Chapter 3: What Do Students Know? Facing Challenges of Distance, Context, and Desire in Trying to Hear Children[1]

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No task is more fundamental to teaching than figuring out what students are learning. Paradoxically, no endeavour is more difficult. In his well-known sociological analysis of teachers and teaching, Lortie (1975) found that questions about the assessment of student learning evoked significant emotional response from teachers. Although the teachers whom he interviewed believed that good teachers inform their practice by closely monitoring students, many despaired of really knowing about the effects of their teaching. Lortie nominates this as one of the endemic uncertainties of teaching, observing that

> ... [a]ll craftsmen must adjust and readjust their actions in line with hoped-for outcomes; must monitor their steps and make corrections as they proceed ... Yet the monitoring of teaching effectiveness, defined as achieving instructional goals, is fraught with complications. (p. 135)

Lortie analyses these complications from two perspectives. Viewed externally, he argues that assessing outcomes is complicated by the lack of shared standards, lack of clarity about cause or influence, multiple criteria and lack of consensus, ambiguity about timing of assessment, and student changeability. Comparing teaching with other occupations, Lortie claims that these complications do not exist in more 'tangible fields' (p. 136). Indeed, he argues, such complications are inherent to the practice of teaching. Lortie also views the complications of assessment from the inside. Interestingly, he reports a bimodal distribution in teachers' feelings about assessing their effectiveness. Over half reported that it was problematic. Yet another third claimed that knowing whether one was doing a good job is relatively easy (p. 142). This divergence suggests that teachers live with the uncertainties differently.

However uncertain it may be, figuring out what students know is central to the everyday work of teachers.[3] Teachers ask themselves many questions in the course of teaching: What should I do next? Is what I am doing working? Does this concept need to be reviewed? Are the girls feeling disconnected from this unit we are doing? What is that student saying? What is he thinking? Teachers are responsible for helping students learn. Minute to minute, day to day, and

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week to week, teachers must constantly 'read' their students, making judgments about how things are going – in general and for particular individuals. They give tests, quizzes, and assignments. They ask questions, observe, listen. Yet what teachers watch, ask, and listen for varies. This variation may underlie differences in how unsettling teachers find the task of assessing what students are learning.

One way of coping with teaching's 'intangibility,' including the uncertainty embedded in assessing outcomes, is to use approaches that evade or mask such complications. One oft-heard shorthand confounds classroom activities with learning. Teachers report what students have 'had,' or what has 'been covered.' Still, most people know that a description of what has been taught is not equivalent to an examination of what has been learned. A consequent desire for accurate 'measures' of learning can be seen in the widespread efforts to make systematic a technology of monitoring students.

Standardized multiple-choice tests, weekly quizzes checked and recorded, progress charts, carefully convergent questioning – all are measures that check students in a manner that controls both variation and sources of ambiguity. Pimm (1987) describes a discourse pattern typical of many mathematics classrooms in which the teacher talks but leaves small openings for students to fill in key blanks:

T: Supposing you started with that shape and you increased its length by two, what would have happened to the area. It would simply have ...
P: Doubled.
T: Doubled – but that's only moving in, increasing in one dimension – if you've got to increase the width by two as well – then you've got to double it again. So you would be doing – first of all, times two to double the length – and then times two again – to double the width again, so altogether you must have multiplied ...
P: By four.
T: By four, by two squared. (Pimm, 1987, p. 53)

In this exchange, the pupil does little more than play a part in the teacher's script. This typical pattern 'allows the teacher to maintain control of the discourse, while focusing attention on particular items along the way. It ... acts as a check for the teacher that the particular pupil questioned has grasped what is being explained' (Pimm, 1987, pp. 53-54).

Enacting such approaches, teachers can feel they know something about what students know. Still, what is being examined about students' learning is constrained – evidence often restricted to short, unprobed, and standardized answers (Cohen, 1995). Much is left unknown that teachers, parents, and the public care about – for instance, the ability to reason, to be imaginative, to construct

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a convincing argument. And under the cover of simpler measures lie the tangled – sometimes wonderful, sometimes terrible – idiosyncrasies that are the products of children's sensemaking.

In a famous case study of a sixth grader named Benny, Erlwanger (1975) probed beneath the surface. Despite the fact that his teacher thought of him as advanced, the understandings lurking under Benny's consistently correct written work were wildly nonstandard. But in the individualized approach to teaching used in this classroom, it was possible for him to continue to progress officially, his personal constructions invisible to his unsuspecting teacher. Dramatic as this case is, it is likely not unusual. When all we ask students is to fill in answers in constrained spaces, we block our access to views of students' thinking.

A different way of coping with the ambiguities of assessment is to see it as ongoing, drawing on multiple and varied sources of evidence. To make the examination of learning more complex is to admit of its subtleties and requires broadening both the objects and methods of assessment. Students are presented with open-ended tasks that have no single constrained 'right answer' and which invite application, imagination, and extension. Such assessments seem more satisfyingly 'authentic,' to have more relevance to the enterprises for which teachers strive to prepare their students (Darling-Hammond, Ancess, & Falk, 1995). At the same time, the open-endedness increases the uncertainty of interpreting and appraising student progress and makes it more difficult to share common standards across students, groups, and settings (Cohen, 1995). Moreover, the context of the task, necessarily constructed by each individual student, make it that much more complex to ascertain what students know. How is the context affecting the student's response in ways that are idiosyncratic to the particulars of the situation? Issues like these challenge core epistemological and psychological assumptions about what it means to 'know' something. Although some researchers claim that students who engage in close reasoning about performance-oriented problems have deep 'insights into the structure of the subject matter' (Hiebert et al. 1996, p. 17), others express reservations about the assessment of abstract ideas which are usable across settings (Anderson, Reder, & Simon, 1996, p. 9). Cohen (1995, p. 9) synthesizes the issues:

> [W]e have little evidence about how well the newer assessments represent students' 'understanding' of mathematics or reading, in part because this assessment technology is very young. It seems reasonable to think that performance assessments could do a better job than norm-referenced standardized tests, but researchers are only now turning their attention to what understanding might be, how it might be assessed, and what satisfactory assessment of understanding might be.

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Despite these uncertainties, enthusiasm is high for focusing on what students *understand* and *can do* through ongoing and performance assessment (Darling-Hammond et al., 1995). Widely eschewed among reform advocates are the traditional measures that constrain the problems of 'measuring' what students know. But this current press for 'authentic assessment' moves central ambiguities of practice to the public stage of school accountability. The new assessments reveal the uncertainties of interpreting what students know and have learned and yet situate such tenuous information in the political context of schooling.

In this chapter, I return to and extend Lortie's (1975) examination of the sources of endemic uncertainty resident in assessment. Writing from the perspective of practice,[4] I investigate challenges inherent in the central task of figuring out what students are thinking and learning. I explore the problem from three angles:

- Teacher knowledge: First, I review briefly recent work on teacher knowledge. What have researchers identified as key elements of teaching knowledge and how do these play a role in hearing and assessing students' thinking and learning? What kind of knowing *is* knowing students, and how well do we as teachers, researchers, teacher educators, and parents understand the nature of what we know (or think we know) about students?
- Challenges embedded in practice: Second, I analyse conceptually what makes the task of assessment so difficult in practice. From the perspective of the teacher, what persistent challenges influence efforts to figure out what students are learning?
- Learning to know about students: In the final section of the chapter, I turn to an examination of what is involved in preparing teachers to hear their students. What do teachers need in order to be prepared to figure out what students know, and what are some ways in which they can develop and enhance such capability?

TEACHER KNOWLEDGE: BEING BOTH RESPONSIBLE TO SUBJECT MATTER AND RESPONSIVE TO STUDENTS

I begin with a brief look at recent developments in research on teaching and teacher thinking where issues of what teachers need to know have occupied center stage. In particular, over the past decade the role of subject matter knowledge has claimed considerable attention among researchers and teacher educators. In a seminal presidential address to the American Educational Research Association, Lee Shulman (1986) referred to subject matter understanding as

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'the missing paradigm' in research on teaching and teacher education. With a sweep of convincing argument, he attracted renewed interest in the nature and role of teachers' subject matter knowledge[5] – and in a new kind of knowledge, *pedagogical content knowledge*. Not only do teachers need personal understanding of the material they *were* teaching, Shulman and his colleagues argued, but they also need to know ways in which key ideas might be compellingly, engagingly, and helpfully represented. Teachers need a repertoire of metaphors and analogies, problems and tasks, pictures and diagrams. They need to be aware of topics with which students often had difficulty, and of common misunderstandings (Shulman, 1986; Wilson, Shulman, & Richert, 1987). In short, they need ways to see into the subject matter through the eyes, hearts, and minds of learners. And their task is to 'transform' the content in ways that make it accessible to students (Wilson et al., 1987) while maintaining its integrity (Ball, 1993b; Bruner, 1960; Lampert, 1992).

Responsibility to the subject matter is only one part of the equation. In making subject matter 'accessible,' teachers also need to know students. Teachers need understandings of students in general – patterns common to particular ages, culture, social class, geography, and gender; patterns in typical student conceptions of the subject matter. But, more to the point, teachers must know *their* students (Peterson, Carpenter, & Fennema, 1989). Face to face with actual children who are particular ages and gender, culture and class, teachers must see individuals against the backdrop of sociological and psychological generalizations about groups. Often to simplify the enormous complexity of teaching, teachers may presume a 'shared identity,' mistakenly assuming that students share their teachers' experiences of school and of the world (Jackson, 1986). Teachers are themselves particulars in the social tapestry and different from their students. They are separated at least by generation and in multiple other ways as well. For example, women teachers will teach boys and middle class teachers poor children, while non-religious teachers teach devout Christians (Delpit, 1988; Paley, 1995). Yet teachers must build bridges across the chasms of difference. In order to make the subject matter accessible to students, the more they know about student experiences, backgrounds, understandings, and interests, the better equipped they will be, so the argument goes.

