Ball, D. L., Lubienski, S., and Mewborn, D. (2001). Research on teaching mathematics: The unsolved problem of teachers' mathematical knowledge. In V. Richardson (Ed.), Handbook of research on teaching (4th ed.). New York: Macmillan.

24.

Research on Teaching Mathematics: The Unsolved Problem of Teachers' Mathematical Knowledge

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Why does it work to add a zero on the right when multiplying by 10, or two zeros when multiplying by 100? Why, when the number includes a decimal, do we move the decimal point over instead of adding zeros? Is zero a number? If it is a number, is it even or odd? What does it mean to divide *by* one-half? What is an irrational number? Is a square a rectangle? What is the probability that in a class of 25, two people will share a birthday?

For all the rhetoric about the centrality of mathematics to the life and progress of this new century, we did not manage to achieve high levels of mathematical competence among most American adults by the end of the past century. Most well-educated adults cannot comfortably answer the questions in the previous paragraph. They cannot make judgments about orders of magnitude, cannot reasonably estimate the likelihood of particular events, and cannot reason skillfully about quantitative relationships.

That many people leave their formal experience of learning mathematics both uninterested in and unskillful with the subject has long been a matter of concern. Problems with mathematics education are not new. Why does formal schooling in the United States manage to help so many people learn to read and to write successfully and yet fail with so many others in developing a similar level of mathematical proficiency?

Although the reasons for this failing and its consequent effect on adults' mathematical proficiency are many, we focus in this chapter on one: our insufficient understanding of the mathematical knowledge it takes to teach well. This insufficient understanding has meant inadequate opportunities for teachers to develop the requisite mathematical knowledge and the ability to use it in practice. Without such knowledge, teachers lack resources necessary for solving central problems of their work—for instance, using curriculum materials judiciously, choosing and using representations and tools, skillfully interpreting and responding to their students' work, and designing useful homework assignments. Because what teachers and students are able do together with mathematics in classrooms is at the heart of mathematics education, the problem we set out to investigate in this chapter is, we argue, central to improving such education.

The ideas in this chapter have been significantly influenced by the thinking and writing of Suzanne Wilson, David Cohen, Magdalene Lampert, Hyman Bass, Lee Shulman, Joan Ferrini-Mundy, Daniel Chazan, Deborah Schifter, and Jeremy Kilpatrick. The authors also acknowledge, gratefully, Mark Hoover's and Eric Siegel's assistance across many phases of the work and Kara Suzuka's help in the final preparation of the manuscript. Work on this chapter was supported, in part, by the Spencer Foundation for the project, "Crossing Boundaries: Probing the Interplay of Mathematics and Pedagogy in Elementary Teaching" (MG #199800202, Deborah Ball). The order of coauthorship is alphabetical. Both coauthors contributed significantly to the development of this chapter along with the lead author.

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Why Focus on Mathematical Knowledge for Teaching?

We chose to focus on teachers' knowledge of mathematics for a chapter about research on mathematics teaching for three reasons. First, the history of research in the past 15 years reveals an overwhelming focus on teachers' knowledge and beliefs. In our perusal of articles published between 1986 and 1998 in 48 educational research journals, we identified 354 articles that dealt specifically with mathematics teaching and learning (i.e., focused on teachers, students, curriculum, or interactions among any of these).[1] Almost half of those pieces focused on mathematics *teachers* alone, and many others included teachers' knowledge and beliefs in their central questions and analyses.[2] As we had expected, teachers and teacher knowledge had been a significant focus of research since the publication of the *Handbook of Research on Teaching, Third Edition* (Wittrock, 1986).

Researchers, who turned inward to investigate teachers' reasoning, have greatly added to our knowledge of what teachers know and believe, as well as to our ideas about the frameworks for asking such questions. Because the present *Handbook* represents an opportunity to take stock of where we are as a field and because, in the last 15 years, so much effort has been aimed at questions about teachers, the focus on teachers in this chapter seemed appropriate.

A second reason for choosing this focus on teachers was less internal to the field, less exclusively about research. We also considered the continuing public concern with improving mathematics education. Amid frequent claims about how to improve teachers (or improve teaching), we found that research, policy, practice, and advocacy are deeply intertwined when it comes to questions of teaching and learning mathematics. Researchers and policymakers, mathematicians and practitioners, politicians and parents are in the mix as they make claims and recommend courses of action. Appraising what we know—and what we do not—as well as where we need to head with respect to teachers' knowledge seemed timely. For example, although 15% of the articles we reviewed focused on teachers' knowledge and beliefs, only 5% probed how teachers' mathematical understanding affected their practice, and only 2% examined how it affected students' learning. We still face a gap in the knowledge needed to guide policy and practice.

A third reason for choosing to focus our chapter on teachers' knowledge of mathematics was theoretical. We saw in this topic the opportunity to integrate and to investigate comparatively what have been very different kinds of work. We saw an opportunity to lay the foundations for a programmatic agenda focused on mathematical knowledge for teaching, an agenda that would combine theory, practice, and empirical inquiry in research on mathematics teaching and teacher learning.

Before we examine the problems associated with understanding, studying, and developing the mathematical knowledge needed for teaching, we step back to situate this chapter in the broader context of contemporary mathematics education. We trace briefly other explanations offered for so many American adults' lack of mathematical proficiency.[3]

Why Are So Few American Adults Mathematically Proficient?

The school mathematics experience of most Americans is and has been uninspiring at best, and intellectually and emotionally crushing at worst. "Mathematical presentations, whether in books or in the classroom, are often perceived as authoritarian" (Davis & Hersh, 1981, p. 282). Ironically, the most logical of the human disciplines of knowledge is transformed through misrepresentative pedagogy into a body of precepts and facts to be remembered "because the teacher said so."

Despite its power, rich traditions, and beauty, mathematics is too often encountered in ways that lead to its being misunderstood and unappreciated. Many pupils spend their time in mathematics classrooms where mathematics is no more than a set of arbitrary rules and procedures to be memorized. Davis and Hersh (1981) describe the pattern of the "ordinary math class" in which many readers of this chapter were likely raised:

> The program is fairly clear-cut. We have problems to solve, or a method of calculation to explain, or a theorem to prove. The main work will be done in writing, usually on the blackboard. If the problems are solved, the theorems proved, or the calculations completed, then the teacher and the class know they have completed the daily task. (p. 3)

When students don't "get it," their confusions are addressed by the teacher's repeating the steps in "excruciatingly fine detail," more slowly, and sometimes even more loudly (Davis & Hersh, 1981, p. 279).

Improving Mathematics Teaching and Learning: Why Has It Been So Difficult?

The past 40 years have seen several waves of mathematics reform, each entailing serious efforts to improve mathematics learning. Each wave has attempted to upgrade what counts as "mathematics" in school, to alter students' mathematical experiences, and to improve their grasp of fundamental ideas and skills. Yet change has been difficult, and many students experience their math classes much like those described in the previous section.

In fact, much about mathematics education has remained the same as it was in 1950 or even 1900. Students still practice pages of sums and products and are still asked to solve improbable

[1] We included in our perusal those journals that contained at least some mathematics education–related research, were national or international in scope, and were accessible through ERIC. The lists used for the annual reviews of research in mathematics education were a starting point for creating the pool of journals. Several entries on the annual review lists did not meet those criteria and were deleted, and a few missing journals that were obviously relevant were added to the list (e.g., the new *Hiroshima Journal of Mathematics Education*).

[2] Fewer than a third of the articles focused on curriculum, and most of those focused on how technology might be used in classrooms. Just over a fifth of the articles focused on students; only 4% (14) of the articles examined interactions of students, curriculum, and teaching.

[3] John Allen Paulos (1988) describes in detail the nature of this "illiteracy" and, in so doing, provides rich examples of what it might mean to be mathematically literate.

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story problems. Students are still told to "invert and multiply" to divide fractions and to use "My dear Aunt Sally" (MDAS) to remember to multiply and divide before adding and subtracting in an expression. Teachers still explain how to do procedures, offer rules of thumb, give tests on definitions and procedures, and provide applications. Those practices in and of themselves are not necessarily unhelpful. However, the prevalence of instruction that consists only of such practices helps to explain why the number of students who leave school as proficient with mathematics as they are literate with English remains small.

No single cause can account for the failure of past reform efforts to change the face of mathematics teaching in American classrooms. Yet the patterns of the "ordinary" mathematics class have dominated. And despite contemporary rhetoric and debate, they continue to prevail. The failure of reforms to penetrate core practice has preoccupied many scholars (Berman & McLaughlin, 1975; Cohen & Ball, in press; Cohen, 1989; McLaughlin, 1990; McLaughlin & Marsh, 1978; Sarason, 1971; Tyack & Cuban, 1995). Dominant explanations for this failure of past reform efforts suggest factors that impede progress. Among the most frequent explanations are the misrepresentation of mathematics; culturally embedded views of knowledge, learning, and teaching; social organization of schools and teaching; curriculum materials and assessments; and teacher education and professional development.

In this chapter, we focus on a problem that underlies all of those explanations: our understanding of the mathematical knowledge that is needed to teach mathematics well. We begin with a brief examination of the other explanations, however, because we argue that each of them contributes to the difficulties in solving the problem that is the focus of this chapter—mathematical knowledge for teaching.

The Misrepresentation of Mathematics

Alfred North Whitehead (1911/1948) saw one root of mathematical illiteracy in the fact that, as mathematics is usually taught, the elegance and power of mathematics are not exposed to students as they are helped to acquire skills and tools. Whitehead described the encounters that most people have with mathematics:

> The study of mathematics is apt to commence in disappointment. The important applications of the science, the theoretical interest of its ideas, and the logical rigour of its methods, all generate the expectation of a speedy introduction to processes of interest. . . . Yet, like the ghost of Hamlet's father, this great science eludes the efforts of our mental weapons to grasp it. . . . The reason for this failure of the science to live up to its great reputation is that its fundamental ideas are not explained to the student disentangled from the technical procedure which has been invented to facilitate their exact presentation in particular instances. Accordingly, the unfortunate learner finds himself struggling to acquire a knowledge of a mass of details which are not illuminated by any general conception. (Whitehead, 1911/1948, pp. 1–2)

Too often, the curriculum and teaching methods used in school inundate students with skills and procedures without allowing them to develop an appreciation for the power of mathematics as a system of human thought. Most adults graduate from school never having experienced any of the power, elegance, and beauty of the subject. Their interest has not been captured, and many conclude that they neither can do nor need mathematics.

Views of Knowledge and Learning

A second explanation for the resilience of the common patterns of mathematics instruction in the United States reflects what Jackson (1986) calls the "mimetic" tradition, which is at odds with the more ambitious instruction that is advocated by reformers. Knowledge is fixed; teachers give knowledge to pupils who store and remember it. This tradition is firmly embedded in Western culture. Cohen (1989) writes that, as many as 300 years ago,

> most teaching proceeded as though learning was a passive process of assimilation. Students were expected to follow their teachers' directions rigorously. To study was to imitate: to copy a passage, to repeat a teacher's words, or to memorize some sentences, dates, or numbers. Students may have posed questions in formal discourse, and perhaps even embroidered the answers. But school learning seems to have been a matter of imitative assimilation. (pp. 42–43)

Cohen (1989) argues that innovative approaches to teaching and learning often embody assumptions about knowledge as well as about teaching and learning that are "a radical departure from inherited ideas and practices" in this culture. These new approaches fly in the face of centuries-old intellectual traditions, not only in mathematics but also in all disciplines. Those intellectual traditions also live outside the school institutions, in the everyday occasions for informal teaching—occasions involved in parenting, for example. The views that knowledge is fixed, that teaching is achieved through transmission, and that teachers are authorities run very deep in U.S. culture. By contrast, Stigler and Hiebert (1999) provide portraits of Japanese mathematics instruction and argue that the differences highlight the ways in which teaching is a cultural activity, embedded in and reflective of the culture in which it takes place.

Organizational Factors

A third common explanation for the failure of mathematics reform involves institutional and organizational factors in schools. Schools are charged with multiple and competing goals. The push to standardize and to avoid risks or experiments is great in light of those pressures (Ball & Cohen, 1999; Cohen & Neufeld, 1981; Goodlad, 1984). Administrators and school board members, who are preoccupied with test scores, put pressure on teachers to emphasize "basic skills"—computation and memorization of "facts." Teachers are generally responsible for a curriculum that is both traditional and warranted by those very traditions.

