The first captures the power offered by multiple perspectives, each with its own integrity contributing to some more comprehensive view of the phenomenon under investigation. The second relies on the interactions of people with very different perspectives, who develop a different and more integrated level of understanding than either could produce separately as a product of their mutual engagement in the issues.

Some formative work of the project is in first mode. Bass independently produces interpretations and analyses of what he sees going on from his perspectives; these interpretations draw on his experience, commitments, beliefs, insights, and lenses as a research mathematician. His interpretations offer interpretations of classroom events, of students' thinking, and of the mathematics at play from moment to moment.[41]

However, because our interest is in probing and elaborating the interplay of content and pedagogy, these preliminary, independent analyses usually constitute drafts for the predominant mode of this collaborative project. This is more in Eisenhart and Borko's second view of collaboration. Dialogue and exchange are critical

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[40] Eisenhart, M. & Borko, H. (1991). In search of an interdisciplinary collaborative design for studying teacher education. *Teaching & Teacher Education*, 7, 137–157.

[41] Different mathematicians will be involved in this work over time, scholars deliberately selected from different areas of mathematics, and with different views of the nature of mathematics or mathematical activity and knowledge. Bass' is a particular constellation of perspectives on mathematics, and hence, on the elementary classroom. Other mathematicians are likely to produce different perspectives. Examining the continuities and discontinuities in the mathematicians' commentaries will be an important part of our second-level analyses.

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components of the work. Negotiating the different interpretations of and perspectives on what is going on will push and extend plausible accounts of classroom episodes: the core analytic work of the project will be dialogic. Such discussions do not necessarily yield joint or single interpretations, but the interactions are a crucial medium of the work. The goal is to produce plausible analyses of teaching and learning that interplay mathematical perspective with pedagogy, with an eye to expand the range of mathematical possibility that might be seen, heard, located, and, in turn, nurtured, in teaching and learning.

Bass's work is designed to help construct a set of first-level analyses of lessons and children's work, interpretations produced out of the interaction with me (and graduate students who work with me, who are themselves experienced K-12 math teachers). While it is true, of course, that there are members of the mathematics education community who are themselves well-trained mathematically, as well as mathematicians who have become involved with issues of mathematics education, the strategy of this research is to involve research mathematicians who have not been involved thus, and whose perspective on mathematics, while varied, emerges from their own regular and sustained work as practitioners of mathematics. The principle here is that signs of the children's and teacher's mathematical activity may appear differently to someone who is engaged in the practice and may have a special eye for mathematical mind, insight, process, or flight of activity.[42] A similar argument might be made for what a teacher might be able to contribute to analyses of college-level mathematics instruction. When I watch university instruction in mathematics, my take on what I see frequently differs substantially from that seen by the mathematicians with whom I am talking. I notice moves made by the teacher, the ways in which he or she extends what kinds and aspects of mathematical knowledge, students' mathematical ideas and involvement, the mathematical discourse and culture of the classroom. The heart of the work I am proposing reverses this by engaging a mathematician's eye on the elementary mathematics classroom, with the aim of uncovering and explicating mathematical issues at play, issues salient to the mathematical eye and unnoticed by others. Note that any group of people could be chosen as collaborators for such joint work, and much of it could be illuminating to analyses of teaching and learning. But mathematicians who, like others, have eyes untrained for elementary school classrooms and elementary school pedagogy, bring eyes that are highly trained for mathematics. Thus, as practitioners of the field, they are uniquely situated to probe mathematics as its seems to be entailed in elementary school teaching.

To provide a snapshot of this work, I shall return for a moment to the example of Shea, my third grade student, and his idea that six can be both even and odd. When he says this, many of the children appeared dubious and most seemed to think he was wrong. Several tried to persuade him of this. One girl, using a pointer on the number line hung above the chalkboard, invoked the even-odd-even-odd pattern. Others cited the working definition of even numbers, assuming he would see that six was even. He agreed that it was. But, he said, "It can fit the definition of odd, too". A girl named Lin tried a third tactic: If you call six odd and even, what do you call ten? Shea paused, and then said, "Thank you for bringing

[42]Hawkins, D. (1974). Nature, man, and mathematics. In *The informed vision: Essays on learning and human nature* (pp. 109–131). New York: Agathon. (Original work published in 1972.)