Much of the recent work on pedagogical content knowledge has advanced our understandings of what teachers need to know about students to inform 'the ways of representing and formulating the subject that make it comprehensible to others' (Shulman, 1986, p. 9). In 'transforming' the content for instruction, teachers examine critically the topic or text at hand and select a way of representing it, adapting particular representation in light of what they know in general about students (Wilson et al., 1987). Wilson and her colleagues describe the interplay between the teacher's understanding of the specific content and of the

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particular students, one step in a cyclical process they call *pedagogical reasoning*:

> Student characteristics that might influence the ways in which material is represented include student misconceptions or misunderstandings of the material; student ability, gender, and motivation. When preparing a lesson on light, science teachers might think about the preconception that students may have about light that will interfere with their learning. Math teachers preparing to teach fractions might consider students' prior knowledge of coins and how best to use that prior knowledge in developing a more sophisticated understanding of the material. *Tailoring refers to adapting the material to the specific students in one class rather than to the student population in general.* (Wilson et al., 1987, p. 120; emphasis added)

But planning and selecting is but one element of teaching well. In order to know if they are effective, and to make adjustments as they go, teachers must assess how students are making sense and what they are learning. This must happen in the course of instruction as well as at landmark points (e.g., the end of a unit). To do this, teachers must do more than *use* knowledge of students. They must, on an ongoing basis, *construct* new knowledge of their students. They must observe, listen, and make conjectures. They must test and revise these conjectural understandings, from moment to moment and week to week.

Seeking to engage students productively, teachers adjust, make mid-course corrections, follow students' ideas (Duckworth, 1987). Dewey (1902/1990) saw the construction of curriculum as inherently constructive and interactive. He urged:

> [A]bandon the notion of subject-matter as something fixed and ready-made in itself, outside the child's experience; cease thinking of the child's experience as also something hard and fast; see it as something fluent, embryonic, vital; and we realize that the child and the curriculum are simply two limits that define a single process. Just as two points define a straight line, so the present standpoint of the child and the facts and truths of studies define instruction. It is continuous reconstruction, moving from the child's present experience out into that represented by the organized bodies of truth that we call studies. (Dewey, 1902/1990, p. 11)

Paley (1986) captures elegantly the shift that this requires of the teacher, from one who presents knowledge to one who supports its construction. Describing

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her own initial efforts to teach in ways that drew on and connected with students, she reflects,

> When my intention was limited to announcing my own point of view, communication came to a halt. My voice drowned out the children's. However, when they said things that surprised me, exposing ideas I did not imagine they held, my excitement mounted ... I kept the children talking, savoring the uniqueness of responses so singularly different from mine. The rules of teaching had changed; I now wanted to hear answers I could not myself invent ... Indeed, the inventions tumbled out as if they had been simply waiting for me to stop talking and begin listening. (Paley, 1986, p. 125)

In sum, teaching in ways that are responsible to content and responsive to students (Ball & Wilson, 1996) requires teachers to work continually to know students. What they need to know cannot be fully specified in advance. Instead, across multiple intellectual, personal, and cultural divides, teachers must work to see and hear students flexibly in the moment and over time. As they work to help their students develop understandings of content, teachers must listen to students – making use of their own knowledge of the content while not being limited by it. A teacher who knows little of the content, or knows it in only narrow and rigid ways, may miss children's often wondrous insights. But, paradoxically, a teacher with considerable depth of knowledge may fail to hear the nonstandard perspective, the novel insight, listening only for 'the answer.' Teachers, argues Hawkins (1974/1972), must be able to notice when children's observations and questions take them 'near to mathematically sacred ground' (p. 113) – that is, to the edges of wonder or to core foundations. In much the same way, teachers must use theories about adolescents and children, males and females, learning and development, social class and culture, as resources, not determinants (Duckworth, 1987). Teachers must know students both in general and in the moment – their knowledge always changing in light of the current and the particular (Ramsey, 1991). In order to be both responsive and responsible, teachers need to live in multiple worlds, moving back and forth between the wider disciplinary culture with its conventions, norms, and theories, and the idea-filled worlds of their students. The teacher cannot belong entirely to any single world, 'but [must be] an ambassador from one to the other' (Lampert, 1992, p. 310).

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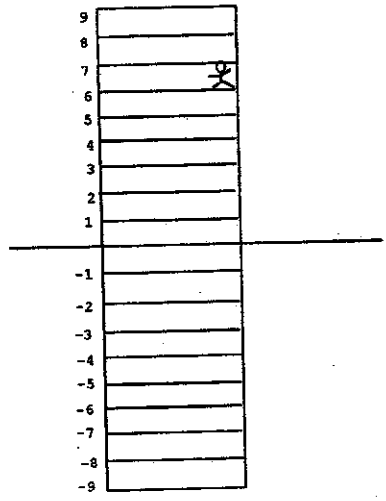
KNOWING STUDENTS

What does it take to know particular students, to construct, use, and revise knowledge of students? Much can of course be learned about students in general. But particular students – not the faceless students of general theory, or even the real and remembered students of previous classes – are the actual terrain of practice. It is *these* students, now, whom teachers must teach. Take the following example from my third grade mathematics class, early one October,[6] where I confronted a typically puzzling incident in which I did not know what a student was saying. It is important to understand that, as an experienced teacher of over 15 years, I did know a great deal about students of this age. I knew theories of child development. I knew what mathematical ideas third graders were likely to have encountered. I came into this episode with considerable knowledge of students. Still, what Mei said puzzled me, and knowing what she was thinking was critical in helping her – and probably, others – to make progress in the lesson. In the midst of a unit in which we had been exploring integers, we were struggling on this particular day to figure out a sensible answer to 6 + (-6). Two answers had been proposed: 6 and 0. Mei, a frequent contributor to class discussions, disagreed with both, because, as she explained, she had come up with the answer 9.

Betsy, a classmate, was incredulous. '*Nine*?' I, too, was puzzled. I asked her if she could show us. She went to the board and, using our paper model of a sky-

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scraper with floors below and above the ground, placed a paper person on the building, on the sixth floor above the ground.



Mei explained, 'And then I, and then you plus six below zero, and six below zero is below zero, so I add three to that three, because when you go three and three is six, so I went up three more, and one, two, three – there. Oops, went too far. Yeah, and then I got nine, so I think it's nine.'

I still had no idea what she meant. Lucy's hand was up and I called on her.

Lucy *(to Mei)*:	Where did the other three go then?
Mei:	Well, see, 'cause it's *below zero*, so I …
Betsy interrupted:	*I* know what you're saying, *I* know what you're saying …
Mei continued:	So we put two in each group in order to make one because it's below zero.
Betsy:	Mei, is *this* what you're saying? (places a paper person on the building). There. And then you're saying six below, and since it's below, that we have to go up to the three?
Mei:	I'm saying that since it's below that we have to take the, you take three and go up three, 'cause it's like *three*, because three plus three is six, and then we

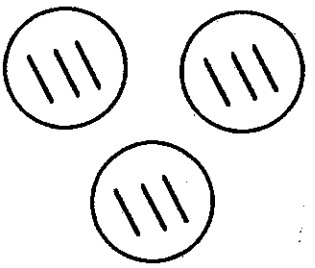
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put two in each group in order to make one, so I added another three more to the six above zero.

Still quite unsure of what Mei was thinking, I pushed. 'I don't quite understand this 'put one, put three in each group in order to make one.' I don't understand that part.'

Mei: Well, see, I said that if we starting on six and we'd plus six below zero, and you start on six, plus – *(Mei starts to make a drawing, Betsy tries to help)* like, three, I start on six above zero and three and three is, three and three is six, and since this is below zero so I say three, and so I say that think that we, since this is six, then if we add six to it, we get twelve above zero, and it's below zero and three and three plus six, well three and three equals six, so we add three more to the six above zero and then I got nine.

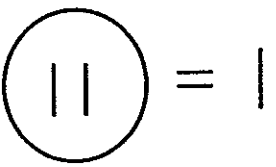
Betsy: Is this what you're doing *(draws three circles with three hash marks inside each one)* – making three groups of three?



Mei: Let me see …

Betsy: Is that what you're doing, or are you making four groups? *(makes another group)*

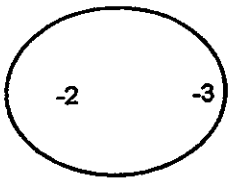
Mei giggled: No! I'm making a group of … This is one of the numbers, these are two of the numbers below zero, and if two of these equals one, and if I have about, like

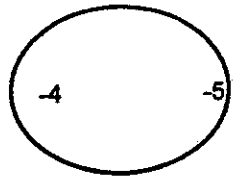


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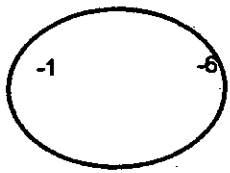
Betsy continued to press Mei. She erased the hash marks and asked Mei what the numbers were: 'Can you put the numbers in yet?' Mei paused, wrinkling her face. Trying to respond to Betsy's question, she put numbers in the circles:

> Okay, let's see, two below zero and three below zero, and this could be four below zero and five below zero, equals one.





> And then I have two already, and that means you will go up to eight. *(She moved the paper person from the 6 to the 8 on the building.)* And then I make one more.



> One below zero and six below zero, so there's one more then to go up, and now I end up on nine.

And she moved the paper person up to the 9 on the building.

What was Mei thinking? What was she doing with her circles and numbers? Despite everything I knew, I could not quite hear in what she was saying a glimmer of anything familiar. I simply could not see in what way (-2) + (-3), (-1) + (-6), and (-4) + (-5) could all equal 1. What was she thinking?

It was only later, in talking with others in my research group,[7] that it seemed most plausible that she had built an explanation on some vague idea that 'two negatives make a positive,' which her mother had told her. With 'six below zero,' Mei may have thought, each 'pair of the negatives' would combine to make a positive. Since we were starting with 6 (as in 6 + [-6]), then each of the pairs of negatives in the 'six below zero' would be added to the 6 above zero. Since there were three pairs of negatives, then there were three 'ones' to add to the six above zero, for a result of nine. Looking at her diagrams, this conjecture was plausible, but not entirely satisfying.

When teachers seek to construct the curriculum in response to students' ideas and understandings, then moment-to-moment appraisals of student thinking are

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critical. This is hard work, bounded by the uncertainty of ever knowing what anyone means, and multiplied by the complexity of classroom teaching in which teachers are responsible for groups of students, not just individual learners (Jackson, 1968; Lortie, 1975). In this case, subsequent reflection on Mei's confusion, and the mess that was created in class that day, led me to rethink what I was doing. No simple adjustment would work. The problem here was not an inherently mathematical one in some respects but a problem emerging from the representation – the building with its floors above and below ground (see Ball, 1993b, for a thorough discussion of my deliberations about representational contexts for integers). Later I could see that ours was a problem of convention, not discoverable somehow, especially not in any meaningful way with the elevator rides as we were using them (Lampert, 1990a). Unlike problems of invention, conventions are not readily discoverable, for they are decisions interior to and agreed upon by a community. We would have to have defined and agreed what a 'negative number ride' meant and then use it that way. But once having taken this path, and not yet fully appreciating our situation, I was stuck in trying to make sense of Mei's thinking.