The structure of the school day means that teachers are isolated from one another and have little time or support for learning or trying out innovations. Time is segmented into 55-minute blocks, content must be covered, and pupils must be prepared not only for tests but also for the next level. Moreover, working

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with groups of 30 children makes experimenting with pedagogy risky for teachers who must also maintain order and routines. Elementary teachers must teach many other subjects in addition to mathematics; secondary teachers must teach many more students.

The incentives to work on more complex content and to press for higher performance are not necessarily present. Teachers often do not have time to plan and organize rich experiences for pupils, and they cannot afford the looseness of more exploratory curricula. They feel pressure to make sure that pupils master required content. For example, the time required for students to "get inside" a topic like measurement may seem to conflict with the time needed to ensure that students also get to everything else. The pull toward neat, routinized instruction is very strong. Teaching measurement by giving out formulas—$l \times w$ = some number of square units and $l \times w \times h$ = some number of cubic units—may seem much more efficient than hauling out containers, blocks, and rulers and having students explore the different ways to answer questions of "how big" or "how much." With focused, bounded tasks, students get the right answers, and everyone can think that they are successful. The fact that these bounded tasks sometimes results in sixth graders who think that you measure water with rulers may, unfortunately, go unnoticed.

Such features of school organization and of teaching conditions are part of the environments within which reforms must operate (Cohen & Ball, in press; Cuban, 1984; Sarason, 1971). Sarason (1971) argues that one important reason the New Math curriculum failed was that the reformers generally failed to take a sufficiently broad perspective of the "regularities" of the school setting and the culture of the school—teachers had little access to professional development or few opportunities to work together on curriculum, for instance.[4]

These regularities of schooling and the culture of the school are among the many other elements of the surrounding environments in which teaching and learning take place: policies, state mandates, assessments, parents, communities, districts, and the like. If the environments of teaching and learning are ignored, then change—if it occurs at all—is likely to be superficial: for example, changing textbooks but not the mode of instruction (Sarason, 1971).

These environments influence the course of innovation through the many ways in which they permeate instruction (Cohen & Ball, in press). This perspective challenges the very concept of "implementation" itself (Cohen & Ball, 1999; Farrar, Descantis, & Cohen, 1980; Giadomenico & Wildavsky, 1984). Sarason would argue that reformers who do not consider what schools are like will operate under the misguided impression that changes can be simply "put into place." Quite to the contrary, innovations are interpreted and adapted by teachers; the intent, the enactment, and the effect of an innovation changes in the translation (Berman & McLaughlin, 1975; Cohen & Ball, in press; McLaughlin, 1990; McLaughlin & Marsh, 1978; Sarason, 1971). The expectation that reforms can be instituted faithfully from the top down is a fantasy that ignores the loose connections between official authority and actual practice. American teachers, in fact, have considerable elbow room at the classroom level and typically "[arbitrate] between their own priorities and the implied priorities of external policies" (Schwille et al., 1983, p. 387).

Curriculum Materials and Assessments

In the past, a fourth explanation for the continuity and conservatism of mathematics teaching has been the conservative nature of the materials from which teachers teach and the confining effects of traditional assessments. Stodolsky's (1988) analysis of elementary mathematics textbooks suggests that concepts and procedures were often inadequately developed, with just one or two examples given, and that the textbooks emphasized "hints and reminders" to students about what to do. For example, in her study, Stodolsky found that area and perimeter (the content of the example in the preceding section) were presented in terms of the formulas—$l \times w$ and $2 \times l + 2 \times w$—with, perhaps, some pictures to illustrate. The books provided practice in calculating the area and perimeter of some rectangles, with reminders to state the answers to area problems in terms of "*square* units." Multiplication by 10 was explained in terms of "adding a zero," algebra texts claimed that vertical lines have "no slope," and rectangles were represented as figures with two long and two shorter sides.

Until the past decade, geometry below tenth grade was scant, measurement was more procedural than conceptual, and probability investigations were relegated to the little "Time Out" boxes on a few random pages. The curriculum analyses conducted by researchers for the Third International Mathematics and Science Study (TIMSS) revealed a portrait of U.S. textbooks as recycling the same topics year after year, covering a vast number of topics, and not developing any topic deeply (Schmidt, McKnight, & Raizen, 1997; Schmidt, McKnight, Valverde, Houang, & Wiley, 1997a; Schmidt, McKnight, Valverde, Houang, & Wiley, 1997b).

In the past 10 years, however, curriculum materials once again have been a major lever of reform. The National Science Foundation invested millions of dollars in new curricula, and many new mathematics curriculum materials have appeared. Their developers, like the predecessors in the 1960s, seek to improve teaching and learning by effecting significant changes in the mathematics curriculum. Whether and how these curriculum materials will fare differently in their aim to permeate teaching and learning remain to be seen.

Some early studies of teachers who are using new curriculum materials suggest that teachers' knowledge and beliefs continue to shape their interpretations and uses of those materials. Significant questions about the interplay of guidance and opportunity to learn will emerge, as before, in the enactment of new materials (Collopy, 1999; Remillard, 1999a, 1999b; Rickard, 1993).

[4] The "New Math" is the name commonly given to the curriculum reforms of the 1960s, which emphasized mathematical structure, and focused on improving the curriculum for college-bound students, but which also spread to middle and elementary schools. See Romberg (1992) for an insightful analysis of the New Math and its successes and disappointments.

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Teacher Education and Professional Development

A final focus of analysis has been the weak impact of professional education on teachers. Critics observe that preservice teacher education typically has a weak effect on teachers' knowledge and beliefs and that whatever prospective teachers learn at the university tends to be "washed out" once they get to schools (Zeichner & Tabachnick, 1981). In fact, it is rather unsurprising that a handful of university courses often fails to substantially alter the knowledge and assumptions that prospective teachers have had "washed in" through years of firsthand observations of teachers. By the time they begin professional education, teachers have already clocked more than 2,000 hours in a specialized "apprenticeship of observation" (Lortie, 1975, p. 61), which not only has instilled traditional images of teaching and learning but also has shaped their understanding of mathematics (Ball, 1988). Because this understanding of mathematics is the mathematics they will *teach*, what they have learned about the subject matter in elementary and high school turns out to be a significant component of their preparation for teaching.

Professional education is also the target of frequent criticism for its fragmented, episodic, and superficial nature. Although a good deal of money is spent on "staff development" in the United States and although there is no shortage of in-service "training" for teachers, most training money is spent on sessions and workshops that are often intellectually superficial, disconnected from deep issues of curriculum and learning, fragmented, and noncumulative (Cohen & Hill, 2000; Heaton, 1992; Little, 1993; Wilson, Lubienski, and Mattson, 1996). Rarely do these inservice opportunities seem based on a curricular view of teachers' learning (Ball & Cohen, 1999). Teachers are thought to need "updating" rather than opportunities for serious and sustained learning about curriculum, students, and teaching.

Because professional development is rarely seen as a continuing enterprise for teachers, it is only occasionally truly *developmental.* One reason is that many people see teaching as mostly commonsense, and they perceive little need for professional learning. Another reason is that teaching has been seen and organized as a career in which sustained learning is not required for adequate performance. Still another reason is that no coherent infrastructure exists for professional development. In other words, no easily identifiable group or agency is responsible for professional development, and so it happens everywhere—and, consequently, lacks consistency and coherence. A final reason is that professional development lacks a curriculum for teachers' learning, a curriculum that considers the practices they are being asked to enact, the mathematical knowledge that such practice entails, and the attention to what they already know and believe—that is, what they bring to the undertaking (Heaton, 2000).

The Role of Knowledge of Mathematics for Teaching

Another factor that is often cited to explain the weaknesses of U.S. mathematics education is U.S. teachers' knowledge of mathematics. Observers note that interpreting reform ideas, managing the challenges of change, using new curriculum materials, enacting new practices, and teaching new content all depend on teachers' knowledge of mathematics. What many assume, however, is that this mathematical knowledge is shallow and relatively transparent, that the problem faced is one of current U.S. teachers' intellectual weaknesses. Few have looked closely at the practice of elementary or secondary mathematics teaching to consider the nature of its mathematical entailments.

Consider the mathematical understanding entailed in teaching the multiplication of decimals. To enhance appreciation of the complexity and subtlety of the mathematical understanding required, we peek in on a classroom where a teacher is helping her students learn to multiply decimals.[5] When she taught multiplication of whole numbers, she used repeated addition as a model (e.g., $6 \times 7 = 7 + 7 + 7 + 7 + 7 + 7$), but she had introduced area representations as well. She begins the lesson on multiplying decimals by reviewing an example with whole numbers:

> "If the unit is one little square," asks the teacher, "how could you show 7×6 with the tiles?" Michael volunteers. He goes over to the overhead, carefully lays out seven tiles, and then makes six rows of them to form a rectangle:



> "I have 7 times 6," he explains, and he counts each tile carefully across the top row: "One, two, three, four, five, six, seven." Then, he points down the right-hand column, "One, two, three, four, five, six."
>
> "How much is that altogether?" asks the teacher.
>
> "Forty-two," he replies, confidently, pointing at the squares filling his rectangular array. He counts the tiles quickly to show that he is right, and several children chime in, "1, 2, 3, 4, 5, 6, 7, . . . 38, 39, 40, 41, *42*!"

The teacher is pleased. She knows that using the area representation will help to demystify the rule for multiplication of decimals: (a) multiply as with whole numbers, (b) then count the number of decimal places in the numbers being multiplied, (c) count over that combined number of places in the answer, and (d) place the decimal point there.

For years, students would ask, "Why does that work? When we add decimals, you tell us to keep the decimal place lined up. Why, when we multiply, do we move the decimal point?" At those times, she had felt a kind of despair at teaching meaningless rules. Using the area model for multiplication, she now thought, would help to make the procedure make sense.

[5] This example is constructed on the basis of a composite of several lessons taught by Deborah Ball.

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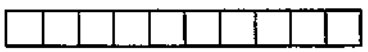
"Good job, Michael. We're going to try something a little harder now," the teacher announces to the fourth graders. She holds up a large square, scored 10 by 10 into 100 little squares. "Instead of the little square being the unit, the *large* square is going to be the unit. So, first, what is the little square now?"

Several children call out, "One hundredth."

"How do you know?" asks the teacher.

Jamie breaks out, "Because there are 100 of them in the big square, so one of them is one hundredth." Other children nod.

The teacher holds up the rod composed of 10 little squares:



"So what is this, then?" she asks. There is a pause.

"It's 10," says one boy. "Wait, no, wait, it's not. It's one *tenth* because there are 10 of those in the whole square."

The teacher nods. "Now try showing this problem with the tiles," she says, and she writes: .3 × .7.

The children go to work at their desks, many using the little squares. Their arrays look like Figure 24.1. The area of these shapes is 21 *hundredths* or .21, as expected. Suddenly, the teacher notices that these arrays are constructed with three *hundredths* or .03 on one side and seven *hundredths* or .07 on the other, because the little squares are each one hundredth. The children were supposed to be multiplying three *tenths* times seven *tenths*. What is going on? Shouldn't they be using the tenths, the long rods, to build the arrays? She looks in the teacher's guide, slightly unsettled, and sees that the arrays look like those her students are making. She feels a little relieved but still wonders what is going on. These rectangles show the right answer, but where are .7 and .3, the original numbers? Looking closely, she sees that one can interpret the arrays as 7 × .03 by adding each column together (.03 + .03 + .03 + .03 + .03 + .03 + .03 = .21). Moreover, she can also see 3 × .07 by counting the rows. What is being represented here? Is it .03 × .07? 3 × .07? 7 × .03? Or is it .3 × .7, as she had posed? This sudden loss of mathematical footing is unsettling.

The teacher sees that, over on the side of the room, two boys have constructed a different proposed solution. Using the long rods, which correspond to tenths when the square is the unit, they have built an array (see Figure 24.2).

She sees that they have built a contiguous array of rods that represents tenths—three groups of seven tenths. But something is definitely wrong here because, in total, this array represents 2.1 (see Figure 24.3), and the product of .7 × .3 is .21, not 2.1. She quickly realizes that these students have indeed worked with tenths, but that the product they have represented is *three* times seven tenths, not *three-tenths* times seven-tenths, hence the answer of 2.1.