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that up. so - I say - ten can be an odd and an even. . ." Stepping back from the rising agitation of the class, whose comfortable understanding of even and odd numbers is being challenged, we ask: What aspects of mathematics are at play here? Analysis suggests that one important aspect has to do with definitions and what definitions do in mathematics.[43] Looking through the lens of "definition", three different definitions of even numbers - all correct - arise in the course of the class, and the definition for odd numbers has added complexity. Knowing this, being able to see this, and also knowing the nature and roles of definitions in mathematics, and the value of children also having some sense of this, changes how one hears the children's discussion. The finer discrimination afforded to the pedagogical ear as a product of the mathematical filter of "definition" changes the picture one gets of the class's discussion. Suddenly the children's moves seem more mathematically germane and even mathematically sophisticated. Another take on what is going on is that it is when Lin proposes the number ten, she is presumably producing it as a counterexample to convince Shea that he is wrong. What happens instead is that hers becomes the moment when Shea's initial insight that some even numbers have a special quality (i.e., as we now start to learn with Lin later, that they have an odd number of groups of two) begins to become generalized.[44] Later the children discover that "Shea's numbers" go on and on - 14, 18, 22, 26, and so on. "They're in a pattern!" exclaims one girl excitedly.

Why does these mathematical interpretations matter? One reason is that preservice teachers watching these children on video rarely talk about definitions or about generalization. Instead they may focus on how polite the children are toward one another. And "definition" and "generalization" are not ideas that they have had opportunity to consider in mathematics. They have certainly learned some definitions, but they have little notion of what a mathematical definition does, its centrality to mathematical discourse, or how it is similar to and different from definitions of terms in history, or in biology. They have probably not had experiences with the nature and importance of generalization in mathematics. Might it turn out that knowing about definitions and how they function in mathematics, or about the work to generalize to a mathematical domain, is one of those elements of mathematical knowledge that matters for teachers?

This project is designed to construct and use the analytic capacity of both parallel and interactive collaboration between mathematics and education. However, in addition, we also explicitly focus on analysis of the collaboration between us. We have set out to bridge the disparate worlds of mathematics and education, and this offers an opportunity to examine self-consciously the course of the collaborative cross-disciplinary effort.

The documentation includes shared analytic memos and documents (such as jointly annotated transcripts), email messages discussing (and often debating) interpretations, and separately or serially annotated pieces of data. In addition, we often audiotape face-to-face work. Other documentation includes tapes of presentations given from this work, especially in settings populated principally by mathematicians, or in settings that involve members of both mathematics and education communities (e.g., presentations at the National Academy of Sciences or its National

[43] I draw here on a pilot analyses that Hyman Bass and I did and presented at an NSF-sponsored meeting of mathematicians and educators in March 1997.

[44] Harel and Tall (1991), ibid.

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Research Council boards). Most of our work takes the form of written exchanges on email or in Word documents. Disagreements, arguments, and other sorts of constructions and exchanges are tracked through the use of colored text to distinguish one speaker from the other, and earlier from later iterations. These tapes and individual and collective written records of our work are used as data for our analyses. We focus on (1) the preservation and separation of individual perspectives; (2) the exploration and mediation of differences in ideas, perspectives, and interpretations; (3) the construction of collective and common knowledge. Ultimately, we would like to understand better what it takes to learn to do collaborations of this kind, and what the likely recurrent challenges as well as profits are.

Research Methods and Analytic Procedures

I turn now to some explanation of the research methods we are developing and with which we are deliberately experimenting. Although we often refer to what we are doing as a kind of "mathematical analysis" of teaching, in fact, the interaction on the project entails a dialogic engagement with both mathematics and pedagogy. The methods required for this work are within the interpretive traditions of research on teaching.[45] In order to understand a phenomenon - in the present case, the nature and role in teaching of teachers' mathematical knowledge – that has eluded more traditional methods of research, we are studying specific instances in detail. One central analytic task is to probe the particulars of the cases we are examining: to uncover mathematical issues that can be seen to figure in particular moments of teaching practice, to seek connections with other moments, and to consider the role such elements of mathematics play in teaching. The other is to identify what is generalizable and what is specific to particular approaches to teaching, or to the specific cases which we are studying. This is done both through logical analysis (focused on identifying elements of mathematical knowledge or insight would be important in any approach to teaching) and through comparisons derived from further empirical analyses of other classroom data. Combing through records of classroom activity, records that document that activity from multiple perspectives (e.g., videotape - as a kind of observational record; children's written work, the teacher's written notes at the time) and represented in multiple forms (e.g., transcripts of classroom lessons, videotapes of those same lessons), we look for signs of mathematical activity, places where mathematical issues appear salient. Using a mathematical eye as a tool of the inquiry, we seek to annotate and index mathematical issues that shape an account of what is happening in the class. One form this seems to take is as interlinear annotation on the "texts" of classroom activity. Another is commentaries on segments of classroom activity, supported with particular evidence from those "texts". We seek to identify patterns, themes, mathematical issues and lacunae, and to support the identification of those with evidence in the records.