Duckworth (1987), writing about the 'virtues of not knowing,' proposes that knowing the right answers is not what really counts 'in the long run.' She argues, 'What you do about what you don't know is, in the final analysis, what determines what you ultimately know' (p. 68). Although Duckworth is writing about children's learning, her insight applies equally to teaching. Not knowing what Mei was thinking, what I did to understand her is what mattered. I let her continue to try to explain her interpretation. Betsy, a classmate, worked along with her, doing what children will often do, engaging in their peers' ideas even when they did not agree. As mathematical explorers, they would try to get inside another's thinking, to see what it might mean. 'Is this what you're saying?' asked Betsy. And, 'Where did the other three go?' asked Lucy. I, too, probed. Together, we assumed that Mei's thinking made sense to her, and that our challenge was to figure out what that sense was.

In the face of the uncertainty, what can a teacher do to see, hear, and interpret as flexibly as possible, assisted but not limited by her own background, habits of mind, and understandings about both children and mathematics? How can a teacher construct understandings of students in the moment, understandings that are robust enough to be able to take the next step in practice? What is good enough evidence for such an interpretation? What problems arise when trying to understand children's sensemaking? In this case, it was not until after class, that I thought I had a possible interpretation of what Mei was thinking. I still was not sure this was right, but it also made me question the task I had pursued. We ultimately agreed to consider 'negative number rides' as rides down the elevator. Hence, 6 + (-6) was defined as meaning a person starts on the sixth floor and take a ride *down* six floors. Even so, I was left humbled by the challenges

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of knowing what students think, and by the elaborate processes they use to construct solutions and understandings, often quite complex to follow.

While working in an environment that presses for demonstrable outcomes, confronting the ambiguities of assessing what students know creates endemic challenges for teachers. In what follows, I explore three such challenges. First, I situate my work and explain my methods of analysis.

USING MY OWN PRACTICE AS A SITE FOR RESEARCH ON TEACHING

The analyses in this chapter draw on my own daily teaching of mathematics in a local third grade classroom between 1988 and 1992. (This is also the school where, prior to the arrangement I describe, I was a regular, full-time classroom teacher from 1975 on.) Over half of the students in this school are from other countries and speak limited English; the American students are linguistically, culturally, and racially diverse and come from many parts of the United States. Sylvia Rundquist, the teacher in whose classroom I was working, taught all the other subjects besides mathematics. She and I would meet regularly to discuss individual students, the class, what each of us was trying to do, and the connections and contrasts between what we were trying to do with the children. We also investigated mathematical ideas, analysed representations generated by the students or introduced by me, assessed the roles played by me and by the students in the class discussions, and examined the children's learning.

In situating a strand of research inside my own practice, I seek to contribute to our collective understanding of what 'teaching mathematics for understanding' entails. My focus is on identifying and examining closely its endemic challenges – that is, issues with which teachers like myself must grapple, and which are *built in* to the intersection of vision and practice. For example, what it is like to try to teach for understanding while also maintaining a broad sense of responsibility to the school district and state goals and objectives? What does it mean to get students to be more centrally involved in deciding whether something is true or reasonable and yet, as the teacher, maintain an eye on accepted knowledge of the field? In this chapter, I take up the special problem of trying to know what students are thinking, a problem of critical importance in a pedagogy that seeks to be responsive to students. This first-person research on teaching seeks to investigate issues that arise in practice *from the inside*. It is a kind of research *into* teaching that is complementary to other approaches to research on teaching.[8]

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The setting.

This chapter draws on data from my teaching during 1989-90. In this particular class, in which we had 22 students, 10 were from the United States and 12 were from other countries – Indonesia, Taiwan, Korea, Nepal, Nigeria, Kenya, Egypt, Ethiopia, Nicaragua, and Canada. Four of the ten U.S. students were African-American. Although no standardized or district testing is done in this school until the end of third grade, informal assessment showed that these students' entering levels of mathematics achievement varied widely in both mathematical skills and concepts.

The mathematics period was approximately one hour long. During this time, we often worked on just one or two problems. My intention was to select problems that would be generative, rich with mathematical possibility. Usually the problems were built from the previous day's work, with an eye to where we were heading. As such, the task of figuring out what students were learning was basic to the task of figuring out what to do next. The class often began with students exploring the problem of the day individually. As part of the class, students kept mathematics notebooks in which they recorded all their work and in which they also wrote about that work. Although I asked students to start by working on the problem alone, I also encouraged them to confer with others. After ten minutes or so (depending on the problem), students moved into small groups and worked together some more. We spent about half the class period in a large group discussion, during which individuals and groups presented their solutions and discussed their ideas.

Data collection and analysis.

Every class period was audiotaped, and most were video-taped as well. Many class periods have been transcribed, working from both video and audio records. I kept a daily journal about my thinking and work – a kind of all-purpose plan book and journal – which provides a record of my thinking on a day-to-day basis, including comments I made about students and their thinking and learning. In addition to the students' own math notebooks, there were also quizzes and homework and all student work was photocopied and saved. To complement what could be learned from their written work, students were interviewed regularly, sometimes informally – sometimes more formally – sometimes in small groups and sometimes alone.

The analysis for this chapter consisted of reading a sample of lesson transcripts, examining student talk and writing, and looking closely at evidence for my interpretations of students' thinking. I examined moves I made at that time to uncover or probe students' thinking and looked at the interpretations I

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seemed to be making. My journal and the students' written work gave me other angles on particular instances. Of particular interest to this analysis were instances where I, as teacher, was unsure about what students meant. Also of interest were instances where I appeared to have been sure of my interpretations at the time, but with the benefit of hindsight, my earlier interpretations now seem problematic. Uninterested in making general claims about my teaching or this class, my research focused on examining what seemed to influence my efforts to know what students were learning and to create better understanding of the general pedagogical task of trying to know students. I used my own teaching as a case that could contribute insights about specific challenges of that task.

CHALLENGES IN TRYING TO KNOW WHAT STUDENTS ARE LEARNING

Hearing students – understanding what they are saying, showing, feeling, doing – seems like the everyday work of communication. However, as I analysed instances where I had puzzled over what a child was thinking, I began to see particular features of the task that seemed to make it complex. These features did not seem specific to me as the teacher or to these children. As I understood them better, I began to see that these features were inherent in the task of listening to children. In this section, I identify three challenges that underlie this task and use instances from my teaching to ground and provide evidence for my claims. The instances reveal not only the temporality of practice and its fleetingness, but also the remarkable density of its intellectual and personal demands from moment to moment.

Content.

All the instances in this section come from a series of classes in January, during which the students and I were working on even and odd numbers – what makes a number even or odd, which numbers were even and which were odd, and patterns in adding even or odd numbers. In the course of this, we were also working on the epistemological issue of what it means to *prove* something in mathematics – pushing past the mere giving of examples to establish the plausibility of a claim to finding ways to show that a claim would hold true for *all* cases. For example, to show that 'an even number plus an even number equals an even number,' students were trying to do more than illustrate it with examples such as $6 + 8 = 14$ or $2 + 10 = 12$. To understand even numbers, they were trying to come up with a way to know if a number was even that did not involve alternately reciting numbers along the number line (e.g., '5 is odd, 6 is even, 7 is odd, so 8 is even') or drawing circles and trying to split them into two equal groups

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Name*	Gender	Race	Country	English proficiency	How long at this school**
Lindiwe	M	African-American	U.S.A./South Africa	Fluent	2 weeks
Nathan	M	White	Ethiopia	Fluent	3 years
Betsy	F	White	Canada	Native speaker	4 months
Daniel	M	Asian	Indonesia	Developing	3 years
Jeannie	F	White	U.S.A.	Native speaker	3 years
Keith	M	African-American	U.S.A.	Native speaker	2 weeks
Tembe	M	African Black	Kenya	Fluent	3 years
Mei	F	Asian	Taiwan	Fluent	2 years
Lucy	F	White	U.S.A.	Native speaker	3 years
Maria	F	Latina	Nicaragua	Beginning	4 months
Mark	M	White	U.S.A.	Native speaker	2 years

* All names are same-gender culturally congruent pseudonyms.
** This column reflects the length of time the child has been in this school as of mid-January. No one had been in this class longer than four months (since September).

Table 1: Third-Grade Class Roster in Mid-January .

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Name*	Gender	Race	Country	English proficiency	How long at this school**
Ofala	F	African Black	Nigeria	Fair	3 years
Devin	M	White	Nepal	Beginning	9 months
Riba	F	White	Egypt	Good	3 years
Harooun	M	Asian	Indonesia	Developing	16 months
Sean	M	White	U.S.A.	Native speaker	2 years
Sheena	F	African-American	U.S.A.	Native speaker	4 months
Tory	F	White	U.S.A.	Native speaker	2 weeks
Cassandra	F	African-American	U.S.A.	Native speaker	16 months

* All names are same-gender culturally congruent pseudonyms.

** This column reflects the length of time the child has been in this school as of mid-January. No one had been in this class longer than four months (since September).

Table 1: Third-Grade Class Roster in Mid-January (Continued).

(this becomes unwieldy with 1,421). Developing this more principled way of understanding what makes any number even or odd would extend the domain of their mathematical reach.

Students.

Although it was the middle of the school year, three students had joined the class less than a week before the instances I report here. Two students had already left the class since September. Table 1 offers a summary of the range of students who were in the class in January. The table illustrates some of this class's diversity. It is but a dim portrait, for there were many more differences among the children. The children's ages ranged over three years, and their home life varied, as did the kinds of experiences they had. Some had siblings, some did not. Some lived with one parent, some with two. Their religious backgrounds varied. I could not assume that my experiences of the world or of learning were similar to theirs. Hence, there was much for me to learn about each of them.

But there were some similarities as well. Most were from middle-class backgrounds, although what this meant varied given the diverse nationalities. They all lived in the identical brick barracks of student housing and had parents who were university or English Language Program students. And they were all in a common class, talking about even and odd numbers together.[9]

What did I know about this content and students in general?