She considers again the area represented by the 21 hundredths (see Figure 24.1). Suddenly she realizes that this array *does* represent three tenths times seven tenths. Each small square has a *side length* of *one tenth*. The side of the unit square has a length of 1, and so the side of each of the little squares is one-tenth of 1, or 0.1 (see Figure 24.4). If each little square has a side length of one-tenth, then it makes sense, she sees, that each square has an area of one-hundredth of a unit, that is, one-tenth times one-tenth (see Figure 24.5).

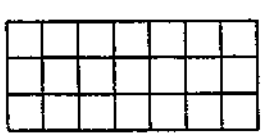


Figure 24.1 A typical arrangement of tiles to show .3 × .7.

Given all that, the rectangle formed by 21 little squares *does* represent .3 length along one side and .7 length along the other, and hence, the area product of .3 × .7. She realizes, with interest, that the same array can be used to represent 3 × .07 or 7 × .03 but that, in these products, the numbers do not correspond to lengths. Instead, those arrays are examples that iterate the addition of rows or columns. In this representation of repeated addition, one number represents the number of iterations while the other number represents the area units being iterated. In 7 × .03, the seven represents how many units of three-hundredths are added together. Three-hundredths corresponds to a unit of area, and seven corresponds to the number of such units.

In the classroom example we have just examined, what was the content? Multiplication of decimals. This teacher knew that 3 × 7 was 21, and she knew that .3 × .7 was .21. She knew that when you multiply tenths, you get hundredths—hence the rule for moving over the decimal point. And her textbook provided guidance for using the area model for helping students understand procedures for multiplying decimals. But in this lesson, she realized afresh how subtle and yet how important the mathematical ideas underlying this lesson were. When one multiplies whole numbers with the area model and counts the unit square as 1, it is easy to overlook the fact that what is being multiplied are the lengths of the sides—which are linear measures—and that what is being produced are areas. In the whole number example, both the length and the area of the little unit are 1 in this model, so one can easily miss this important distinction. And yet this distinction becomes much clearer with the multiplication of tenths because then the areas of the little squares are hundredths, and their side lengths are, therefore, tenths. If all one does is count the areas, it can appear that what is being multiplied are hundredths, yielding hundredths as well.

Knowing this web of ideas is important for teaching the multiplication of decimals. Understanding the ideas connected to a particular mathematical topic matters in any approach to teaching, whether the teacher provides the explanation and asks students to practice, or whether, as in this example, she engages students in tasks and discussions of those tasks. Without question, being able to multiply whole numbers and decimals constitutes necessary but insufficient knowledge for teaching that skill well. This teacher needed to have a refined and explicit understanding of the meaning of multiplication, combined both as iterated addition and as the product of two linear units that yield an area, together with the correspondences between these models. She needed to understand how the model she was using worked and how it corresponded to the symbolic form of .3 × .7 = .21. She also needed confidence in her own mathematical knowledge to be able to analyze the problem and its representation "on the fly" when she realized that she was unclear about what was being modeled with the blocks. In this case, the

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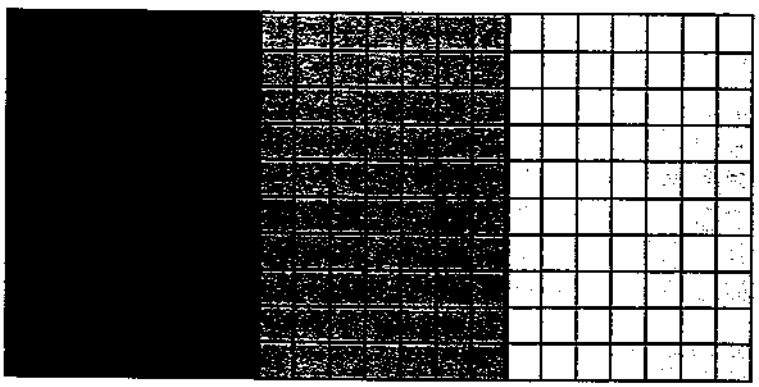


Figure 24.2. A representation of 3 × .7 using three groups of seven rods.

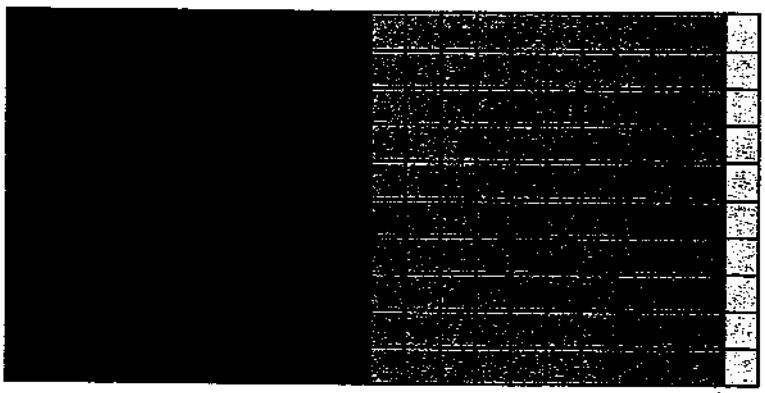


Figure 24.3. Three groups of seven rods reorganized as two large squares and one tenth of a large square.

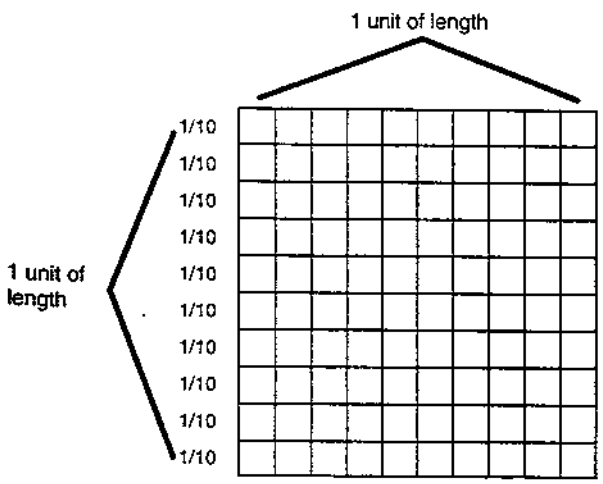


Figure 24.4. The relationship between side *lengths* of the little squares and the large square.

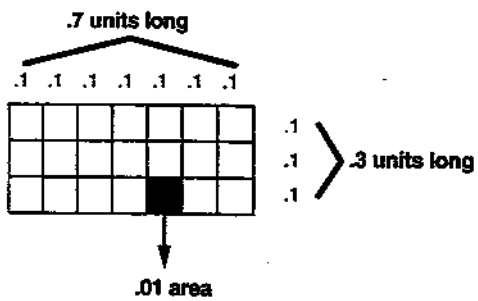


Figure 24.5. The relationship between side *lengths* (in tenths) and *area* (in hundredths) for .3 × .7.

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teacher needed to be able to reason through her confusion as the lesson unfolded so she could keep the lesson on track mathematically.

With this glimpse of mathematics teaching in mind, we return to our problem. The search to delineate what it is that teachers *do* need to know in order to teach has been a challenging one. In this chapter, we trace the history of the efforts to determine the relative role of teachers' mathematical knowledge in teaching, to explore changing ideas about what counts as mathematical knowledge for teaching, and to consider what we need to learn.

Teacher knowledge is not a new issue and is not one tied exclusively to reform. Questions about the knowledge that teachers need to know, how it affects their teaching, and how they can be helped to develop it have preoccupied professional developers, policymakers, and researchers alike for more than four decades. Teachers' knowledge of mathematics was one of the first variables investigated by researchers on teaching in the 1960s. Those researchers asked questions about how teachers' knowledge influenced the teachers' effectiveness (Begle, 1972, 1979). And teachers' mathematical knowledge has continued to be a much-discussed problem in the contemporary debates about the improvement of mathematics teaching and learning.

In 1999, Liping Ma published *Knowing and Teaching Elementary Mathematics,* a book that has attracted renewed and much wider interest in this issue. In her study, Ma compared Chinese and U.S. elementary teachers' mathematical knowledge. Producing a portrait of dramatic differences between the two groups, Ma used her data to develop a notion of "profound understanding of fundamental mathematics," an argument for a kind of connected, curricularly structured, and longitudinally coherent knowledge of core mathematical ideas.

Ma's book has broadened and reignited discussions around teachers' mathematics knowledge and its contributions to instruction. Research mathematicians, deeply concerned about what some perceived as the morass of K–12 mathematics education, have taken note. They were suddenly struck with how central teachers' mathematical knowledge is to effective teaching and to improvement. Roger Howe (1999) reflected the reactions of many mathematicians when he wrote:

> [Ma's work] gives us new perspectives in the problems of improving mathematics education in the United States. For example, it strongly suggests that without a radical change in the state of mathematical preparedness of the American teaching corps, calls such as those contained in the NCTM Standards, are simply doomed. (p. 884)

Although many researchers, teachers, policymakers, and teacher educators have recognized the importance of teachers' mathematical knowledge, understanding the nature and role or mathematical knowledge for teaching—plus developing it effectively—remains an unsolved problem for the improvement of mathematics teaching and learning. We move next to trace where the field has been in seeking to define and solve this problem. We describe what we know—and do not know—from research, and we frame directions for where work on this problem needs to head.

Mathematics Knowledge for Teaching: A Problem for Practice, Policy, and Research

First, we take a look back. In the previous edition of *Handbook of Research on Teaching,* Romberg and Carpenter (1986) surveyed research in mathematics education and worried about what they saw as a disconnect between research on teaching and research on learning. They noted that, despite big gains in understanding students' mathematical conceptions and development, the findings from research on teaching were unaligned with questions arising from research on learning.[6] While researchers on learning were producing portraits of the complexity of developing conceptions of rational numbers, or area, or multidigit subtraction, research on teaching was investigating whether students involved in cooperative learning within small group settings were indeed behaving cooperatively and were more productive than students in noncooperative small group settings. How research on teaching related to mathematical work was often invisible, leaving unprobed many questions about the critical relationships between teaching and learning.

Looking ahead to the next 10 years, the authors urged researchers to integrate studies of learning with studies of teaching to "bridge the learning–teaching gap" (Romberg & Carpenter, 1986, p. 868). Mathematics should be at the center of such studies, they urged, with increased refinement and detail when considering the role of content in teaching and learning. Without this increasing integration of mathematical content and without greater coordination between research on teaching and research on learning, knowledge useful for improving instruction would continue to be elusive. Romberg and Carpenter's argument was an important one. Our examination of the literature suggests that researchers *have* turned more toward instruction, as Romberg and Carpenter had urged.

Solving the Problem of Mathematical Knowledge for Teaching: From Teachers to Teacher Knowledge

The assertion that teachers' own knowledge of mathematics is an important resource for teaching is so obvious as to be trivial. Commonplace are statements such as: "A thorough grounding in college-level subject matter and professional competence in professional practice are necessary for good teaching. . . . The data are unequivocal: students learn more mathematics when their teachers report having taken more mathematics" (American Council on Education [ACE], 1999, p. 6). The ACE report goes on to cite evidence that earning a college degree in mathe-

[6] One example was the robust finding that the amount of time devoted to mathematics instruction was found to be positively related to student achievement. Time on task, argued Romberg and Carpenter (1986), was inadequate as a useful measure without data on what students were engaged in doing. Academic tasks differ markedly in mathematical scope and cognitive demand, and task enactment in class further differentiates the nature of the work. As such, the amount of time spent engaged in work was an imprecise measure of learning opportunity and was certainly too imprecise to be useful for improving classroom learning.

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matics, being certified in mathematics, and being mathematically skillful "all contribute to effective teaching of mathematics" (p. 6). The warrant is, however, much weaker than the ACE authors would like.

The claim that teachers' knowledge matters is commonsense. However, the empirical support for this "obvious" fact has been surprisingly elusive. "No one questions the idea that what a teacher knows is one of the most important influences on what is done in classrooms and, ultimately, on what students learn," assert Fennema and Franke (1992, p. 147). And yet, as those authors explain, no consensus exists on the mathematical knowledge that is required to teach. This lack of consensus is a problem for policymakers and teacher educators alike. If we are to identify what teachers need to know or to learn, then an important matter is how much knowledge of mathematics it takes to teach. It matters for teacher certification requirements, it matters for teacher education policy and practice, it matters for recruiting and hiring teachers, and it matters for curriculum development.