In interaction with me and other members of the research group, our written and oral exchanges over the mathematical analyses enable us to meld mathematical issues to pedagogical patterns, issues, and events that we identify as intersecting

[45] Erickson, F. (1986). Qualitative methods in research on teaching. In M. Wittrock (Ed.), *Handbook of research on teaching* (pp. 119–161). New York: Macmillan.

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with them. These emanate from analyses of teaching and learning from the perspective of practice.[46] For example, in the case of the segment of class where Shea's classmates were seeking to convince him that six could not be odd, there is a set of pedagogical issues intertwining. What to do with the fact that the clock is ticking while Shea's unusual assertion lies on the table, slowing the class's progress on the day's planned work? Are Shea's classmates putting him on the spot? Is this worthwhile to allow to go on? From a pedagogical perspective, how to respond to an unexpected student assertion or idea is a recurrent challenge. What pedagogical moves are available and how does an assessment of the mathematical issues entailed shape considerations of whether to allow the discussion to continue, or to step in and clarify, to offer particular kinds of mathematical guidance or input? If one sees Shea as fundamentally confused, and the definitions of even and odd becoming unhelpfully muddied for the class, then one set of teacher moves seems implied. If one sees Shea - and shortly, his classmates with him - as fascinatingly edging onto a piece of very clever mathematics (e.g., a conjecture about a particular kind of even numbers, those of the general form $2n$ where n is an odd number, that is, those congruent to 2 mod 4), then the teacher's choices lead in a different direction. Interpretations of students, and hence the pedagogical issues, vary as a function of the mathematics inferred in their talk, drawings, writing, and emphases. It is into these sites where mathematics and pedagogy intertwine that we will seek to press mathematical and pedagogical analyses in an effort to create interpretations of the mathematics entailed in the moments of elementary classroom life.

An explicit assumption of this work is that understandings of classroom teaching and learning of mathematics can be enhanced through the melding of mathematical and pedagogical perspectives. In this project, we are explicitly cultivating the articulation of complementary analyses of practice from the perspective of mathematics and work to intertwine the two, as a means of improving our lens on the interplay of content and pedagogy. In so doing, we will be developing approaches to the study of teaching in the tradition of Schwab's "arts of the eclectic", or polyfocal conspectus:[47] No single theoretical perspective, or interpretive standpoint, can account comprehensively for the intricacies inherent in the particularities of practice. Each emphasizes some elements and obscures others. Schwab argues for the need to join and reconcile different partial perspectives, in order to understand aspects of practice seen better through the interaction of perspectives. This project sets out specifically to develop methods of melding pedagogical perspective with mathematical ones, along the way continuing to articulate each as well as the means to intersect them. We seek, in Schwab's terms, "tentative bridges built between useful parts of bodies of knowledge in the course of their application to practical problems"[48] And in "bodies of knowledge", we mean to include ideas and theories as well as habits of mind and kinds of questions.

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[46]See, for example, Lampert, M. (in press). Studying teaching as a thinking practice. J. Greeno and S.G. Goldman (Eds.) *Thinking practices.* Hillsdale, NJ: Lawrence Erlbaum and Associates; Ball, D. L. (in press) Working on the inside: Using one own practice as a site for studying mathematics teaching and learning. In Kelly, A. & Lesh, R. (Eds.). *Research design in mathematics and science education.* Dordrecht, Netherlands: Kluwer.

[47]J. J. Schwab. (1978). The practical: Arts of the eclectic. In I. Westbury & N. Wilkof (Eds.), *Science, curriculum, and liberal education: Selected essays* (pp. 322–364). Chicago: University of Chicago Press. (Original work published 1961.)

[48]Ibid., p. 330.

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These two perspectives - mathematics and pedagogy - are far from comprehensive in what they can illuminate about teaching and learning. But just as cognitive science has made enormous strides in the last twenty years in offering cognitive analyses of children's mathematics, for example, this project will contribute to our understanding of what can be seen through mathematical lenses applied to the cognitive and social activity of classroom teaching and learning.