I had taught even and odd numbers many times before. Personally, I am fascinated with number theory, and I enjoy the work of pattern-finding, explaining, and proving. I knew the formal definitions – $k2$ is an even number (k groups of 2) and $k2 + 1$ is odd – and I knew that the domain of these definitions was the integers. I also knew that a useful representation of even numbers for young children was to take a set of objects and define as even those quantities that could be divided into two equal groups without having to split any in half, even though this representation was effective mainly for natural numbers, elusive for 0, and inapplicable for negative numbers.

I was aware that third graders often could correctly label numbers as either even or odd but that their understandings were often more recognitional than principled. That is, they knew that 1 was odd and 2 was even, much as they knew that A was a capital letter and a lower case. Presented with a number such as 74, many would be unsure: 7 is odd and 4 is even – so what *is* 74? Those who said it was even would likely explain it on the basis of the maxim, 'You only

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have to look at the ones place.' These things I knew from other students, but I had never read any research probing children's understandings of even and odd numbers. I doubt it exists. As such, the knowledge I had about students in general and their understandings of this content was constructed from my own experience. I had also never talked with anyone else about students' understandings of even and odd numbers, so I knew little of what other teachers had encountered. My knowledge was personal – and idiosyncratic, for all I knew.

Challenge #1: Listening Across Divides

The first challenge is rooted in the fact that teachers are faced with learning what people quite different from themselves think, say, and mean. Children are different from adults. The divide of age is great; eight-year-olds see and think differently from adults. They talk, gesture, move, and think in ways that are special and not always familiar. And children may differ from their teachers in other ways as well – in gender, class, culture, religion, and native language, to name a few. How children think, talk, and represent their ideas is shaped by their varied identities and experiences. Trying to hear children challenges teachers with trying to listen across a gulf of human experience and meaning.

One day, near the end of class (1/17/90), I asked two girls – Betsy and Jeannie – to tell the rest of the class about a disagreement they had had. Jeannie explained that she thought that zero was 'even *and* odd' but that Betsy thought it was even. 'Why do you think 0 is both even and odd?' I inquired. Jeannie said she wasn't sure why but that that was what she thought. Betsy said she had not been convinced by Jeannie and that she had asked her parents and they had said that 'if you have zero things, zero things, and you cut it in two, you have zero stuff on both sides.' A student proposed that there was 'half on both sides,' and another offered, 'There's *air* on both sides,' but Betsy dismissed this firmly, 'We're talking about numbers, not the environment.' I asked if others wanted to comment. Sean raised his hand.

Sean: Uh huh, I just, I don't, I disagree. Well, I agree *and* disagree because um, because I think it's true that you can cut zero in half and there'd be zero on both sides, but that would, there, but I don't, I still don't, I don't think that um, that um, you can ever, you can ever have zero to begin with. Because, like, what um, like, like, how, how would you get zero then?

He continued:

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> Lucy said that um, if you, if you, if you had um, if you had one below zero (-1) plus one (1), she said it would be zero, but I disagree and said it would be um, it would be 1 *above* zero, and it is.

He seemed to be claiming that (-1) + 1 = 1 instead of 0. What was he saying, and why? It seemed somehow related to the discussion of whether 0 was even or odd, because it had to do with exploring what you get if you 'split zero in half.'

I didn't know exactly what Sean was thinking or what he understood, but I could feel among the other children the rising tide of emphatic disagreement that occurs most often when one child says something that conflicts with something that other children think we have already agreed upon. And we had already agreed months ago that (-1) + 1 = 0. In fact, we had come close to proving the general claim that any number plus 'that same number below zero would always equal zero.' I asked Sean to show us on the number line how he thought about (-1) + 1.

He went up to the board, and used the pointer to work with the number line that stretched above the chalkboard. Starting at (-1) he moved one spot to the left, ending up on (-2). 'That's *backwards*!' called other children.

I didn't know what he was thinking. And I was therefore unsure of what to do or say next. It might be that he was merely making a clerical error, a misstep of direction, and that no fundamental or conceptual confusion existed. But it could also be that he was struggling to make sense of adding to a negative number. For eight-year-olds, I knew that there was something troubling about the fact that, to the left of 0 on the number line, smaller numbers have larger magnitudes. They are farther from zero, and, in this sense *more*. From this perspective, 5 and -5 are actually equal, for they are both at a distance of 5 units from 0. But -5 is also less, and in fact this is part of what had been puzzling for the children So why did Sean choose to work on (-1) + (1)?

Listening across a divide, teachers face a problem similar to most forms of cross-cultural communication. It centres on problems of understanding what students mean with their words, pictures, gestures, and tone. Students often do not represent their thinking in ways that match adult forms. They use nonstandard terms, draw pictures, and make analogies. They get apparently right answers for unsound reasons, and apparently incorrect answers for sound reasons. Their speech comes in parcels of expressive talk or few words. What is required to listen both generously and critically? Awash in eight-year-old speech, some of it in halting English, scrutinizing drawings unclear from lack of technical skill as well as from undeveloped thinking – teachers must seek to 'read' and 'hear' their students.

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The students had moved into a space of the number system outside the domain to which their working definition had thus far applied. It seemed to make little sense to them to talk of 'splitting in half' the number zero. Moreover, they had developed no conventions for dealing with negative numbers; hence, their definition did not provide them with adequate guidance. I did not know what Sean was saying. I was not even sure if he was saying he thought zero was even, like Betsy, or both, as Jeannie was. It sounded as though he was disagreeing with Betsy's explanation that zero was even because if 'you cut it in two, you have zero stuff on both sides.' And his explanation? That one below zero plus one was not zero. Why?

I could guess. One conjecture might be that Sean was using this point to show that zero could not be even. If there are not two numbers that can be added to equal 0, then it is not even. Another hypothesis might be that Sean was feeling argumentative and was enjoying this exchange, that he was trying to say something that would provoke controversy. A third is that Sean was not really thinking about whether 0 is even or odd but was interested instead in whether it can be divided in half and was wrestling with whether that made sense or not. Yet another was that Sean was mixing up classes of numbers – positive/negative with even/odd. While thinking about even and odd numbers, perhaps he had fallen unwittingly into the category of positive and negative.

None of these hypotheses felt – or feels – satisfying. None seemed compelling. Not knowing what he meant, it was difficult to know what to do next. Was Sean misunderstanding something? If so, would it be a good move to engage everyone in what he was saying or preferable to talk with him later? Was he just playing around? Was he making a novel observation that would be intriguing to explore?

And this instance was no unusual event. From minute to minute, students say things, or say them in ways, that are unfamiliar to adults. A few days later, Ofala sought to explain what she thought an odd number was.

Ofala:	I just wanted to say if, um, you wanted an odd number, you just have to take one off of it.
Ball:	Why would that work?
Ofala:	Um, because usually odd numbers are … like, *like* … *(picks up chalk)*
Ball:	*(interrupting)* Cassandra, can you hear Ofala okay?
Ofala:	This, kind of like have ones in the middle? Like five *(she draws:*



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and there's a one in the middle, or, um nine: *(she draws)*



Mei: I think it's better with circles.

Ofala: *(she circles groups of two in her drawing of nine hash marks)* This two together, this two together ... this two together, and this two together.



There's one left. And even numbers – like six – *(she draws)*



– you can't get *anything* in the middle. There isn't one left.

Ofala, like many of her classmates, often used drawings and movements interspersed with words.[10] Her sentences would be better represented like this:

Um, because usually odd numbers are ... like, *like* ... this, kind of like have ones in the middle? Like five



here and there's a one in the middle, or, um nine.



Even with this representation, the reader cannot see Ofala, moving gracefully as she speaks, waving her slender arms, pointing and emphasizing. The drawings and the motions, it seemed, were not illustrations of the statement. They were an integral part *of* it. She, and many of the others, would make their drawing or movements *as* they spoke, rarely using a diagram or picture that was already on the board. Hearing the children meant watching. 'Reading' them meant integrating multiple media: facial expression and diagrams, words and body language.

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Ofala's definition was not the textbook's. Structurally it was very close to the algebraic one – $k2 + 1$. (Note: k two's, not two k's.) Or so I thought at the time. I now see that Ofala was actually saying two things here and coupling her words with two sets of drawings.[11] One idea she had was that 'odd numbers are the ones with ones in the middle.' This she represented with a linear arrangement of units. Her idea was to consider the bilateral symmetry of the row: With a central reflection, odd numbers are those where a 'middle one' remains. But she never carried this idea out. Her drawing showed the linear arrangement only, but without representing a test for a 'middle one', e.g.:



Ofala's idea about 'odd numbers have ones in the middle' was never demarcated or used. Mei's simple comment, 'I think it's better with circles,' seems to have influenced Ofala to abandon the 'one in the middle' idea, and shift to



And I, trying to keep my eye both on what Ofala and each of her classmates were thinking *and* on making sure that we developed a more robust idea of even and odd numbers, did not at the time notice that she was actually proposing two solid – and distinct – ideas. I completely missed the power of her linear arrangement. When she drew the circles, she switched back to circling in groups of two, and proposed that what matters is whether there would be any left. But at the end, she also leaned back to her more linear image, and explained that 6 is even 'because you can't get any in the middle.' She used her long slender fingers to motion that there were none left, but she did not concretize her representation as she did for the circle diagrams.

Did Ofala simultaneously understand oddness in two different, both correct, ways? Circling by twos and asking if you have any left is an appropriate representation for even or odd numbers. Arranging in a line and establishing that 7 will have one in the middle is another. Ofala was often not easy to understand. Her speech was soft, halting, and falling. She was highly mobile and physically graceful, and movement was a pronounced channel of her communication. So, although she did not mark the linear arrangement of units with chalk, she did mark the lines with gestures. Still, I completely missed the two distinct ideas, and only picked up the standard, $k2 + 1$. Steered by Mei ('it's better with circles'), and unassisted by me, Ofala emphasized the more standard representation. I did not even notice her first and very interesting idea recede. And, my

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own unfamiliarity with the mathematical distinction she was conjecturing also made it harder to hear her complex contribution.