Three approaches have dominated efforts to solve the problem of teachers' mathematical knowledge. One approach is a policy response; two approaches involve research. The policy response to resolving this problem of what teachers need to know has been to work logically and to create documents and lists of what teachers should know. Some documents are general; others, much more specific. The general documents (ACE, 1999; Interstate New Teacher Assessment and Support Consortium [INTASC], 1993; National Board for Professional Teaching Standards, 1991; National Commission on Teaching and America's Future, 1996) simply call for teachers to know their content well and are not particular to mathematics. They extol qualities of the requisite knowledge: it should be deep, connected, and conceptual and should include knowledge of the epistemology of the field.

In the more specific documents, experts identify the mathematical knowledge by considering the topics in the curriculum and by assuming, quite reasonably, that teachers need to know that which they have to teach (e.g., Conference Board of the Mathematical Sciences, 2000; Leitzel, 1991; National Council of Teachers of Mathematics, 1991). Those documents produce lists of mathematical topics. As they recognize the need for teachers to have mathematical perspective about the connections among ideas, such lists usually identify topics beyond the curriculum. This approach to specifying knowledge for teaching is rooted primarily in policy deliberations and often does not reflect research evidence.

Although this approach has been an important one, in our analysis, we focus on two other approaches that researchers have used to solve or at least to better understand the problem of mathematical knowledge for teaching. How the problem has been defined has influenced the manner in which it has been approached by researchers. Those approaches are distinguished by more than differences in their orientation and focus: they also entail particular stances on measurement and teacher education. Each approach has relied on measuring mathematical knowledge for teaching in particular ways and is associated with particular perspectives on how to help teachers develop the requisite knowledge for teaching.

The first research approach, which centers on looking at characteristics of *teachers,* is associated with the assumption that knowledge of and skill with mathematics content is essential to teaching. It assumes, further, that indicators such as the courses taken, degrees earned, or certification received will represent the requisite knowledge and that teacher knowledge can be measured using such relatively straightforward indices. Thus, research in this domain focuses largely on description and analysis of teacher characteristics, particularly the amount of mathematics that teachers have taken.

The second research approach, which focuses on teachers' *knowledge,* builds on the first approach in that it acknowledges the importance of the content of teachers' mathematical knowledge, but it also prominently includes a qualitative focus on the nature of teachers' knowledge. This approach is based, in part, on the notion of pedagogical content knowledge as an intertwining of knowledge about how ideas might be represented, how students learn, and what they find difficult (Shulman, 1986, 1987; Wilson, Shulman, & Richert, 1987). This second approach to research does not replace but builds on the first: the first emphasizes *teachers;* the second emphasizes teachers' *knowledge.* Although the second approach has a strong contemporary following among researchers and teacher educators, the first approach is attractive to policymakers. We argue in this chapter that each approach has merit. We propose, further, that these two approaches be reconnected in the next steps to solve the problem of what mathematical knowledge is necessary to teach well.

An alternative approach, with which the chapter concludes, builds on both lines of prior work—on *teachers* and on *knowledge*—but shifts to a greater focus on *teaching* and on *teachers' use of mathematical knowledge.*

Looking at Teachers

In the earliest approach to answering questions about the mathematical knowledge that is necessary for teaching, researchers attempted to validate empirically the common maxim that the more mathematical knowledge teachers have, the more mathematical knowledge their students will have. This foundational assumption led researchers to count the courses teachers had taken and to analyze relationships between the extent of their mathematics course work and their students' learning. The approach to measuring teachers' knowledge is straightforward: count the courses taken, the credits earned, and the degrees attained. Each of those totals is considered to be a representation of teachers' mathematical knowledge. This approach still holds considerable interest (e.g., Darling-Hammond, 1999; Monk, 1994). Similarly, in terms of teacher preparation, this approach to teacher knowledge leads to stipulations of credit and course requirements but pays little attention to content, scope or the nature of mathematics. Teachers' exposure to and development of mathematical content is assumed to equip them with the resources needed for effective teaching.

Perhaps best known among the early work in this approach was Edward Begle's (1979) meta-analysis of studies conducted between 1960 and 1976 that probed the effects of teacher variables on student performance. Convinced that mathematics education was suffering from too much opinion and not enough careful empirical work, Begle initiated the National Longitudi-

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nal Study of Mathematical Abilities as a forum for examining factors that affect mathematics learning. In offering his analysis of teacher effects, Begle (1979) commented:

> There is no doubt that teachers play an important role in the learning of mathematics by their students. However, the specific ways in which teachers' understanding, attitudes, and characteristics affect their students are not widely understood. In fact, there are widespread misconceptions, on the part not only of laypeople but also of mathematics educators, about the ways in which teachers influence mathematics learning by their students. (p. 27)

Begle's analysis of the relationship between the number of courses teachers had taken past calculus and student performance produced a surprise to many. He found that the extent of teachers' mathematics course taking produced positive main effects on students' achievement in only 10% of the cases and, perhaps more jolting, negative main effects in 8%.[7] This latter result has been the topic of much speculation. It is an interesting result because the negative main effects are not easily explained by the ubiquitous criticism that mathematics courses past calculus are not relevant to teaching or that taking courses is a poor proxy for teachers' actual mathematical knowledge. Those claims support finding no effects.

But how might such variables be associated with negative effects? One explanation might rest with the increasing compression of knowledge that accompanies increasingly advanced mathematical work, a compression that may interfere with the unpacking of content that teachers need to do (e.g., Ball & Bass, 2000; Cohen, in preparation). Another explanation might be that more course work in mathematics is accompanied by more experience with conventional approaches to teaching mathematics. Such experience may imbue teachers with pedagogical images and habits that do not contribute to their effectiveness with young students (Ball, 1988). What underlies those negative main effects is not known but is worth further exploration.

Begle (1979) also examined other teacher knowledge variables. For example, his analysis of the relationship of teachers' course taking and their students' performance revealed that majoring or minoring in mathematics yielded positive main effects in 9% of the cases and negative main effects in 4%. A less-often reported result was that the greatest number of positive main effects was produced from the analysis of the relationship between the number of credits in mathematics *methods* courses and student performance—49, or 23%. This analysis yielded only 12 negative main effects (5%).

Begle (1979) concluded from these results that the belief stated as "the more a teacher knows about his subject matter, the more effective he will be as a teacher" demanded "drastic modification" (p. 51). On the basis of his finding, he claimed that advanced mathematical understanding contributed little to teacher effectiveness. He noted that, although "it seems to be taken for granted that it is important for a teacher to have a thorough understanding of the subject matter being taught," the question of what should be meant by "thorough" was never examined in these studies (p. 28). He concluded that future analyses of the amount of mathematics taken by teachers were not likely to produce significant effects, and that further research of this sort was not needed until the ways in which teacher knowledge might affect student achievement were better understood. A clue may have been buried within his own results—the relative strength of the relationship between credits in mathematics methods and student performance.[8]

The interest in teachers has continued, however. In 1996, the National Commission on Teaching and America's Future (NCTAF) released its report, which proposed a series of strong recommendations for improving the nation's schools. It consisted of "a blueprint for recruiting, preparing, and supporting excellent teachers in all of America's schools" (p. vi). Asserting that what teachers know and can do has the most important influence on what students learn, the report argues that teachers' knowledge affects both students' opportunities to learn and their actual learning. Teachers must know the content thoroughly to be able to present it clearly, to make the ideas accessible to a wide variety of students, and to engage students in challenging work.

The report's authors cite a number of studies showing that teacher knowledge makes a substantial contribution to student achievement. They argue that "differences in teacher qualifications accounted for more than 90% of the variation in student achievement in reading and mathematics" (Armour-Thomas, Clay, Domanico, Bruno, & Allen, 1989, cited in National Commission on Teaching and America's Future, 1996, p. 8). Challenging some reformers' designs, the NCTAF report takes the position that neither curriculum materials, standards, nor assessments can make an impact unless teachers understand them and can make productive use of them with students. It is a document decidedly focused on teachers and on the importance of their qualifications.

To investigate empirically the relationship of teachers' subject matter preparation and student achievement, Monk (1994) analyzed data from the Longitudinal Study of American Youth (LSAY).[9] Teacher preparation was measured by a survey administered to teachers in which they reported the number of mathematics and science courses they took as undergraduates and in graduate programs. Surveys were used to document teachers' overall education level (bachelor's degree, master's, or master's plus additional credits) and years of teaching experience. In general, Monk reached a similar conclusion to that reached by Begle—namely, that the number of mathematics courses makes a difference, but only up to a point. In his analysis, he found that five courses in mathematics, independent of the specific content covered, was the threshold beyond which few effects accrue.

Interestingly, Monk's analysis also showed that each under-

[7] A positive main effect would mean that more credits of mathematics were associated with greater student performance; a negative main effect would mean that more credits of mathematics were associated with lower levels of achievement.

[8] Darling-Hammond (1999) claims that education course work matters more than content knowledge, especially in terms of predicting teacher performance: 16.5% of the variance in student achievement is accounted for by teachers' course work in education.

[9] The Longitudinal Study of American Youth (LSAY) was conducted in the late 1980s and early 1990s with high school sophomores and juniors. Student achievement data were based on items developed by the National Assessment of Educational Progress (NAEP).

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graduate mathematics education course a teacher takes is associated with a .04% gain in student achievement. Monk also investigated more traditional measures of teacher preparation such as having a mathematics major and holding a master's degree. As with earlier studies, whether a teacher had majored in mathematics had no effect on student performance. Whether a teacher had completed additional mathematics course work at the graduate level had either no effect or a negative effect on student performance. Monk notes that many of the relationships that he did find were nonlinear, suggesting that effects are diminished over time or that there are some threshold effects. In addition, where teacher characteristics are associated with pupil gains, the gains are of a rather minor magnitude. Finally, as in Begle's analysis, Monk uncovered significant effects of courses in undergraduate mathematics pedagogy. Such courses, he found, "contribute more to pupil performance gains than do courses in undergraduate mathematics" (p. 130).[10]

Much of the work examining the relationship between teacher characteristics (educational level, courses taken, intelligence, etc.) and student performance is not focused on mathematics. One widely cited example is the work done by Ferguson (1991). In his analysis of factors that predict student achievement, he found that, when he controlled for family and community background influences, teachers' scores on the state certification exam (Texas Examination of Current Administrators and Teachers [TECAT]) accounted for about 20–25% of the variation in students' average scores on the state competency examination in mathematics and reading. Moreover, between third and seventh grade, teachers' scores also predicted changes in students' average scores over time.

In interpreting those results, Ferguson is careful to note that the analyses do not reveal what high-scoring teachers do that affects their students' learning. It may be, he acknowledges, that teachers with high scores attended better teacher education programs, or they may simply have better thinking skills that influence their pedagogical reasoning or their articulateness as teachers. Ferguson's analysis is still rather remote from the specific question of mathematical knowledge for teaching because the TECAT measures basic literacy skills, which offer neither a reliable proxy for mathematical history nor an estimate of knowledge. Still, this study provides support for the continuing assumption that teachers' ability and achievement are factors in their effectiveness, and it has been widely used to support the importance of teachers to students' achievement (Darling-Hammond, 1999).

Although many of the studies in this approach have been criticized for equating simple indicators such as course taking with mathematical knowledge and although the equivocal findings of those studies seem to lack promise for guiding our thinking about the knowledge needed for teaching, it is impossible to entirely dismiss this approach to the problem. It is intuitively logical and inherently reasonable to believe that a teacher's knowledge and mathematical proficiency, as developed through opportunities for formal study, do matter. Still, the correspondences are often difficult to map.