A number of other issues arise in considering issues of analytic method in this work. First is an issue of sampling and data. How do we decide what lessons to examine, what materials to analyze from the extensive body of data available to this project? At the outset, two considerations appear crucial; others may emerge as the work evolves. The first is that, in order to probe a range of mathematical issues, and the interactions they create in and around the pedagogical ones, we will analyze lessons from every major content unit of the school year (permutations and combinations, integers, addition and subtraction with regrouping, relations and functions, number theory, probability, multiplication and division, and fractions). The issues may vary across content domains; the insights about mathematical entailments may differ. For instance, the issues raised through the course of my class's pursuit of Shea's claim about the number six appear to have some different contours than some of those in the probability unit, where the nature of mathematical knowing differs in some possibly important ways from that in number theory. Since our claims will be related to the mathematics of the required curriculum, drawing from the whole of the grade level curricular terrain seems pertinent. Second, because we are concerned with ways in which the mathematical and pedagogical intertwine, we will also want to select lessons that appear to have rather different pedagogical structures and problems. For instance, a day where the lesson follows from a clearly designed task posed by the teacher at the beginning of class may raise some significantly different mathematical entailments than does a day like the one with Shea, where the teacher's agenda is heavily determined by an unexpected student question or idea.

A second issue is how to ventilate the work with multiple perspectives on "the mathematics". Acknowledging that there is variation in how mathematicians see and conceive the nature of mathematical knowledge, mathematical activity, and the major structures of the body of knowledge and method, we want to enable an interplay of mathematical perspective. Our main strategy for this is to involve a few mathematicians, deliberately choosing people whose research is in different areas and whom we think hold distinctive views on these matters. We do not pretend that we are being somehow comprehensive with this strategy; we merely approach this with the caution not to be too narrow and in a serious attempt to uncover at least some alternative perspectives. Given this approach, we will be interested in where there is overlap and where considerable difference among the mathematicians, with regard to their mathematical analyses of and commentaries on the teaching.

A third issue has to do with the nature of the claims we intend to make from this work and how the methods we will employ will allow us to warrant our claims. Here, too, we can say some things about what will be entailed in this important issue, but it is also the case that the development of method is explicitly part of this work, and we aim to be reading and exploring both within methods of interpretive educational scholarship, but also in other fields, such as different modes and traditions of content analyses, textual interpretation and analysis, discourse analysis, art criticism, and anthropology. One outcome of this project will be an explication of method for the

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construction of mathematical (or other subject matter or disciplinary) perspectives on teaching, and for the intertwining of those with what we are calling pedagogical perspectives. At this point, it seems that the project will produce three possible kinds of claims. First are claims about mathematical issues at play in a given lesson or set of lessons, and their interplay with pedagogical issues (e.g., what Shea is up to, what his efforts seem to be, and how those shape what happens for the teacher and the other students in class, as well as what could happen). Second are claims about the kinds of sites or tasks of teaching where mathematical issues appear to arise (e.g., in interpreting and deciding what to do with unexpected student comments, such as Shea's announcement about the number six) - in other words, not just what the mathematical entailments of teaching appear to be, but where they seem to be located. And third are claims about the mathematical entailments of elementary teaching, as a product of our secondary analyses of the first order analyses. In looking across the episodic analyses, drawn systematically from the classroom records as described above, we will analyze patterns and find ways to organize themes and patterns that might helpfully reconstruct our current sense of the particulars, the categories, and the overarching frameworks for thinking about what is mathematically entailed for teachers in teaching elementary mathematics.

To conclude, it is important to underscore two points. One is that much of this work builds on methods of analysis familiar by now in recent interpretive research on teaching. Second is that this work also precipitates a need for new developments in method, in the cultivation of something we might be able to call "mathematical analyses" of teaching and learning. In complement to this, this research will need to continue to consider the development of methods for intertwining pedagogical and mathematical perspectives where the perspectives draw on different orientations and even different disciplinary training. This challenge is implied in Eisenhart and Borko's two views of interdisciplinary collaboration. Such development of method seems essential if we are to learn to improve our lens for examining the role of mathematics in teaching math, and for interpreting and improvising pedagogical content knowledge - or perhaps, pedagogical content knowing - in practice.

SCHOOL OF EDUCATION UNIVERSITY OF MICHIGAN ANN ARBOR MI 48109
E-mail address: dball@umich.edu

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