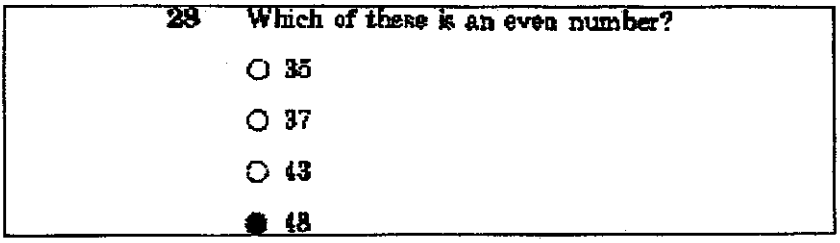


Figure 1: Sean's response to the item on even/odd numbers on the standardized test.

With more conventional approaches to teaching and assessment, children may represent their ideas in more standard, more adult – and, hence, more familiar – terms. Indeed, teachers and curriculum developers often work to constrain children's expressions, channelling their thinking in ways that will help them remember things correctly (Edwards, 1993). In the third grade mathematics workbook that I was *not* using there were six pages on even and odd numbers. Students were asked to 'circle the even numbers' from a list of 15 numbers between 1 and 40. No zeroes, no negative numbers, no invitations to conjecture and hence complexify. And on the standardized test which my students took in late April, three months after these complicated class discussions, there was one item on even and odd numbers. Figure 1 shows the item, reprinted from an actual test booklet.

Not surprisingly, no one in the class missed this item. In its constrained opening, the children's varied and idiosyncratic expressions had no space. In such openings, teachers are protected from the puzzling, the novel, the unfamiliar in students' thinking and expression. In such openings, too, teachers are protected from the problems with how children sometimes think, as well as from their insights and capacity for deep, complex thought. The constraints typical in much teaching and assessment channel students' ideas in conventional directions, perhaps supporting the development of standard knowledge but also masking the existence of nonstandard understanding. What does it tell me that all the students got this item correct? Some may have gotten it right because they know that 40 is even and 8 is even, others because they know the rule of thumb that all you need to look at is the ones place and 8 is even, so 48 is even. Maybe some got it right because 4 is even and 8 is even, and would not have chosen *38* of-

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fered as the correct choice. Even these alternatives represent quite different understandings of even and odd numbers, covered over by the simplistic item response.

When schools constrain children's responses to familiar structures, the divide is masked. Students find ways to fit their ideas into the structured openings, their thinking shaped into recognizable forms, but that may not represent their meanings. I turn next to the role of context in shaping students' thinking.

Challenge #2: Listening through the Multiple Influences of Contexts

The second challenge to understanding what children understand is rooted in the contextual nature of human sense-making and presents a challenge to the kinds of claims we want to be able to make about what students are learning. Despite the fact that schools tend to treat knowledge as something that can be transmitted and then measured, understanding is variable, and not nearly as stable or externally consistent as we pretend. Children can think one thing under one set of conditions and quite another under other conditions. They can seem to 'get it' one day and have little clue the next. Contexts matter, and new insights can, at times, confound that which had previously seemed clear, or change that which had seemed solid. When Sean had thought zero was even and later been unsure, it is not necessarily that he forgot. Mei had been adding and subtracting negative and positive numbers accurately. When she encountered 6 + (-6), what was different to her? Why did she stop, think, and come up with an answer of 9 when she had added (or so I thought) many other similar problems? What was different in this particular case?

Edwards (1993) argues that talk (or other expressions) cannot be interpreted as 'windows on children's minds.' We are not answering the question, 'What does this child *really* think?' Instead, by listening and watching, we are asking, 'What is going on *here*?' We seek to make sense of what, given this social and psychological context, a child is thinking *here*, now. Yet schools and school talk support a shared illusion that claims we make about students are general and independent of time, person, task, and setting.

During the first part of class one day, the children were working on conjectures which they had formulated about adding even and odd numbers – for instance, (1) even + even = even; (2) odd + odd = even; (3) if you add an odd number of odd numbers, you will get an odd number; (4) any number of even numbers will always add up to another even number. Children looked for examples of the conjecture they were investigating and were trying to find a way to prove that the statement would 'always be true.'

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During the first part of class, the children were working with partners. I leaned over Cassandra, who was exploring pairs of odd numbers. She had written:

$$35 + 35 = 70$$

When I asked her what this showed, she said it was an example for 'odd plus odd equals even.' I asked how she knew that, and she said that 35 was an odd number and that 70 was even. I knew that 70 was a challenging example for the children. Since their understanding of the structure of numbers was just developing, many would not be sure that the 7 represented 70, and hence, be even. Some would claim that the 7 was odd, and the uncertainties about what 0 was would lead them to be unsure about 70. I was curious what Cassandra understood, so I asked how she knew that 70 was even. She said that 70 came after 69 on the number line, and that 69 was odd, and numbers go 'even, odd, even, odd' – an explanation that had consistently been her favourite. Then, using the current class definition of even numbers, she said that 70 could also be split into two equal parts – 35 and 35 – without 'having to use halves.' When I asked if there was any other way to show that 70 was even, she added that since 0 was even, and you only have to look at the 'last number,' then 70 was even. I was impressed at her versatility and confidence. I resolved to call on her later during the whole group discussion so that her ideas could be available to others and also to give her an opportunity to shine with her thinking. I knew that Cassandra liked to be centre stage but did not always find a constructive way to take that part. This seemed to me an ideal opportunity since she had developed such elaborate explanations.[12]

In the large group discussion, I called on Cassandra to share her work on the conjecture, 'odd + odd = even.' She went to the board and carefully carried out 35 + 35 and explained her answer but then said nothing about its relationship to the conjecture. Then I probed gently, 'Okay, Cassandra, why did you do that? How does that fit with the conjecture?'

Cassandra:	That these two are odd numbers and it ends up with an even number.
Ball:	Cassandra, if somebody wanted you to prove that 70 was even, what would you say?
Cassandra:	Um, I would say that I would agree with them.
Ball:	No, if they wanted you to *prove* it. If they said, 'I don't believe you that it's even, what would you do?'
Cassandra:	Um, because um …
Ball:	What were you thinking about? Maybe you can tell us what you were thinking about.

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Cassandra: All right. Because 69 is odd and you would probably guess that 70 would be even.[13]

I noticed that she did not use the number line or any more detailed explanation, so I asked her why that followed.

Cassandra: Because you would say 69s odd, then 70s even.

Recognizing this as her first, and preferred, theory, I pushed her.

Ball: What if they didn't believe that explanation? Could you give them a different one?

Cassandra: Um, let's see ...

Ball: Is there another way to prove that 70 is even?

Cassandra: Um ... I don't think so.

I was disappointed at the receding of her ideas.

Ball: You showed me three ways two minutes ago. Think about what you showed me.

Cassandra: Um ... what?

Cassandra either could not or would not continue. I was not sure what to make of this. Perhaps I had imagined her clarity in our private discussion earlier; perhaps I had assisted more than I realized in the production of her explanations. I could not remember vividly enough. Maybe her ideas about this were less stable than I had thought. Or possibly she was unnerved in front of the class. What did Cassandra understand about evenness and the different explanations we had been exploring as a class? Should I base this on what I remembered her saying to me when we talked alone? On what she did in the whole group discussion? Later, looking at her written work, I saw that she relied most on the alternation of even and odd numbers without establishing that a given number was even or odd. For example, when asked whether 72 was even or odd, and how you would prove it, she wrote, 'Even. Because 71 is odd, so 72 is even.'[14] In response to a question about definitions of even and odd numbers, she was able to use the pic-

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tures – although she did not explain them – and also the recognition of particular numbers:

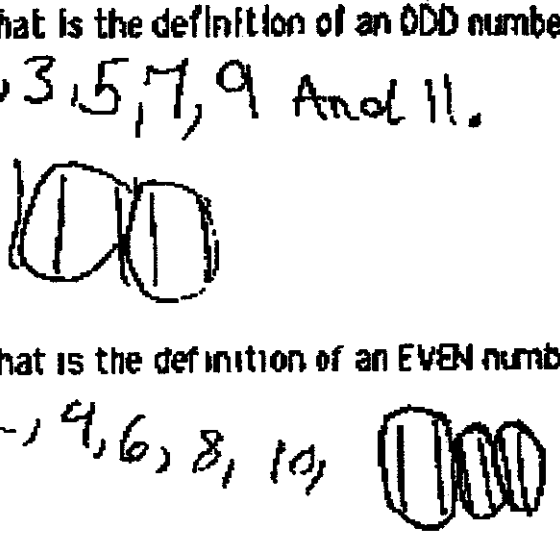


Figure 2: Cassandra's written definitions of even and odd numbers (2/1/90).

Although it may appear from the pictures that she had a structural sense of even and odd numbers, what Cassandra meant by the pictures, or how she understood them, was not clear from her writing. What was more clear was that she could name certain even and odd numbers (i.e., odd numbers are 1, 3, 5, etc., and 2, 4, 6, and so on are even). What then should be made of her more elaborated explanations with me in an individual conversation in class? Did she understand in a more fleeting sense? Connected with me, without the social distraction of presenting before the group, was the depth of her understanding more visible? Or was it that I was doing part of the understanding for her? Did the nature of the specific mathematical question affect her understanding? In class, she sought to reproduce that, but the presentation was also to establish that 35 + 35 did equal 70. Did the social context and the task itself unsettle her understanding, or was it that my presence earlier had strengthened it? In her written work, was it the lack of social and personal connection, the isolation of writing, that did not draw her out to elaborate her thinking? Or was it that these represented her current understanding, unaffected by relationships with me or with her

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classmates? Which snapshot of Cassandra provided the best picture of her thinking? Or did they all?

Lots of questions; few clear or definite answers. Perhaps this is why teachers may avoid delving deeply into questions of what students know. The questions seem to multiply exponentially. And, given the fleeting and intangible nature of knowledge, no answer seems satisfying. To assume that there is a 'true' picture of a child's understanding is to blot from consideration the many elements that contribute to understanding being more fragile than is convenient to assume. We would like to think that understanding can be a state, or a destination, that we can observe and certify. That it is so sensitive to context – to social relations, to time and tiny filigrees of consideration – is to admit of an uncertainty in the very core of teaching and learning, one understandably unappealing for either teachers or researchers to acknowledge.

Cassandra's thinking illustrates the complexities within one individual child. As if this was not already sufficiently confusing and challenging, it is important to bear in mind that such attentiveness to what students think and understand takes place in a context where one must be concerned with not one but many individuals. But there is a multiplication involved, for sometimes the sensitivities of student thinking are matters of the collective, located in the shared assumptions of the group. A discussion later on this same day in class, January 26, was one such example. In the past, the children's way of approaching these conjectures had been to find many examples of the statement and to assume that if they supported it with many examples, that this was sufficient to claim that it was true for all cases. This notion was beginning to come apart. In the middle of class on this day, Jeannie raised her hand.