For example, how might research that focuses on teachers inform our earlier example of teaching decimal multiplication? Knowing how many mathematics courses a teacher had taken does not enable us to predict whether the teacher would be able to untangle the mathematical complexities that arise in this lesson. Simply counting courses does not permit an examination of whether the teacher has the mathematical knowledge needed for this lesson. In fact, even if the data permitted us to learn exactly what courses a teacher had taken, it would not be easy to identify what sort of course would offer opportunities to learn about the nature of the number system and its properties, as needed in this example and its adjacent lessons. Some of this content might arise in a modern algebra course or in a course in real analysis, typically studied by advanced undergraduate mathematics majors, and yet, most of the evidence suggests that advanced courses have no effects or even negative effects. Probably the best course would be an outstanding fifth grade mathematics class, complemented by a later study to extend and make more explicit a global and overarching perspective on the number system. In fact, typical advanced mathematics courses do not provide this perspective nor do typical courses for teachers.

A focus on teachers and their mathematical qualifications and experience affords certain clarity for policy; it leaves obscured, however, the nature of teachers' mathematical knowledge. This chasm between data on teachers and questions about teacher knowledge has given rise to the second approach to solving the problem of the mathematical knowledge that matters for teaching— an approach that preserves the focus on teachers but that adds a closer, more probing analysis of mathematical knowledge.

Looking at Teachers' Knowledge

Dissatisfied with studies of teachers that did not probe the nature of the knowledge being measured, many researchers were convinced that teachers' knowledge of mathematics content mattered in ways that were masked by counting numbers of courses. They turned to a closer probing of mathematical knowledge rather than measuring second-order indicators of knowledge. Some of the precipitants for this shift were the substantial reforms being advocated. Kennedy (1997) explains the contextual factors that also influenced the field:

> Because the main goal of [current] reformers is to instill a deeper understanding in students of the central ideas and issues in various subjects and to enable students to see how these ideas connect to, and can be applied in, real world situations, it therefore makes sense to require that teachers themselves also understand the central ideas of their subjects, see these relationships, and so forth. (p. 6)

A host of descriptive studies of teachers' knowledge appeared. Most were oriented to this more probing approach. One significant distinction is that those studies focus on specific mathematical topics rather than on global conceptions of math-

[10] "A statistical test of the difference between the coefficients estimated for undergraduate mathematics courses and undergraduate mathematics education courses indicated . . . that the difference is significant at better than the 0.01 level" (Monk, 1994, p. 130).

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ematical knowledge. A second is that many of the studies are qualitative, using interviews to probe teachers' knowledge. A third and, perhaps, most important distinction has been the opening up of the idea of *mathematical knowledge for teaching*. Unwilling to accept general notions of mathematical background as sufficiently equivalent to the mathematical knowledge needed for teaching, researchers have explored the knowledge that they thought would matter in teaching.

In developing methods for this work, researchers grappled with the question of what would constitute conceptual understanding for teachers (Ball, 1991a, 1991b; Kennedy, 1997; Ma, 1999; Shulman, 1986; Wilson, 1988; Wilson, Shulman, & Richert, 1987) and have designed new ways to probe teachers' knowledge. The analytic work of defining mathematical knowledge for teaching went hand in hand with the development of new approaches to measuring teacher knowledge (Grossman, Wilson, & Shulman, 1989; Kennedy, Ball, & McDiarmid, 1993; Post, Harel, Behr, & Lesh, 1991; Wilson, 1988). How researchers conceived of content knowledge for teaching shaped the tools they created to probe it. Those new measures, in turn, have profoundly influenced what the studies have produced.

Central to this work has been a focus on propositional and procedural knowledge *of* mathematics—that is, understandings of particular topics (e.g., fractions and trigonometry), procedures (e.g., long division and factoring quadratic equations), and concepts (e.g., parallelism and infinity) as well as the relationships among such topics, procedures, and concepts (R. Davis, 1986; Hiebert & Lefevre, 1986; Skemp, 1978). This *substantive knowledge of mathematics* is most easily recognized by others as "subject matter knowledge," or "content knowledge" (Ball, 1991a; Grossman, Wilson, & Shulman, 1989).

A second dimension, however, has been a focus on knowledge *about* mathematics (Ball, 1991a). This focus includes understandings about the nature of knowledge in the discipline—where it comes from, how it changes, and how truth is established. Knowledge about mathematics also includes what it means to know and to do mathematics, what is the relative centrality of different ideas, what is arbitrary or conventional versus what is necessary or logical, and what is key to having a sense of the philosophical debates within the discipline.[11]

Researchers working in this approach often use methods that probe teachers' knowledge and that situate the questions in and around questions that might arise in teaching. Such questions respond to a student's confusion, explain a mathematical procedure whose meaning is buried inside rules of thumb, or consider the connections among ideas. Those questions create conditions where teachers have to make explicit their understanding of the mathematical ideas and procedures behind the questions.

An overwhelming majority of the studies associated with this approach to teacher knowledge have been conducted with preservice elementary teachers. Three reasons help to explain the preponderance of work on preservice teachers. One is that researchers have easier access to prospective teachers who are right there on campus. Second, researchers, usually teacher educators themselves, are curious about their own students' knowledge. Finally, ethical and relational issues around "testing" teachers have plagued researchers interested in examining experienced teachers' knowledge. However, some studies have focused on the mathematical knowledge of practicing teachers. Why research has focused on elementary teachers reflects a continuing assumption that content knowledge is not a problem for secondary teachers, who, by virtue of specialized study in mathematics, know their subjects. However, research on secondary teachers repeatedly reveals the fallacy of this assumption (e.g., Ball, 1988, 1990b; Even, 1993; Steinberg, Haymore, & Marks, 1985).

An overview of those studies of teachers' mathematical knowledge—elementary and secondary, preservice and experienced—reveals pervasive weaknesses in U.S. teachers' understanding of fundamental mathematical ideas and relationships. Although numerous mathematical topics have been investigated, many have not been studied with any depth.

We summarize here the research that is focused on the topics of whole number multiplication and place value, division, rational numbers, functions, geometry and measurement, and proofs as a representative sample of the research that has been conducted.

MULTIPLICATION AND PLACE VALUE

Research that studies teachers' knowledge of multiplication has focused primarily on the algorithm for performing computations with multidigit numbers as a strategy to probe teachers' understanding of the place value concepts that underlie the procedure. By and large, studies show that the teachers interviewed were able to accurately perform the multiplication algorithm to achieve correct answers. However, they often were not able to provide explanations of the place value concepts that underlie the algorithm. For example, Ball (1988) interviewed 19 prospective elementary and secondary teachers on the topic of multidigit multiplication. She posed the following task:

> Some eighth-grade teachers noticed that several of their students were making the same mistake in multiplying large numbers. In trying to calculate
>
> $$\begin{array}{r} 123 \\ \times\ 645 \\ \hline \end{array}$$
>
> the students seemed to be forgetting to "move the numbers" (i.e., the partial products) over on each line. They were doing this:
>
> $$\begin{array}{r} 123 \\ \times\ 645 \\ \hline 615 \\ 492 \\ 738 \\ \hline 1{,}845 \end{array}$$
>
> instead of this:

[11] Schwab (1961/1978) refers to these kinds of understandings as knowledge of the "substantive" and "syntactic" structures of a discipline—in this case, the discipline of mathematics.

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```
  123
×  645
  615
 492
 738
79,335
```

> While these teachers agreed that this stacking was a problem, they did not agree on what to do about it. What would you do if you were teaching eighth grade and you noticed that several of your students were doing this? (p. 123)

This question is prototypical of many of the research tasks used in this approach to studying teachers' knowledge. Ball (1988) explains:

> Since place value (and its root idea, grouping) is a fundamental mathematical idea and since pupils often find it difficult, it seemed a critical area of teachers' knowledge to investigate. I embedded place value concepts in two different interview tasks: one a classroom scenario focused on student difficulties with the multiplication algorithm, the other a structured planning–teaching–assessment exercise on subtraction with regrouping ("borrowing"). Although place value is the underlying foundation of these conventional procedures, adults can perform the procedures competently without thinking about place value at all. I wanted to examine how well the prospective teachers' adult operational knowledge of numbers equipped them to help pupils make sense of the meaning of written numerals and the meanings of these operations with them. (p. 49)

Consider what is going on inside this question. The algorithm for multiplying large numbers is derived from the process of decomposing numbers into "expanded form" and multiplying them in parts. To understand this process, one must understand decimal numerals as representations of numbers in terms of hundreds, tens, and ones—namely, that in the numeral 123, the 1 represents one *hundred*, the 2 represents two *tens*, and the 3 represents three *ones*. In this example, 123 × 645, one first multiplies 5 × 123,

```
  123
×   5
  615
```

then 40 × 123,

```
  123
×  40
 4,920
```

and then 600 × 123.

```
  123
×  600
73,800
```

In the final step, one adds the results of these three products

```
  123
×  645
  615
 4,920
73,800
```

In effect, one is putting the results of the parts, or steps, of the operation back together—that is, 645 × 123 = (600 × 123) + (40 × 123) + (5 × 123). Few people write their computations out this way. Most "shortcut" it by writing

