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in the early grades focuses on arithmetic concepts, calculations, and algorithms, and, then, as they enter secondary school, pupils are suddenly required to understand and write proofs, mostly in geometry. Substantial empirical evidence shows that this curricular pattern is true in many countries.

Needed is a culture of argumentation in the mathematics classroom from the primary grades up all the way through college. However, we need to know more about the difficulties pupils encounter when they are confronted with proof and the challenges faced by teachers who seek to make argumentation central to the mathematics classroom. The epistemological difficulties that confront students in their first steps into proof can be compared to those faced by scientists in the course of developing a new theory. At the beginning, definitions do not exist. It is not clear what has to be proved and what can be presupposed. These problems are interdependent, and researchers (like students) find themselves in danger of circular reasoning. In the infancy of a theory, a proof may serve more to test the credibility or the fruitfulness of an assumption than to establish the truth of a statement. Only later, when the theory has become mature (or the student has come to feel at home in a domain), can a proof play its mathematical function of transferring truth from assumptions to a theorem.

All in all, work is needed in three areas with regard to the teaching of proof. We need (1) a more refined perception of the role and function of proof in mathematics, including studies of the practices of proving in which active mathematicians engage (epistemological analysis), (2) a deeper understanding of the gradual processes and complexities involved in learning to prove (empirical research) and (3) the development, implementation and evaluation of effective teaching strategies along with carefully designed learning environments that can foster the development of the ability to prove in a variety of levels as from the primary through secondary grades and up to college level (design research).

We begin in Section 2 with an analysis of what mathematical proof might involve in the primary grades. Section 3 gives results of a longitudinal study on the development of proving abilities in grades 8 and 9. Section 4 is based upon an empirical investigation of college level teaching and shows how the natural habit of referring to an example can be used as a leverage into the teaching of proof, and section 5 discusses the idea of 'physical mathematics' as an environment for the teaching of proof.

2. What does it take to (teach to) reason in the primary grades?[1]

Although the teaching and learning of mathematical reasoning has often been seen as a focus only beginning in secondary school, calls for improvements in mathematics education in the U.S. have increasingly emphasized the importance of proof and reasoning from the earliest grades (NCTM, 2000, p.56). While some may regard such a focus on reasoning and proof secondary to the main curricular goals

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in mathematics at this level, we consider reasoning to be a basic mathematical skill. Yet what might 'mathematical reasoning' look like with young children, and what might it take for teachers to systematically develop students' capacity for such reasoning? These questions form one strand of our research on the teaching and learning of elementary school mathematics.

We define 'mathematical reasoning' as a set of practices and norms that are collective, not merely individual or idiosyncratic, and that are rooted in the discipline (Ball & Bass, 2000, 2002; Hoover, in preparation). Mathematical reasoning can serve as an instrument of inquiry for discovering and exploring new ideas, a process that we call the *reasoning of inquiry*. Mathematical reasoning also functions centrally in justifying or proving mathematical claims, a process that we call the *reasoning of justification*. It is this latter on which we focus here.

The reasoning of justification in mathematics, as we see it, rests on two foundations. One foundation is an evolving *body of public knowledge* — the mathematical ideas, procedures, methods, and terms that have already been defined and established within a given reasoning community. This knowledge provides a point of departure, and is available for public use by members of that community in constructing mathematical claims and in seeking to justify those claims to others. For professional mathematicians, the base of public knowledge might consist of an axiom system for some mathematical structure simply admitted as given, plus a body of previously developed and publicly accepted knowledge derived from those axioms. Hence, the base of public mathematical knowledge defines the grain size of the logical steps which require no further warrant, that is acceptable within a given context. The second foundation of mathematical reasoning is *mathematical language*—symbols, terms, notation, definitions, and representationsand rules of logic and syntax for their meaningful use in formulating claims and the networks of relationships used to justify them. 'Language' is used here to refer to the entire linguistic infrastructure that supports mathematical communication with its requirements for precision, clarity, and economy of expression. Language is essential for mathematical reasoning and for communicating about mathematical ideas, claims, explanations, and proofs. Some disagreements stem from divergent or unreconciled uses of terminology, whereas others are rooted in substantive and conflicting mathematical claims (Crumbaugh, 1998; Lampert, 1998). The ability to distinguish these requires sensitivity to the nature and role of language in mathematics.