Jeannie:	Me and Sheena were arguing together, but we didn't find one that didn't work. We were trying to prove that, um, Betsy's conjecture – um, that you *can't* prove that Betsy's conjecture always works.
Ball:	Go on, Jeannie. Say more about why you think that.
Jeannie:	Because, um, there's, um, like numbers go on and on forever and that means odd numbers and even numbers, um, go on forever and, um, so you couldn't prove that all of them aren't.

I was amazed that the two girls had extended the problem to a sophisticated epistemological issue ('we were trying to prove that you *can't* prove the conjecture'). I had wondered how young children could begin to make a shift toward establishing general mathematical validity of a claim, rather than merely illustrating a claim with finitely many verified cases. Thus far, all the children's work had been at this level of example-giving; their work more like inductive science than mathematics. Yet, Jeannie and Sheena seemed suddenly to have

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made two leaps: First, they were, for the first time, referring to the *set* of the integers as a mental construct, rather than to individual integers. Second, they seemed to be recognizing that giving examples could not establish general claims about this infinite set.

Touched by the magnitude of their new insight, I asked what others thought of this idea. Ofala immediately raised her hand and, looking at her intricate notebook page, filled with sums, said that she thought 'it can always work,' because she had tried eighteen of them and so she thought it could always work. I did not ask her what the 'it' was here – Betsy's conjecture, or the issue of method that was being challenged by Jeannie and Sheena. Ofala did not seem troubled by their objection. Many others nodded. Mei looked thoughtfully around the room at several posted ideas that had started in the class as conjectures at some point earlier in the year, and that the class had explored and decided to hold as true, based on their proofs. Suddenly Mei realized that Jeannie and Sheena's concern had the potential to unseat that earlier work.

> Mei: ... with those conjectures, we haven't even tried *them* with all the numbers that there is, so why do you say *those* work? Well, we haven't even tried all *those* numbers that there ever could be ... we haven't tried, we haven't ... she said that it's not always true. Then why do you say that is, that those conjectures is not always true? *(She waved with her hand, indicating the lettered posters with the class's earlier agreements – such as, from October, 'Any number plus that same number below zero equals zero.')*

Jeannie paused, thoughtfully, and then replied, 'I wasn't thinking about it.' And Sheena added, 'We were just thinking about it today.' In a moment of inspiration, Mei's insight took Jeannie and Sheena's one step further. challenging the certainty and validity of all the children's previous work on conjectures.

I asked what other people thought about Jeannie and Sheena's claim. Sean, still very comfortable with the inductive example-based approach, said he thought you could prove that conjectures were always true, because he had been adding pairs of odd numbers and he kept getting even numbers for the answers and so he was convinced. Several others agreed and gave examples from their work. Riba nodded. She had tried sixteen examples and 'they all turned out to be even.' 'But that's not all the numbers in the world,' demurred Sheena, pressing her new realization.

The class's prior agreements about what would constitute a proof were beginning to disintegrate, but the children were in different places with respect to the girls' insight and subsequent challenge. What could I say about the children's understanding about the role of examples in establishing that a conjecture 'always works'? The collective understanding was coming apart, on the

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way to one of the central powers of mathematical reasoning – the capacity to reason abstractly and construct general claims about entire classes of objects. Some students had little glimpse yet of what Jeannie and Sheena were recognizing; others' ideas were being challenged or unsettled, and others also glimpsed the big shift in thinking that this insight would provoke.

On a subsequent day, some of the children began trying to prove that some of their conjectures would hold true for *all* even and odd numbers. They tried to show that an odd number plus an odd number would always have to add up to an even number. They used drawings like this:

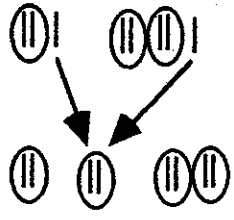


Figure 3: Drawing used to prove that any odd number plus any other odd number would always add up to an even number.

They seemed to intend these drawings to represent the general case of two odd numbers: Arguing that any odd number would always have one left over after grouping it by twos (using the currently accepted class definition), they showed, triumphantly, that two leftover ones would always be able to be combined; hence, yielding an even sum for the combination. Proud as these children were of their argument, many of their classmates were understandably unable to make the leap required to see the drawing in Figure 3 as something other than 3 + 5. They agreed with the claim that 3 + 5 = 8, and that 3 and 5 were both odd, and their sum, 8, even, but they could not see this argument as a support for a more general claim about *all* odd numbers. Moreover, many were not yet entirely – or even a little bit – convinced that there was any problem with using many examples to prove that the conjecture was true.

The limitations in their mathematical lexicon were creating a problem of understanding in context. The argument they wanted to make was an abstract one. Had they been able to represent an odd number comfortably as $2k + 1$, they might have been able to argue that $(2k + 1) + (2h + 1) = (2k + h) + 2 = 2(k + h + 1)$, which would be, by definition, even. The symbolic form, after all, paralleled the intention of those who made the drawings in Figure 3, and which grounded the general conclusion for many of their classmates. Still, it was not clear that

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understanding that odd numbers are the ones with one left over after you group by twos was an abstraction yet for many of the children. Instead, some of the children understood the idea only as a linked set of examples, while others glimpsed something more patterned and general. Understanding what Jeannie, Sheena, and Mei were seeing requires first of all to see odd numbers as a set, with the common property of having one left over once divided by two. The particularity of numbers were needed for the children to grasp this idea, and yet the idea paradoxically also requires a conceptual lift-off from the particular. Unsure of the complex role of context in individual children's current thinking, I did not try at this juncture to offer symbolic notation that might have helped transport them into the world of mathematical proof.

Still, this major mathematical insight challenged what had been a common agreement in the sufficiency of an empirical approach to mathematical proof. Although arguments of this kind occurred in the group setting, individual children continued to believe in examples while others switched to efforts to establish the generality of claims. The rules of knowing and proving seemed ironically less stable than they had been before the girls' mathematical discovery.

The variability in students' thinking, shaped by context and moment, can be seen as the failure to attain the 'solid' understanding to which teachers aspire for their students. It can be frustrating, even discouraging, to try to figure out what students think. However, what appears as an instable fragility may be merely the reflection of the highly contextualized nature of knowing and learning. I move next to the emotional demands of trying to figure out what students think and the challenges that arise in caring that students learn.

Challenge #3: Listening With and Through Desire

A third challenge for teachers rests in the basic commitment of the role. Teachers want students to understand. One part intellect, three parts emotion, teaching involves hope and wish, prayer and care, worry and craving. I wanted to feel that my third graders understood even and odd numbers well and that they were developing some ideas about how a definition and a conjecture are different. Being a teacher is centrally about helping others learn, and feeling successful is wrapped up in accomplishing this. Teachers are disappointed when they confront their students' confusions, missing pieces, distorted understandings. They care about their students. After investing time and effort in a particular student, a teacher *wants* to hear right answers, sensible reasons, creative ideas. Teachers ask leading questions, fill in where students leave space, and hear more than what is being said because they so hope for student learning. The desire to have been effective, to have helped, is strong and affects pedagogical hearing in ways

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that shape the assessment of what students know. Teachers also want students to feel good about their own accomplishments and may shy away from pushing students for fear of being seen as critical and, subsequently, as silencing.

Desire is also related to a fear for the fragility of understanding. Desire not only makes interpretation more complicated, it also can instill the hesitation to probe. On hearing a student say something right, fear of destabilizing the child's thinking can hold a teacher back from probing. Maria, who was often shy in class and whose English was just emerging, raised her hand to offer an example for the conjecture 'an odd plus an odd equals an even':

Ball:	What about other people? Maria, did you try some? How many did you try?
Maria *(barely audible)*:	I just want to show one.
Ball:	You want to show just one? Okay. Okay, do you want to tell them why it, what that is?
Maria:	21 plus 21 equals 42.
Ball:	What does that show?
Maria:	That 42 is an even number.
Ball:	Cassandra, can you listen to Maria right now please? Go ahead, Maria.
Maria:	That 42 is an even number.
Ball:	And 21?
Maria:	21 is an odd number.
Ball:	Comments? Cassandra?
Cassandra:	I agree with her.
Ball:	Anyone disagree with Maria? Thank you, Maria.

Her short turn over, we moved on. I had protected Maria and myself from saying anything wrong or from getting confused. I *wanted* her to understand, to get it 'right.' In wanting it, I arranged the context so that she did. It is true that there were many good reasons not to press Maria here. She volunteered for few turns in class, and even in small group work she was quiet. She volunteered on this day, and I was pleased. I wanted it not to be an uncomfortable experience; I hoped it would encourage her to do it more often.

Because Maria talked less in class, though, I was far less sure about what she was understanding. Leaving her answer unprobed did not threaten the possibly fragile nature of her feelings or her understandings. I did not risk confusing or unsettling her. But I also had less information on what she was thinking. Although I heard that she could correctly identify 21 as an odd number and 42 as an even, I did not know how she understood the definition of even or odd numbers. Without questioning her, I could not help her take the next step.

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Desire also shapes teachers' responses to students' ideas. After Ofala expanded the class's definition of odd numbers, I 'helped' by rephrasing more straightforwardly what (I thought) she had said:

Ball: So, you're saying the even numbers are the ones where you can group them all by two's and the odd ones are the ones where you end up with one left over?

Ofala had not said this precisely. What she had said was, 'I just wanted to say if, um, you wanted an odd number, you just have to take one off of it ... because usually odd numbers are ... like, like ... this, kind of like have ones in the middle? Like five



and there's a one in the middle, or, um nine.



This two together, this two together ... this two together, and this two together.'



There's one left. And even numbers – like six



you can't get *anything* in the middle. There isn't one left.' She had actually probably been saying two rather different, and both potentially valuable, things. However, as a consequence of my desire for Ofala to have a successful turn, and for the class to have a stable definition of odd numbers, I leaned into her contribution, heard it as correct, and restated it in my own terms to imbue it with still more stability.