```
  123
× 645
  615
 492
 738
```

This shortcut, in effect, hides the conceptual base of the procedure. Furthermore, people can learn the shortcut without learning the conceptual foundation of the procedure. This procedure, which depends conceptually on place value and the distributive property, is thus a strategic site for exploring knowledge of place value.

The teachers in Ball's study (1988, 1990a) explained the algorithm, particularly the need to begin the second partial product in the tens place rather than the ones place, using language such as "lining it up correctly," "moving the numbers over," "adding a zero," and "using zero as a place holder." In some cases, the preservice teachers clearly had only partial or incomplete understanding of the role of place value in multiplication. In other cases, it was less clear whether they lacked understanding or whether they lacked the ability to make that understanding explicit for the interviewer.

Using this same task, Ball studied practicing teachers and obtained similar results. Later, Ma (1999) reanalyzed these same data and reported that the teachers tended to give "pseudo-explanations" for the algorithm, using much of the same language as the preservice teachers in Ball's (1988) study. For example, teachers spoke of the "tens place" or the "hundreds place" but were referring primarily to the location of digits, not to the essential features of place *value*. Some of the teachers even used nonmathematical symbols, such as asterisks, for place-holders in the algorithm. When Ma used this same question with 72 Chinese teachers, however, she found that they provided explanations of the multiplication algorithm that used the distributive property to explain the role of place value in the partial products. Using the same questions, Ma's interviews produced portraits of teacher knowledge that were distinctly different from those revealed in many of the previous studies.

Studies of secondary teachers revealed, to some people's surprise, that these teachers' knowledge was much thinner and less robust than most assumed, despite the fact that they—unlike their elementary counterparts—usually major in mathematics. Indeed, in studies at several major universities, mathematics majors stumbled over similar concepts and had difficulty justifying similar procedures and solutions as did elementary teachers (e.g., Ball, 1988, 1990a; Ball & Wilson, 1990; Steinberg, Haymore, & Marks, 1985). Studies comparing the mathematical knowledge of prospective elementary and secondary

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teachers show that secondary teachers' conceptual knowledge of elementary mathematics is not significantly stronger than that of their elementary counterparts.

In particular, Ball (1991b) compared the mathematical knowledge of preservice elementary education majors and preservice secondary mathematics education majors using the multiplication problem noted above. Although the mathematics majors were more successful at obtaining correct answers than the elementary majors, they were not adept at explaining the reasons behind the rules they invoked, and their knowledge was not connected across various contexts. Thus, Ball concluded that although the secondary preservice teachers who were mathematics majors had successfully completed a number of advanced mathematics courses, this academic preparation did not provide them with "the opportunity to revisit or extend their understandings of arithmetic, algebra, or geometry, the subjects they will be teaching" (p. 24). This finding lends further credence to the claim noted earlier that simply requiring teachers to take more mathematics classes without paying attention to the specific content of those classes is unwarranted.

DIVISION

Research on teachers' knowledge of division has revealed that teachers use a predominantly partitive (sharing) conception of division (Ball, 1990a, 1990b; Graeber, Tirosh, & Glover, 1989; Simon, 1993; Tirosh & Graeber, 1990). Because those teachers did not tend to use the quotitive (measurement) interpretation of division, they often were not able to reason through problems involving division by zero, division of fractions and decimals, or dividing a smaller number by a larger number. For example, Simon (1993) posed the following problems to 33 prospective elementary teachers:

> Write three different story problems that would be solved by dividing 51 by 4 and for which the answers would be, respectively,
> (a) 12 ¾ (b) 13 (c)12
> You should have three realistic problems. (p. 239)

> Write a story problem for which 3/4 divided by 1/4 would represent the operation used to solve the problem. (p. 240)

In the first task, 74% of the story problems generated by the prospective teachers reflected a partitive interpretation of division, and 17% of the problems reflected a quotitive interpretation. The teachers found part (b) of the task to be the most difficult, and only those who produced quotitive problems were successful in obtaining a realistic problem for which the answer would be 13. For the fraction division task, 73% of the prospective teachers gave inappropriate problems. The most common incorrect problem reflected a multiplication situation. When asked to generate a word problem involving division by a fraction, teachers across many studies provided a problem that represented a multiplication situation instead (Ball, 1990a; Ball & Wilson, 1990; Borko, Eisenhart, Brown, Underhill, Jones, & Agard, 1993; Ma, 1999). Researchers hypothesize that a teacher's tendency to rely on partitive models for division may lead to conceptual difficulty when using fractional divisors. (For a thorough explanation of the meaning of division, particularly in the context of fractions, see Ball, 1990a.)

Few teachers (at any level) were able to convincingly explain why division by zero is undefined. In some cases, teachers were not aware that division by zero presented a significant issue. For example, Ball (1988, 1990a) found that 12 of 19 teacher candidates (elementary and secondary) relied on a memorized rule to answer and to explain the calculation 7 divided by zero. Of those 12 responses, five invoked the wrong rule, saying that 7 divided by zero was zero. Two teacher candidates could not remember a rule at all and were unable to provide an answer to the question.

Simon (1993) found that not one of the 33 prospective elementary teachers in his study was able to explain why the steps of the long division algorithm result in a correct solution. Their explanations focused on the computational aspects of the algorithm (e.g., checking to see if the number remaining is smaller than the divisor) and made no references to the iterative process of making groups of a particular size and removing a certain number of those groups.

In another study, Tirosh and Graeber (1990) found that elementary preservice teachers incorrectly applied their procedural knowledge of division of whole numbers to situations involving division of decimals. For example, the majority of participants thought that division resulted in a quotient that is smaller than the dividend, even when dividing by a decimal between zero and one. When the participants carried out the correct computational procedure and arrived at a quotient larger than the dividend, some were still inclined to rely on their previous experiences with whole numbers and to dismiss the results of the algorithm, perhaps because they lacked an understanding of the equivalence-preserving effect of moving the decimal point in the divisor and the dividend.

RATIONAL NUMBERS

Studies that focus on teachers' knowledge of rational numbers suggest that teachers are considerably less confident and less successful in the area of rational numbers than they are in the domain of whole numbers. Post et al. (1991) administered to 218 practicing middle school teachers a written survey containing both content items and pedagogical items related to rational numbers. Follow-up interviews were conducted with 15 teachers. Examples of content items included the following:

> What number is one-third of [the] way from 1/2 and 7/8? (p. 188)

> What will happen to the value of the fraction a/b if "a" is increased four times and "b" is halved? (p. 188)

> Given 9 circles of equal size, which represent 3/4 of [the total], determine [the number] of circles in 2/3 of [the] same unit. (p. 189)

> Marissa bought 0.46 of a pound of wheat flour for which she paid $0.83. How many pounds of flour could she buy for one dollar? (p. 193)

Participants were asked to solve approximately 60 of these

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types of problems on the written survey. For six items, they were asked to provide thorough explanations of their thought processes and solution procedures. Some of the items also asked for an explanation of how the information could be taught to children. The interviews probed the participants' responses on both the written survey and the pedagogical items.

Post et al. (1991) found that 10–25% of the teachers struggled with basic computation with fractions. Approximately half of the teachers were unable to find a decimal from two given decimals, to find a number one-third of the way from two given fractions, to solve a simple proportion, and to solve word problems given on the National Assessment of Educational Progress tests for 13-year-old and 17-year-old students. The overall mean for the written test was between 60% and 70%. A significant number of teachers (20–30%) scored below 50% on the written instrument.

The follow-up interviews showed that even when the teachers were able to provide computationally sound solutions to problems, they were unable to provide pedagogically sound explanations for their solutions. Although the authors did not provide a detailed explanation of what they considered a pedagogically adequate explanation, they did delineate the types of responses the teachers gave in general terms. For example, in the proportional reasoning problem given above, only 10.5% of the teachers gave an adequate explanation. Many gave partial or sketchy explanations that were on the correct path, while others gave explanations that were garbled, confusing, or difficult to follow. Still others gave vague explanations or gave no explanation at all. Approximately 30% of the teachers gave only procedural explanations, and 28% gave explanations that made partial use of both procedural and conceptual explanations. Two-thirds of the teachers did not use diagrams in their explanations, and only 5% used estimation to check their answers.

Other studies showed that teachers tended to overgeneralize their knowledge of whole numbers when working in the domain of rational numbers (Tirosh, Fischbein, Graeber, & Wilson, 1999), which led to misconceptions and impoverished ideas about rational numbers. For example, Leinhardt and Smith (1985) observed classroom teachers who thought that multiplication always resulted in a product that is larger than either of the two factors (true for whole numbers but false for rational numbers). Those teachers also failed to recognize the use of the identity element for multiplication when creating equivalent fractions. Thus, in teaching their students to make equivalent fractions or to simplify fractions, they told their students that the resulting fraction was actually larger (in the case of equivalent fractions) or smaller (in the case of simplified fractions) than the original fraction.

FUNCTIONS

Functions are a fundamental topic in the secondary mathematics curriculum and are one more mathematical area that has received considerable focus in this approach to research on teachers' knowledge. Even (1993) used questionnaires and interviews to assess 162 prospective secondary teachers' understanding of functions. The prospective teachers came from eight universities and were in the final stage of their preparation programs. Those teachers thought that functions should be represented by algebraic equations and that the graphs of functions should be smooth and continuous. Although most of the participants knew that the vertical line test could be used to determine if a particular graph represents a function, most also thought that all functions must be one-to-one and onto. Few teachers seemed to understand that functions need not satisfy these properties, nor were they able to explain why it is necessary to be able to distinguish between functions and relations.

GEOMETRY AND MEASUREMENT

Several studies have used the van Hiele levels to characterize teachers' knowledge of geometry. For example, Mayberry (1983) found that more than 70% of the preservice elementary teachers in her study performed at either the recognition or the analysis van Hiele levels before instruction. Similarly, Swafford, Jones, and Thornton (1997) found that 80% of the practicing middle school teachers in their study were functioning at or below the van Hiele level of informal deduction.

A particular area of geometry and measurement that has been investigated is teachers' understanding of the relationship between area and perimeter. Such studies suggest that teachers frequently assume that there is a direct relationship between area and perimeter so that as perimeter increases, area increases also (Ball, 1988; Ball & Wilson, 1990; Baturo & Nason, 1996; Heaton, 1992; Ma, 1999). Further, teachers often do not use appropriate units when computing area and perimeter, commonly failing to use square units when reporting measures of area (Baturo & Nason, 1996; Heaton, 1992; Simon & Blume, 1994).

PROOF

A few studies have investigated teachers' knowledge of proof. The evidence from those studies suggests that teachers are prone to accept inductive evidence, such as a series of empirical examples or a pattern, as being sufficient to establish the validity of a claim (Ball & Wilson, 1990; Ma, 1999; Martin & Harel, 1989; Simon & Blume, 1996). Several researchers (Ball, 1988; Ma, 1999; National Center for Research on Teacher Learning, 1993) presented teachers with the claim that, as the perimeter of a rectangle increases, the area increases. The claim was framed as a conjecture that was developed by a student and was accompanied by drawings of two rectangles that fit the conjecture—one 4 cm long and 4 cm wide and another 8 cm long and 4 cm wide. Teachers were asked how they would respond to the student.

In Ma's (1999) analysis of this item, among 23 U.S. teachers and 72 Chinese teachers, 9% of the U.S. teachers and 8% of the Chinese teachers accepted the claim on the basis of those examples with no further verification. Another 22% of the U.S. teachers said that they would need to look in a book to determine if the claim was true and placed no burden on themselves to investigate further. More than half (56%) of the U.S. teachers wanted to see more examples to investigate the claim. Although they recognized that one example was not sufficient to establish the validity of the claim, they did not seem to realize that a

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finite number of examples would also be insufficient. Among the Chinese teachers who investigated the claim, either by using examples or by using the formulas for area and perimeter, 22% erroneously concluded that the student's conjecture was correct. Thirteen percent of the U.S. teachers and 69% of the Chinese teachers were able to explore the problem and produce counterexamples to show that the conjecture was false.

Many teachers seemed to have little propensity to search for alternative solutions or to verify their solutions using alternative methods (Baturo & Nason, 1996; Ma, 1999). However, Ma (1999) found that the Chinese teachers tended to provide multiple solutions to problems, often searching for more efficient and elegant solutions without external prompting.

Simon and Blume (1996) studied a group of 26 preservice elementary teachers in a mathematics content course and documented the process by which the group developed as a mathematical community. They noted tendencies on the part of the preservice teachers that were similar to those described for the groups studied by Ma. Namely, the preservice teachers were not initially inclined to question the validity of explanations offered by classmates, and they tended to accept inductive evidence as sufficient justification. Simon and Blume paint a vivid picture of the challenges and opportunities of trying to engage this group of students in mathematical arguments. In particular, they noted that each participant's knowledge of the mathematics at hand affects his or her interpretation of peers' use of justification. The preservice teachers' mathematical knowledge limited their ability to make sense of peers' arguments.

Revising Conceptions of Teacher Knowledge: Pedagogical Content Knowledge

This rapid growth of research on teacher knowledge yielded more than mountains of data on teachers' understanding of particular topics. One of the most significant contributions within this closer focus on teacher knowledge has been a powerful and convincing new conception of subject matter knowledge for teachers called "pedagogical content knowledge" (Grossman, 1990; Shulman, 1986, 1987; Wilson, Shulman, & Richert, 1987). With the introduction of *pedagogical content knowledge* to the lexicon of research on teaching and teacher education, Lee Shulman and his colleagues called attention to a special kind of teacher knowledge that linked content and pedagogy. In addition to general pedagogical knowledge and knowledge of the content, argued Shulman and his colleagues, teachers need to know things like what topics children find interesting or difficult and what the representations are that are most useful for teaching a specific content idea. Pedagogical content knowledge is a unique kind of knowledge that intertwines content with aspects of teaching and learning.