We have been tracing the development of mathematical reasoning in a class of Grade 3 students (ages 8 and 9) across an entire school year using detailed and extensive records of the class: videotapes of the daily lessons, the students' notebooks and tests, interviews with students, and the teacher's plans and notes. By comparing the class's work at different points in time, we are able to discern growth in the students' skills of and dispositions toward reasoning. We offer two brief examples here. Early in the school year, the teacher presented the problem, 'I have pennies (one-cent coins), nickels, (five-cent coins), and dimes (ten-cent coins) in my pocket. Suppose I pull out two coins, what amounts of money might I have?' The children worked to find solutions to this problem: 2, 6, 10, 11, 15, and 20. The teacher asked the students whether they have found all the solutions to the

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problem, and how they know. Some students seemed uncertain about the question. Other students offered explanations: 'If you keep picking up different coins, you will keep getting the same answers,' 'If you write down the answers and think about it some more until you have them all.' The students believed they had found them all, but it was because they could not find any more. Their empirical reasoning satisfied them. Moreover, they had neither other ideas nor methods for building a logical argument which would allow them to prove that this problem (as worded) had exactly six solutions. They also did not have the mathematical disposition to ask themselves about the completeness of their results when working on a problem with finitely many solutions.

In contrast, consider an episode four months later. Based on their work with simple addition problems, the third graders had developed conjectures about even and odd numbers (e.g., an odd number plus an odd number equals an even number). They generated long lists of examples for each conjecture: $3 + 5 = 8$, $9 + 7 = 16$, $9 + 9 = 18$, and so on. Two girls, amidst this work, argued to their classmates: 'You can't prove that Betsy's conjecture (*odd + odd = even*) always works. Because, um, $\cdots$ numbers go on and on forever, and that means odd numbers and even numbers go on forever, so you couldn't prove that all of them work.' The other children became agitated and one of them pointed out that no other conclusion that the class had reached had met this standard. Pointing to some posted mathematical ideas, the product of previous work, one girl questioned: 'We haven't even tried them with all the numbers there is, so why do you say that those work? We haven't tried those with all the numbers that there ever could be.' And other children reported that they had found many examples, and this showed that the conjecture was true. But some were worried: One student pointed out that there are 'some numbers you can't even pronounce and some numbers you don't even know are there.' A day later, however, challenged by the two girls' claim, the class arrived at a proof. Representing an odd number as a number than can be grouped in twos with one left over, they were able to show that when you add two odd numbers, the two ones left over would form a new group of two, forming an even sum:

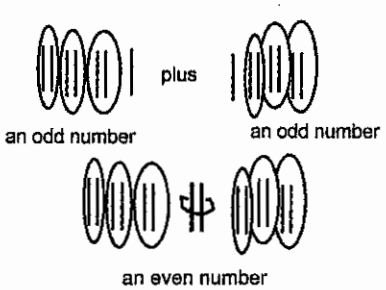


Figure 1: odd + odd = even

This episode illustrates the important role for definitions. Having a shared definition for odd and even numbers enabled these young students to establish a

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logical argument, based on the structure of the numbers. As one girl explained to her classmates: 'All odd numbers if you circle them by twos, there's one left over, so if you ··· plus another odd number, then the two ones left over will group together, and it will make an even number.' The definition equipped them to transcend the barrier that 'numbers go on forever' because it afforded them the capacity for quantification over an infinite set. Moreover, this episode shows the students having developed in their ability to construct, inspect, and consider arguments using previously established public mathematical knowledge.

Our research on the nature, structure, and development of mathematical reasoning has made plain that mathematical reasoning can be learned, and has highlighted the important role played by the teacher in developing this capacity. Three domains of work for the teacher have emerged from our analyses. A first concerns the selection of mathematical tasks that create the need and opportunity for substantial mathematical reasoning. The two-coin problem, for instance, did not originally require students to find all the solutions. Asking this transformed an ordinary problem into one that involved the need to reason mathematically about the solution space of the problem. The second domain of teachers' work centers on making mathematical knowledge public and in scaffolding the use of mathematical language and knowledge. Making records of the mathematical work of the class (through student notebooks, public postings, etc.) is one avenue, for it helps to make that work public and available for collective development, scrutiny, and subsequent use. This includes attention to where and in what ways knowledge is recorded, as well as how to name or refer to ideas, methods, problems, and solutions. Making mathematical knowledge and language public also requires moving individuals' ideas into the collective discourse space. A third domain of work, then, concerns the establishment of a classroom culture permeated with serious interest in and respect for others' mathematical ideas. Deliberate attention is required for students to learn to attend and respond to, as well as use, others' solutions or proposals, as a means of strengthening their own understanding and the subsequent contributions they can make to the class's work.

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3. The complexity of learning to prove deductively[2]

Deductive mathematical proof offers human beings the purest form of distinguishing right from wrong; it seems so transparently straightforward — yet it is surprisingly difficult for students. Proof relies on a range of 'habits of mind'— looking for structures and invariants, identifying assumptions, organising logical arguments — each of which, individually, is by no means trivial. Additionally these

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