Wanting students to 'get it right,' teachers rephrase, reshape, and reconstruct students' talk into more standard form (Edwards & Mercer, 1989). Was Ofala really saying what I said she was saying? I now am quite sure that she was saying both more and less. She was saying more in that she seemed to be making use of a linear arrangement, thereby having a 'middle' position, which conferred a geometric perspective on the number, and illustrated a particular aspect of oddness. She was saying less in that she did not articulate what I said when I

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transformed her words into a standard definition ('odd ones are the ones where you end up with one left over,' or $k2 + 1$). On one hand, a teacher can function as a learned listener of her students: Having come to know a particular child, the teacher can make sense of some of what that child is saying that a stranger would never hear. Sharing history with a class, both teacher and students can 'fill in' around idiosyncratic expressions. Ofala was a child for whom skilled listening was invaluable. An observer to our class would find Ofala particularly complicated to understand; the challenge of unfamiliarity heightened in listening to her. Up to this point, Ofala had mixed pictures and words to display odd and even numbers. I took her contribution and turned it all into words. I wanted the class, not just Ofala, to have access to this idea, to hold it in words, and so I sought to underscore her idea, reinforcing it with my succinctness. Whereas Ofala expressively showed one aspect of what she meant, using both narration ('this two together, this two together') and pictures,



I formulated the idea succinctly and less colourfully. Worrying about its fragile expression, and thinking it mathematically valuable, I did not want it to slip away. My words, I thought, would bolster it. My desire for particular ends overlaid what she said, adding and refining it, and leaving behind her second idea that odd numbers are 'the ones with something in the middle.'

On one hand, perhaps this is appropriate. That I wanted the students to develop their understandings is central to the role of teacher. On the other hand, my desire to move Ofala and her classmates in a particular direction affects what I hear them say, what I make of their expressions, and what I do next. I likely wanted Ofala to have said what I say she said. I wanted her to have made this distinction that if you divide a number into groups of two what matters mathematically is whether there is one left over or not. Inserting myself into her thought is entangled with my wanting to hear her as right. To have success with students is one of the few sources of reward in teaching (Lortie, 1975). Consequently, the incentives are high to hear students as right, to fill in the gaps in what they say, to read their words over-generously.

Listening is a balancing act: Hearing generously is critical to teaching well – listening sympathetically, leaning *into* rather than *away from* students, to believe rather than doubt (Elbow, 1986). It is not always unhelpful to put words in students' mouths. The language proffered can be a valuable scaffold for students' developing ideas. But it is also true at times that when teachers rephrase students' ideas, they take over, and obscure the students' own thinking. In many of the instances in this chapter, it would be possible to interpret and make sense of what students said, to come up with some explanation that would exclude a

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discrepant piece of evidence or interpret it more hopefully. How can teachers balance their desire for their students to have learned with the need to scrutinize the evidence carefully and skeptically? How can teachers be receptive and sympathetic listeners, working hard to see how students are making sense, and yet still be able to push and strengthen students' thinking and not settle for approximations?

PREPARING TO KNOW WHAT STUDENTS ARE THINKING

Trying to figure out what students are learning is central to teaching. However, although it is not popular to admit it, knowing with any certainty what students know is fundamentally impossible. Teachers listen across divides of human experience, through the vagaries of context, and with intense desire that their students succeed. Still, this of course cannot mean abandoning the quest to know students' thinking. In the fleeting moments of classroom talk, teachers must infer what students mean and interpret what they can do. Confronted with students' drawings and writings, teachers attempt to delve the layers of students' writing and drawing, and the relationships of writing and drawing to ideas and thinking. Teachers must convince administrators, parents, and the public of what students are learning, all the while reassuring themselves of what they are accomplishing.

What are the elements that interact in teachers' thinking as they listen and try to understand what students think? One element is knowledge of the particular ideas themselves, the content and the contexts of children's work. In the examples in this chapter, knowing flexibly what it means for a number to be even or odd makes a difference in listening to children's ideas. When Ofala talks of numbers that are odd having 'one left,' knowing that one mathematical representation of odd numbers is as groups of two with one left (one interpretation of $k2 + 1$) helps sharpen my hearing. When she simultaneously speaks of odd numbers having 'one in the middle,' it helps that I understand divisibility by two as meaning either dividing into *two groups (2k)*:

Ofala: This, kind of like have ones in the middle? Like five *(she draws:*



and there's a one in the middle, or, um nine: *(she draws)*



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or into *groups of two* (*k2*):

Ofala: *(she circles groups of two in her drawing of nine hash marks)* This two together, this two together ... this two together, and this two together.



Ofala was doing both, and I can track her thinking better for my own mathematical thinking. In the first instance, Ofala's idea seemed to be that suggested by the following drawing, which she did not, however, produce:



Some of the essential subject matter knowledge is epistemological rather than topical. For instance, appreciating the role of definitions in mathematics, and respecting the requirements of a functional definition would be important to listening to the children's attempts to capture what makes a number odd or even. At one point, when Sean tried to argue that 1 was even because it could be split into two parts (1/2 + 1/2), understanding the incompleteness of the definition on which he was relying helped me to interpret what he was thinking.

And some aspects of subject matter understanding are about relations, connection, and development. Children's ideas are continually in motion, evolving, changing, and adapting. They bump into the boundaries of their current domains and assumptions, and press on these to accommodate their ideas. These discontinuities can closely resemble the paradigm shifts that occur with disciplinary communities, as entire perspectives, and hence, ideas and methods, change (e.g., Kitcher, 1984; Kuhn, 1962). When Jeannie and Sheena realized that they were unable to prove something about *all* numbers, they precipitated a major paradigm shift, first for themselves, and later for their classmates. Familiarity with how this has happened to these ideas within the discipline can help teachers to understand the mobility of children's thinking (Bruner, 1960; Kline, 1970). A broader perspective on the mathematical connections between topics divided by conventions of the school curriculum can also enhance listening, when children work across those traditional boundaries without realizing it. For example, I was unattuned to the geometric potential of Ofala's linear arrangement, and the illumination it made possible for our exploration of number theory.

In the episodes discussed in this chapter, what I understood sharpened my hearing; what I did not know constrained it. For instance, when Jeannie raised

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her concern that there was no way to prove whether Betsy's conjecture (that odd + odd = even) would be true for any numbers, I was able to hear that she was edging toward a fundamental element of proof – that is, the need to establish that something is analytically and not merely empirically true. I was also able to understand that she was not yet thinking of ways that that could be done, but that she did have a sense that numbers were infinite. In the case of Ofala, however, I missed one of two main parts of Ofala's contribution about what makes a number odd. While I noticed that she was saying that odd numbers were those numbers that, when grouped by twos, have one left over, I completely missed the potency of her other idea – that an odd number, arranged linearly, and reflected about a central line of symmetry, would have 'one in the middle.' My own understanding was such that I heard these two as the same: as an argument about 'one left over.' I did not recognize the elegance of the geometric perspective she was offering and, as such, missed one main component of her thinking.

At the same time, mathematical understandings are only part of the work. Knowing about eight-year-olds made a difference – what they are like, how they think, what they are likely to have had experience with, how they are likely to read, write, and draw. Knowing specific students mattered as well. For example, knowing Ofala also helped to hear her. I knew to watch, to pay attention to her drawings, and I understood her drawings as processes, not products. That is, Ofala made pictures as parts of her explanations, not as illustrations of them. Like those of some other children in the class, it was in their construction, not the pictures' final form, that the most vivid expression could be found. With other children, the pictures were only sensible once they were completed; these children had long periods of quiet as they prepared a drawing amid an explanation. I developed sense of how to interpret Sean's lanky body motions and Ofala's graceful hands. Knowing the rhythms of their communication and talk made a difference in hearing them. Across a year of video-taping class every day, and struggling with issues of camera location, and mikes and other supports for recording the sound, I discovered that technical acuity was but a minor part of hearing and seeing children. Instead, really hearing and seeing them was a an active accomplishment of comprehension.

As the instances in this chapter illustrate, the challenges to figuring out what children know are great. Facing two or three dozen students at a time, teachers must make ongoing, usable estimates about what individual students know. They must observe and listen, and they must interpret their observations, reach conclusions, and act. The interpretations on which they act are no more than conjectures – uncertain, provisional, evolving. Yet, in the moment, teachers' assertions have the epistemological status of knowledge and must function as such (Scheffler, 1965). That is, teachers make decisions and design moves based on what they know, and they must treat that knowledge as likely to be

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true. What they must know is much more than what they can know in advance; they must know *in* the context of their interactions with children.

Teachers' knowledge of students is most often private, and as such, lacks the support of community. When mathematicians, or scientists, or historians make claims, they first convince themselves of what they think, but soon they are in the position of trying to convince colleagues and others of the validity of what they are claiming (Lampert, 1992). Doing this locates their efforts to know in the context of shared opportunity to examine evidence and shared standards for appraising that evidence. When the children in my third grade class made assertions, they sought to explain and justify their ideas, and in so doing subjected their conjectures to the scrutiny of others, who raised questions, offered alternatives, and suggested additions. But teachers, on this most central aspect of their work, most often work alone.

Interactions of Method and Knowledge in Figuring Out What Students Think

Despite these challenges inherent in listening to and interpreting children's thinking, to conclude that knowing students is an impossible task, and hence is not worth attempting, misses the point. Hence, the question: How can teachers improve their capabilities to figure out reasonably what their students are learning? In the face of uncertainty, what underlies skilled interpretation and better conjectures, and what are promising avenues of developing these? In an essay on 'The Nature of Method,' Dewey (1916/1944) brings together knowledge and its construction. He argues that, in order to become a skilled artist of any practice, there is an interplay of ideas and 'classical methods,' threaded with flexibility and imagination:

> Part of ... learning, a very important part, consists of *becoming* master of the methods which the experience has shown to be more efficient in like cases of getting knowledge. (p. 171)

At the same time that Dewey highlights the discipline of methods already used, he also recognizes their limits without 'an animating idea.' Exploring what he calls 'intellectual methods,' he considers physicians' engagement in diagnosis and treatment:

> To be used intelligently, existing practices, however authorized they may be, have to be adapted to the exigencies of particular cases ... [methods] should indicate standpoints from which to carry on investigations; they economize a survey of the features of the particular case by suggesting the things to be particularly looked into. (p. 171)

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Figuring out what students think seems like a similar investigative endeavour.[15] On one hand, knowing as much as possible about 'like cases' – other children of similar ages and circumstance engaged in other encounters with the mathematical ideas – can help fuel the interpretations a teacher can venture. When my students fell into a discussion of whether zero was even or odd, it certainly helped that I had heard many third graders before them wrestle with this question. It helped further that I myself had thought about zero, and about its relation to the definitions of even and odd. Still, as the instances in this paper demonstrate, many situations arise in the course of teaching which are not like anything the teacher has heard before. I had never heard a child claim that 6 + (-6) = 9 nor reason about negative numbers as Mei did. What gives teachers leverage on the complexities of interpreting students in such situations is similar to Dewey's idea of the blending of past knowledge with the skilled personal investigation of a novel situation at hand. Past experience – the teacher's own, or that of others – can guide, but not cover, the range of ideas and expressions that children will produce. Beyond the idiosyncrasies of individual manner is method, more than 'extemporized inspirations' (Dewey, 1916/1944, p. 17), comprising instead methods for constructing new knowledge. Such methods depend both on study of children and experiments with probing and interpreting them.