The introduction of the notion of pedagogical content knowledge brought to the fore questions about the content and nature of teachers' subject matter understanding in ways that the previous focus on teachers' course taking had not. It also led to the crucial insight that even expert personal knowledge of mathematics often could be surprisingly inadequate for teaching. Knowing mathematics in and for teaching requires one to transcend the tacit understanding that characterizes much personal knowledge (Polanyi, 1958). It also requires a unique understanding that intertwines aspects of teaching and learning with content.

Consider our earlier example on multiplication of decimals. The teacher had to know more than how to multiply decimals correctly herself. She had to understand why the algorithm for multiplying decimals works and what might be confusing about it for students. She had to understand multiplication as iterated addition and as area, and she had to know representations for multiplication. She had to be familiar with base-ten blocks and to know how to use them to make such ideas more visible to her students. Place value and the meaning of the places in a number were at play here as well. She needed to see the connections between multiplication of whole numbers and multiplication of decimals in ways that enabled her to help her students make this extension. She also needed to recognize where the children's knowledge of multiplication of whole numbers might interfere with or obscure important aspects of multiplication of decimals. And she needed to clearly understand and articulate why the rule for placing the decimal point in the answer—that one counts the number of decimal places in the numbers being multiplied and counts over that number of places from the right—works. In addition, she needed an understanding of linear and area measurement and of how they could be used to model multiplication.[12] She even needed to anticipate that a fourth-grade student might ask why one does not do this magic when adding or subtracting decimals and to have in mind what she might say. All this represents an intertwining of content and pedagogy, or pedagogical content knowledge.

The kind of knowledge and sensibility described above highlights quintessential elements of pedagogical content knowledge. Such knowledge is not something a mathematician would have by virtue of having studied advanced mathematics. Neither would it be part of a high school social studies teacher's knowledge by virtue of having teaching experience. Rather, it is knowledge special to the teaching of mathematics. Pedagogical content knowledge—representations of particular topics and of how students tend to interpret and use them, for example, or ideas or procedures with which students often have difficulty—describes a unique subject-specific body of pedagogical knowledge that highlights the close interweaving of subject matter and pedagogy in teaching. Bundles of such knowledge are built up over time by teachers as they teach the same topics to children of certain ages and by researchers as they investigate the teaching and learning of specific mathematical ideas.

Liping Ma's (1999) work contributes to the notion of pedagogical content knowledge by describing something she calls "profound understanding of fundamental mathematics." Ma describes the "knowledge packages" that are part of the knowledge of the 72 Chinese elementary teachers whom she inter-

[12] This point is not a trivial one. See Simon and Blume (1994) for a detailed explanation of the conceptual complexity underlying the area model for multiplication.

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viewed. Those knowledge packages constitute a refined sense of the organization and development of a set of related ideas in an arithmetic domain. The Chinese teachers whom Ma studied clearly articulated ideas about "the longitudinal process of opening up and cultivating such a field in students' minds" (Ma, 1999, p. 114). Their knowledge packages consisted of (a) key ideas that "weigh more" than other ideas in the package, (b) sequences for developing the ideas, and (c) "concept knots" that link crucially related ideas. Ma's notion of knowledge packages represents a particularly generative form of and structure for pedagogical content knowledge. Key in her notion of mathematical knowledge for teaching is a kind of culturally situated and curricular structuring of the content that readies it for teaching by identifying central ideas and their connections, as well as the longitudinal trajectories along which ideas can be effectively developed.

This approach to the study of teacher knowledge takes several important steps toward solving the problem of mathematical knowledge for teaching. First, it fills in gaps left when the focus is on only teachers' credentials. This approach probes the intricacies of mathematical knowledge to explore and identify what there is to know inside the school curriculum, inside the ideas and procedures that teachers must help others learn.

Second, the idea of pedagogical content knowledge substantially improves our understanding of the knowledge required for teaching. The concept implies that not only must teachers know content deeply, know it conceptually, and know the connections among ideas, but also must know the representations for and the common student difficulties with particular ideas. That concept makes clear that knowledge of mathematics for teaching encompasses more than what is taught and learned in conventional mathematics courses. The concept of pedagogical content knowledge helps to explain Begle's (1979) and Monk's (1994) findings that courses in mathematics methods had more effect on student performance than did typical mathematics courses. Because the curriculum of methods courses often focuses on pedagogical content knowledge, taking such courses may offer teachers the opportunity to develop this kind of knowledge.

Third, this approach to teacher knowledge has produced a constellation of new methods for probing what teachers know and, in so doing, has yielded portraits of highly developed teacher knowledge such as that contributed by Ma (1999).

However, this approach also leaves gaps in efforts to solve the problem of mathematical knowledge for teaching. One kind of gap emerges from the limits of repeatedly documenting teachers' lack of knowledge—or even, in some cases, what they do know. Such portraits do not necessarily illuminate the knowledge that is critical to good practice. What they do reveal is the content and structures of teachers' mathematical knowledge, and they help to substantiate the idea that teachers' knowledge is different from mathematicians' knowledge in some significant ways.

The second gap centers on the remaining distance between studies of teacher knowledge and of teaching itself. To understand the mathematical work of teaching would require a closer look at practice, with an eye to the mathematical understanding that is needed to carry out the work.

We turn our attention now to those gaps and to a promising approach with which we can address them in the next, and final, section of this chapter. It is the recognition of those gaps that has pressed the field toward a new approach to solving the problem of mathematical knowledge for teaching, an approach built on and extending the previous two research approaches.

Redefining the Problem: From Teachers' Mathematics Knowledge to Knowledge of Mathematics in and for Teaching

Before moving ahead, we will recapitulate the two approaches that have been dominant in research on teachers' knowledge. First, looking at teachers and their characteristics, as well as asking questions about how their qualifications relate to student learning, leaves a gap. What knowledge of mathematics do they have? What is the nature of that knowledge? Without this finer probing of the nature of teachers' knowledge, efforts to solve the problem of the mathematical knowledge that could enable better teaching and learning will run aground. Little evidence can be found to support the contributions of mathematics course work or mathematical credentials to teacher effectiveness. And even when such evidence emerges in a few cases, it is not helpful in deciding what mathematics teachers might need to learn.

A second approach to the problem, then, engages researchers in closely analyzing the nature of teachers' mathematical knowledge. However, such an analysis is still conducted at some distance from practice. Findings about what teachers know, as well as how they think about mathematics teaching and learning or about particular mathematical topics, still describe teachers and teachers' knowledge more than they describe teaching. Studying teachers, although valuable, is incomplete because results deduce teaching's mathematical demands from teachers' accounts of what they think or would do. Without our knowing whether the teachers interviewed are actually effective, however, what they report remains in some significant ways unwarranted.

The problem of the mathematical knowledge needed in and for practice remains, therefore, unsolved in significant ways. Given all the work in this area, why do we make this claim? First, many studies reveal portraits of U.S. teachers—experienced and preservice, elementary and secondary—whose fundamental understandings are shallow. On the one hand, discussions of those findings sometimes become mired in criticism and despair about the current teaching force. Observers are tempted to think that the solution is to recruit "smarter" people into teaching rather than to appreciate the subtle and complex content knowledge required for teaching (and to acknowledge the fact that most adults—not just teachers—find these subject matter questions difficult to answer). On the other hand, taking seriously the significance of the findings yields a substantial challenge: How can teachers' mathematical knowledge be developed in ways that would more closely correspond to what Ma (1999) found among Chinese teachers or would more satisfactorily "fulfill" the questions that researchers are using?

Although those questions are important and the issues are serious, no one question completely penetrates the central problems of practice. Although teaching depends on knowledge,

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knowing is not synonymous with teaching. Approaches that focus on teachers rather than on teaching can foreclose opportunities to examine closely the mathematical territory and demands of the work.

Consider, for example, Lampert's research on teaching (e.g., Lampert, 1986, 1989, 1990, 1992). Her writing, not about the teacher but about practice, reveals vividly the mathematical reasoning involved in choosing and using particular representations, in managing a complex classroom discussion, and in designing a problem or in figuring out how to formulate a good question. Her work shows some of the mathematics needed in teaching as well as where and how those mathematics have to be put to use.

A key distinction is worth making: whether studies of knowledge are or are not studies of knowing *in* teaching. New subtleties are added to the problem of mathematics knowledge: namely, what matters ultimately is not only what courses teachers have taken or even what they know, but also whether and how teachers are able to *use* mathematical knowledge in the course of their work.

Knowing Mathematics in Teaching

We turn first to instances in which researchers have been doing more than talking with teachers about mathematics or even about teaching and learning. Instead, researchers have been watching teachers who seek to use mathematical knowledge in the course of their work. What emerges from these observations is that being able to talk about mathematics is different from doing it.

Despite the fact that the work on teachers' knowledge has developed innovative means of probing teachers' knowledge of mathematics for teaching—through scenarios and situated examples—responding to such grounded situations is not fully equivalent to the on-line work of teaching. How it is different is only vaguely emergent, but the deeply cognitive orientation of research on teacher knowledge does not adequately take up the mathematical problem solving that teachers do as they engage in the activities of their work.

What becomes clearer across those studies is that studying what teachers know, or defining the knowledge that teachers should know, is insufficient to solving the problem of understanding the knowledge that is needed for teaching. What is missing with all the focus on teachers is a view of mathematical knowledge in the context of teaching. It is toward this aspect of the problem that the field is currently heading. A few examples help to illuminate this direction and to show what sort of new knowledge is needed.

Swafford, Jones, and Thornton (1997) report on 49 middle school teachers who completed a summer course in geometry where the work was situated in the practical situations in which they would use the mathematics. Those teachers showed significant gains in their geometry content knowledge as measured by a pretest and posttest. Among the teachers, 72% increased one van Hiele level, and more than 50% increased two van Hiele levels. Eight teachers who were selected for classroom observations and interviews during the following school year reported that they spent more time on geometry, were willing to try new ideas, were more likely to take risks to enhance student learning, and were more confident in their abilities to elicit and respond to higher levels of geometric thinking as a result of their experience in the geometry course.

Similarly, Sowder, Philipp, Armstrong, and Schappelle (1998) describe a 2-year teacher enhancement and research project with five middle school teachers in the areas of rational numbers, quantity, and proportional reasoning. Like Swafford and her colleagues, these researchers documented increases in the teachers' content knowledge and changes in their classroom practice. Sowder et al. report that as the teachers' content knowledge increased and deepened, the teachers were more willing to try new mathematics with their students, saw their students as more capable mathematically, encouraged and expected more conceptual explanations of material, became less dependent on prescribed curricula, and tended to probe students' thinking more often. This result suggests that there may have been something that these teachers were able to do in their work with students that was shaped by their new knowledge or by the experiences that they had in the project. In addition to the teachers' changed attitudes toward the content and toward their students, growth in students' knowledge of these topics was documented with pretest and posttest data. The evidence from student data suggested that students' knowledge of fractions did increase. The authors suggest that this increase is linked to the teachers' increased content knowledge.

Getting still closer to teaching practice, studies of the Cognitively Guided Instruction program (Carpenter, Fennema, Peterson, & Carey, 1988) have encouragingly shown that teachers who know problem types for addition and subtraction, the relationship between the two operations, and common strategies used by students in solving such problems are successful in engaging students in rich discourse about mathematics and in facilitating students' constructions of multiple solutions to problems (Fennema & Franke, 1992). Two things seem important here. One has to do with what mathematical knowledge matters (classification of problem types within addition and subtraction) and the other has to do, mostly implicitly, with uses to which teachers must put such knowledge (managing productive discussions in class and helping students develop multiple solutions to problems).

Making use of mathematical knowledge in the practice of teaching is challenging. Some work has probed deeply enough to begin to illuminate that challenge. Thompson and Thompson (1994), for example, highlight the crucial role played by language. They vividly describe the situation of one teacher who, although he understood the concept of rate himself, was restricted in his capacity to express or discuss the ideas in everyday language. He was satisfied with computational language for his own purposes, but when this language did not help students understand, he was not able to find other means of expressing key ideas.

Another clear example is cogently articulated by Schram, Wilcox, Lanier, and Lappan (1988). Reporting some success in helping preservice teachers expand their conceptual understanding during the mathematics content courses, those researchers found that the preservice teachers developed a deeper understanding of many of the facts, formulas, and rules that

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they had previously memorized. This deeper understanding resulted from taking a course that emphasized problem solving, reasoning, discourse, group work, and the use of multiple representations. However, when researchers followed the teachers into their first year of teaching, they found that many of the preservice teachers did not attempt to work with such ideas in their own practice. The experience of the carefully developed mathematical courses in which the teachers had participated did not automatically equip them with the understanding or skill to do similar work with children. The researchers (Wilcox, Lanier, Schram, & Lappan, 1992) were led to conclude the following:

> Disciplinary study is necessary to develop in novice teachers a set of intellectual tools and a disposition to engage in mathematical inquiry themselves. But disciplinary study alone may be insufficient . . . to develop in beginning teachers the knowledge, skills, and beliefs to conceive of teaching as something other than telling or as more than a matter of technical competence. (p. 23)

This example lends further credence to the argument that a number of different types of knowledge interact when a teacher makes decisions. Although the teachers possessed some level of desirable substantive knowledge of mathematics, they lacked adequate knowledge of mathematics as a discipline, as pedagogical content knowledge or as both to enable them to teach mathematics in ways consistent with current reform efforts. Clearly, knowing mathematics for oneself is not the same as knowing how to teach it.

In one of the most astonishing—but not anomalous—cases depicting the challenges of using mathematical knowledge in teaching, Eisenhart, Borko, Underhill, Brown, Jones, and Agard (1993) describe the case of a middle school student teacher, Ms. Daniels, who was asked by a child to explain why the invert-and-multiply algorithm works. Ms. Daniels tried to create a word problem for three-fourths divided by one-half by saying that three quarters of a wall was unpainted. However, there was only enough paint to cover one-half of the unpainted area. As she drew a rectangle to represent the wall and began to illustrate the problem, she realized that something was not right. She aborted the problem and her explanation in favor of telling the children to "just use our rule for right now" (p. 198).

Despite having taken two years of calculus, a course in proof, a course in modern algebra, and four computer science courses, Ms. Daniels was unable to provide a correct representation for division of fractions. She, in fact, provided a representation—orally and pictorially—of multiplication, rather than the division, of fractions. What is important to note here and what distressed the research team is that Ms. Daniels had had opportunities to develop conceptual understanding of division of fractions, including different interpretations of division (quotitive, partitive), as well as representations of the concept. The researchers had, in fact, documented Ms. Daniels' learning about division of fractions and had interviewed her about the topic. That she had such difficulty making sense of division of fractions when she encountered it in practice was a sign of the need to redefine the problem of teacher knowledge. Researchers needed to turn from looking at teachers and their knowledge to looking more closely at teaching and to probing the mathematical entailments of practice (Ball, 1999).

In yet another case, Heaton (2000) studied her own mathematics teaching practice as she worked to change her approach. Her work offers insight into the challenges of knowing and using mathematics in context. For example, in a lesson involving composition of functions, Heaton asked her students to identify patterns they saw in tables as she hoped to help them articulate a generalization about the composition of two functions. When her students began identifying random, interesting occurrences rather than meaningful patterns, Heaton realized that she was unsure how to decide what constituted a mathematically relevant pattern and that she did not know how to use the idea of composition of functions with this problem. In reflecting on the lesson, Heaton came to realize that because she did not really understand how composition of functions was relevant to the rest of the elementary mathematics curriculum, she had no purpose or direction for her lesson or for the questions she asked during her lesson. Thus, she was unable to help children make sense of each other's ideas or to ask questions that would lead to productive discourse. Heaton's story offers a clue: knowing mathematics in practice is not simple; yet, knowing mathematics is what ultimately matters.

These examples reveal the additional—and central—problem of mathematical knowledge in use. They suggest a set of important questions for research that would enable a closer connection of knowledge and practice, the very connection that prompted work on this problem in the first place. A recent finding provides a glimpse of the important distinction between knowing how to do mathematics and knowing mathematics in ways that enable its use in practice.

In their analysis of data from the National Education Longitudinal Study of 1988, Rowan, Chiang, and Miller (1997) report strong positive correlations between teachers' responses to items that are designed to measure the use of mathematical knowledge in teaching and their students' performance.[13] Although teachers' mathematical knowledge has rarely shown any significant effects on student performance, this analysis provides some confirmation that understanding the *use* of mathematics in the work of teaching is a critical area ripe for further examination. It is not only what mathematics teachers know but also how they know it and what they are able to mobilize mathematically in the course of teaching (Cohen & Ball, in press). Although less easily quantified than other indices such as the type and number of courses taken, it is this pedagogically functional mathematical knowledge that seems to be central to effective teaching.

Ultimately, teachers must be able to know and use mathematics in practice, not merely do well in courses or answer pedagogically contextualized questions in interviews. This conclusion

[13] These items were developed at the National Center for Research on Teacher Learning, Michigan State University. See Kennedy, Ball, and McDiarmid (1993).

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suggests the need to redefine the problem from one about *teachers* and *what teachers know* to one about *teaching and what it takes to teach.* What mathematical knowledge is actually entailed in teaching? How is it used?

What Is Mathematically Entailed by Teaching the Multiplication of Decimals?

To begin to unpack this question of practice and its mathematical demands, we return once more to our example of teaching the multiplication of decimals. We note the mathematical depth of the task and its surrounding territory of investigation. What mathematical demands are created by the enactment of this lesson? Underlying this basic procedure—the multiplication of decimals—are several critical mathematical notions: the importance of the idea of the units (often tacitly understood) that numbers carry, the transitions between linear and area representations of number, and the difference and correspondences between the area and iterated addition models for multiplication.

One demand of teaching this topic is to be able to handle the predictable questions. Why *does* the rule for placing the decimal point in the answer—that one counts the number of decimal places in the numbers being multiplied and then counts over that number of places from the right—work? Why does one not do this "magic" when adding or subtracting decimals?

Another mathematical entailment of teaching this topic is to be sure-footed in its complex substantive and representational territory. Whereas discussions of pedagogical content knowledge have made representations a central component, this small example illuminates the mathematical work of using representations in particular contexts. Using and managing the mathematical correspondences and pedagogical challenges of these geometric representations of multiplication, and of place value, requires considerable mathematical depth and flexibility. As our teacher contemplated the area representations that her students were constructing, she noticed that the rectangles were constructed with seven hundredths along the top and three hundredths along the side, and she had to think, "How did this arrangement correspond to the problem .3 × .7?" (see Figure 24.6).

When she saw that this same representation could be used to model 3 × .07 or 7 × .03, she had to make a clear distinction between numbers interpreted as products of lengths in the .3 × .7 version and as iterations of area units in the 3 × .07 or 7 × .03 version. Looking at her students' construction using the rods that represented tenths (of area units), she had to be clear with herself that they were modeling *three* times seven-tenths, a different problem. Using the representation competently requires more than knowing how to show it oneself.

In addition, our teacher was aware that the tenths could be used to model .3 × .7 by superimposing the two rectangular arrays as pictured in Figure 24.7.

Although she did not use it in this instance, this representation was part of her repertoire. Because the problem asks about multiplication of tenths, a student might try to construct a representation starting with tenths. Understanding this representation helps the teacher be prepared to discriminate between this representation and the one depicted in Figure 24.2. Like the

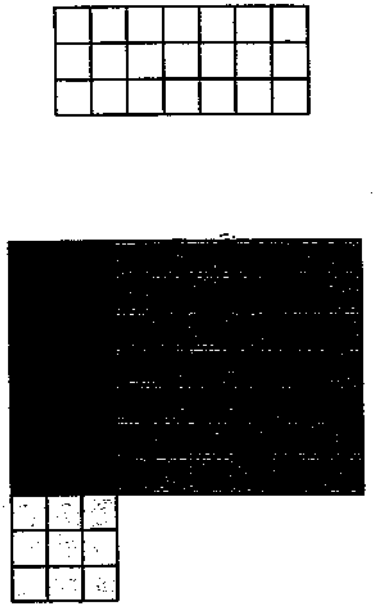


Figure 24.6. A typical arrangement of tiles to show .3 × .7.

basic area model represented in Figure 24.2, this figure shows a rectangle with sides having lengths of .7 and .3 (see dark shaded region in Figure 24.7). However, unlike Figure 24.2, which is constructed directly out of hundredths, Figure 24.7 starts with tenths, corresponding to the problem .7 × .3.

Crisscrossing the tenths affords a view of the interpretations "seven-tenths of three-tenths" and "three-tenths of seven-tenths." (Focusing on the three tenths, one can see the overlapped area as demarcating seven tenths of the three tenths. And focusing on the seven tenths, one can see the overlap as marking off three-tenths of the seven tenths.) This model (Figure 24.7), like the other one (Figure 24.2), affords a view of both the iterated addition interpretation (e.g., seven-tenths of three tenths), as well as the area model (e.g., constructing a rectangle with side lengths of .7 and .3).

Being able to manage the confusions into which students stumble, to keep one's mathematical bearings, to have a sense of the relationships between the multiplication of whole numbers and the multiplication of decimals, and to be skillful in putting the base-ten blocks into flexible use—each of these activities requires particular kinds of mathematical knowledge and ways of reasoning. Looking inside classroom lessons, plus developing mathematical topics over time, helps to unearth the mathematical entailments of practice.

Starting with Practice in Order to Uncover Knowledge

This redefinition of the problem of mathematical knowledge for teaching reverses the usual approaches. The other approaches begin with knowledge, or with teachers, to identify the knowledge needed to teach. Most approaches identify the mathematical knowledge that is needed for teaching by analyzing the curricula that teachers use or by interviewing teachers to learn

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what they know. After one has decided what teachers need to know, the problem has been to try to get teachers to know those things. When they do, another familiar problem arises: knowledge and practice are not seamlessly connected. On the one hand, teachers cannot always use what they know. On the other hand, what they know may not fully accommodate the demands of their work in context. For both of these reasons, researchers working in this third approach would begin instead with an examination of practice itself.

This approach would entail mathematical analyses of core activities of mathematics teaching. Those core activities include things such as figuring out what students know; choosing and managing representations of mathematical ideas; appraising, selecting, and modifying textbooks; deciding among alternative courses of action; and steering a productive discussion. Identifying the mathematical resources entailed by these teacher activities would be an important step in this approach.

For example, teachers design tasks. What does it take to discriminate between different possible examples of the multiplication of decimals? Consider, for example, the differences between assigning 0.2×0.5 versus 0.3×0.7 in our example. Similarly, what difference would it make for students' work if the teacher chose 2.3×3.7? Considering with precision the often subtle ramifications of different numerical parameters in a mathematical task is an example of crucial mathematical problem solving in the context of practice.

What mathematical insights does such work take? What are the mathematical ideas that it requires, and what is the nature of the reasoning entailed? Clearly, this work involves more than being able to multiply decimals oneself. Again, the focus in this definition of the problem is on teaching practice, its sites of work, and its demands, as well as on what mathematical knowledge it entails, and how it is used—not merely on what teachers do or do not know.

What mathematical work is entailed in other core activities of teaching, such as sizing up students' ideas and responding? Responding is also a complex—and not wholly generic—task. It requires being able to hear and interpret what the student is saying, and it includes being able to skillfully probe in cases where the student's idea is not clear. It may entail designing and posing a question, answering a query, or pointing out an incongruity or connection. That this is mathematical, not simply pedagogical, reasoning becomes clearer the more closely one looks. When a teacher decides to ask, "How do you know that those are hundredths?" or "Is that the only solution to this problem?" or "How would you convince someone else of what you are seeing?" then she is engaged in mathematical work, is considering the problems and ideas at hand, and is working to orchestrate more than her own thinking about the subject matter.

This approach to research on mathematical knowledge in and for teaching would also respond to the fact that teaching is a practice embedded with both regularities and endemic uncertainties. That it involves regularities gives rise to certain sorts of knowledge used to anticipate, prepare for, and manage the predictable. For example, some topics—such as arithmetic with integers, probability, and fractions—are quite often difficult for students. Certain ways of approaching these topics—particular representations and methods of development—can help mediate such difficulties. Often-used mathematical tasks can be mapped by the range of typical approaches used by students of a given age (Stigler & Hiebert, 1999). Being prepared for these regularities of practice is enabled by pedagogical content knowledge, which involves bundles of understanding that combine knowledge of mathematics, of students, and of pedagogy. Pedagogical content knowledge is a special form of knowledge that bundles mathematical knowledge with knowledge of learners, learning, and pedagogy. Those bundles offer a crucial resource for teaching mathematics because they can help the teacher to anticipate what students might have trouble learning and to have ready alternative models or explanations to mediate those difficulties.

One big challenge of teaching is to integrate many kinds of knowledge in the context of particular situations. That there are patterns in and predictability to what students might think, and that there are well-tried approaches to develop certain mathematical ideas, can help researchers manage this challenge. However, possessing a body of such bundled knowledge may not always equip the teacher with the flexibility needed to manage the complexity of practice. Teachers also need to puzzle about the mathematics in an unanticipated idea or formulation proposed by a student, to analyze a textbook presentation, to decide to change the numerical parameters of a problem, to make up homework exercises, and to consider the relative value of two different representations in the face of a particular mathematical issue. These recurrent activities of teaching entail a kind of mathematical understanding that is pedagogically useful and ready, not simply bundled in advance with other considerations of students or learning or pedagogy.

Looking closely at teaching brings to the fore the nature of the work, both in planning and reflecting, as well as in the moment-to-moment work of enactment in class. An endless barrage of situations—of what can be freshly understood as mathematical problems to be solved in practice—entails an ongoing use of mathematical knowledge that is not visible until probed at the level of practice. What it takes *mathematically* to manage these both routine and nonroutine activities of practice is the focus of this third approach to understanding the relationship between mathematical knowledge and teaching. It is the need to unearth this kind of *pedagogically useful mathematical understanding* that gives rise to the need for this new work on knowledge of mathematics for teaching. Such a mathematical perspective on the work of teaching can extend what we currently understand about teachers, about the knowledge that they have and need, and about the role of such knowledge in practice. It is a timely shift from teachers to teaching that will push the field ahead in solving this central but complex problem. By making practice the focal point, researchers can bring mathematics and teaching into closer connection in ways that would be able to contribute to the improvement of teacher education, policy, and practice.

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