In this last section, I turn to three emergent approaches in professional development, all of which are centred on developing both knowledge and ways of knowing: discussing cases of student thinking, using redesigned curriculum materials, and investigating artifacts of teaching and learning. Each of these has promise for equipping teachers with the intellectual resources likely to be helpful in navigating the uncertainties of interpreting student thinking. Each of these represents an area in need of development and research. What the three share is a focus on learning to make sense of students' thinking in the fleeting moments of real-time practice. If figuring out what students think is a task localized in the particulars of moments, children, and content, and for which all knowledge cannot be had in advance, we need to design ways to support teachers' preparation for knowing students 'on-line.' Interesting, also, is the different potential each of these three approaches seems to hold for mediating the challenges of distance, context, and desire.

Discussing cases of student thinking.

The first approach focuses on developing a case knowledge of student thinking and the methods of its examination (e.g., Barnett, 1991; Schifter, Russell, & Bastable, in press). Working from written cases of episodes of children's mathematical work, teachers work together to examine, conjecture, and explore alternative interpretations of the children's thinking. At times, case discussions

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will focus on the mathematics at hand, at other times more on children's language and drawings, at other times on teacher questions (see Barnett & Sather, 1992). An important component of this approach is group discussion and analysis of written episodes, which has the possibility of moving teachers beyond the isolation of professional practice and the limits of idiosyncratic interpretation. Collectively, teachers can discuss mathematical and learning themes that arise across classrooms, comparing evidence and inference, and expanding their individual repertoire of interpretations and method.

Schifter et al. (in press) have engaged teachers with whom they work in authoring cases of student thinking. This design, which involves teachers in collecting and organizing information about particular children, has promise for helping teachers develop more finely tuned capacities for listening and watching children. Schifter and her colleagues (in press) quote several participating teachers who have remarked on the contribution that this simple opportunity provided. For example, one commented, 'When I knew I needed to write down what the children were saying, I started to pay attention in a different way.'

Looking and listening more closely is one component of this approach. The disciplined examination and interpretation of the collected information is another. But not to be overlooked in this approach is the expansion of case knowledge, of exposure to the range of possible ideas children might have. Such case knowledge enables teachers to expand beyond the happenstance of their own experience, to students they have not taught, and to analyses they have not made, and to animate the subsequent conjectures they can make. By broadening the cases of student thinking around particular mathematical topics, teachers may be able to hear students better through the multiple contexts of mathematical work. The analysis of several students' thinking on a specific mathematical topic may help teachers to parse better what students are working on and develop more detailed maps for listening and looking.

Using redesigned curriculum materials.

Curriculum could constitute a second resource for helping teachers prepare for hearing and making sense of students. Teachers' guides are well-positioned to guide teachers' listening and their interpretations of what they hear. Curriculum materials are centrally situated in teachers' practice, for most elementary teachers use teacher guides and texts extensively (Schwille et al., 1983). Moreover, unlike many resources for teaching, curriculum materials aim to guide practice at the close level of the daily lesson.

Despite this, curriculum materials do not usually provide much close, detailed information about students. Instead, teachers' guides traditionally emphasize content and ways to present the content, leaving teachers on their own to

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interpret what students produce in response to lessons and worksheets. Teachers' guides usually also lack practical discussions of students' likely thinking. And rarely do curriculum materials offer examples of student work accompanied by comments on assessment of such work. Similarly absent are the kinds of concrete guidance for teachers about ways to respond to particular student comments, questions, or interpretations.

In a poignant episode and analysis from her own teaching, Heaton (1994) describes her own lack of preparation for hearing students' answers to a problem she had so carefully prepared from the teacher's guide. The lesson focused on composition of functions and engaged students in the construction of a simple function table:

> As the teacher's guide suggested, I varied the process of filling in the table with numbers by asking some students to give ending numbers first, followed by an explanation of how they generated them. The teacher's guide also suggested that I call on someone different to offer these explanations ... I held high hopes for the next part of the lesson in which I planned to ask students to notice patterns. (pp. 127-128)

In the ensuing minutes, children produced examples of patterns that Heaton did not understand and did not seem the point of the lesson, and she began to falter. 'Hearing Pili's [one of the students] idea, I thought to myself, what sort of patterns were we looking for? Definitely not ones like this' (p. 129). She grew frustrated as the students gave examples of 'patterns' that did not seem mathematical, but for which she was unprepared. The teacher's guide had prepared her insufficiently to anticipate, hear, or manage what students might say in response to the problem and questions of the lesson.

> I wanted to be open to the idea that my students' ideas might vary from the teacher's guide. How was I to know which of my students' responses were reasonable variations ... and which were not? How would I figure out how to be responsive to these variations? ... *The script in the teacher's guide was designed with questions for me to initiate, but I felt frustrated with what little help it offered me in figuring out what to do next in the situation, especially when it seemed that what my students were saying did not match what the developers ... anticipated they might.* (p. 137, emphasis added)

Reflecting on her hopes for the teacher's guide to help her learn to teach mathematics in ways that were both deeper mathematically and more responsive to students, Heaton considers what might make curriculum more educative for teachers. Following Bruner (1960, p. xv), who argues that curriculum 'must be

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first and foremost a curriculum for teachers,' that curriculum materials can only affect pupils if they 'change, move, perturb, and inform teachers,' Heaton outlines elements central to practice that curriculum materials could be better designed to offer. She suggests that teachers' guides could, in addition to providing likely student responses, provide guidance for interpreting those responses:

> Verbatim quotes of likely student responses are not very useful without some understanding of *why* these might be typical responses. What mathematics might someone understand who would say these things? (p. 375)

What is needed, according to Heaton, is information for teachers to hear, assess, and make use of students' responses, all the while keeping their eye on the mathematical goals at hand.

Note that this sort of information and guidance are of a different kind than the usual content of teachers' guides, which typically concentrates on what teachers do and say. Including more information about students' thinking, alternative plausible responses to students' ideas, and examples and discussions of student work would be much less matters of what to do and say, and would instead be resources for the interactive work of constructing curriculum in classrooms.

Recently, mathematics curriculum materials have been developed that are designed to provide teachers with substantial resources for probing, hearing and reading, interpreting, and responding to likely student work (see, for example, Lappan, Fey, Fitzgerald, Friel, & Phillips, in press; Russell & Rubin, 1994). Such material has the potential to reduce the challenges of distance by better preparing teachers to hear what students might say. Plans for mathematical work, accompanied by well-developed material on students' thinking and ways to hear and respond, can provide teachers with better preparation for the territory in relation to both content and learners. If such development is grounded in pilot work in appropriately varied settings, curriculum materials have the potential of covering more of the variation in students ideas and expressions, hence for reducing some of the inherent distance between teachers and students. Such material is well-situated to provide such insights along with commentaries that relate student thinking to the mathematical goals of the curriculum.

Investigating artifacts of teaching and learning.

In preparing to know what students know, a third approach entails creating contexts for shared examination of unnarrated children's work, talk, and ideas. Dif-

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ferent from the case work described above, this approach engages teachers in direct examination of information from a classroom – video-tapes of classroom lessons, and interactions and children's written work and drawing on the chalkboard and on paper (e.g., Lampert & Ball, 1995). Teachers can examine closely a video-tape of a class discussion and analyse what different students seemed to be thinking. To do this, they would have to listen closely, striving to hear and see what students are doing. Could they examine closely what Ofala was presenting, for instance? Watching her body's motions, her hands, her drawings, and listening to her nonlinear and expressive talk, they could try to make sense of what she was saying. Unassisted by the interpretation of a case writer or a curriculum developer, they would be in the position of practice, leaning toward children and trying to hear. Similarly, teachers could look at a set of student essays and compare the kinds of comments they might write and why.

Such work is particularly well-designed for confronting the uncertainties of practice precisely because the information is unstructured and uninterpreted. Investigating puzzling comments or unclear writing, teachers would seek to make sense and to compare what they see and hear. Because the actual interactions and products are available, teachers have the opportunity to go back to the level of direct examination of the phenomenon.

That teachers cannot interact with the students provides an opportunity that they do not have in real time, in their own classrooms. That these are not their students creates a buffer of reflection that could support learning to make interpretations on-line. Unfettered by the desires to help their own students learn, or the sense of anxious responsibility for effecting learning, teachers are freed for the direct and focused examination of student thinking. They are left with the challenges of distance and context and may be able to consider and analyse more open-mindedly.

DIRECTIONS FOR FURTHER INQUIRY

Understanding better what makes knowing students – a central task of teaching – so complex creates an opportunity. It can seem daunting to think that the very thing for which teachers are most centrally responsible is so fundamentally impossible. Yet, its difficulty is also at the heart of what can make teaching intellectually challenging and engaging work, and what makes teaching so professionally demanding.

The ideas of this chapter suggest several lines of continued inquiry that can contribute both to our collective understanding of teaching and its challenges and the consequent issues posed for teachers' professional development.

First are continued questions about the challenges of knowing students and what students are learning. One issue crucial to understand better is how differ-

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ent approaches to and contexts of teaching affect the task and expectations for assessing student thinking. For example, within direct instruction, what are the issues entailed in knowing what students are learning different? Are teachers interested in different outcomes, hence face different issues in ascertaining what their students know? Or are the problems of knowing what students have learned plagued by similar challenges? Another important variable is subject matter. I have contextualized these analyses in the topics of mathematics teaching and learning. How are the issues similar and different in other subject areas, or in more integrated, thematic teaching? Do teachers of English literature face comparable challenges? How do the pedagogical problems of knowing students vary when the goal is to read *Canterbury Tales* rather than to develop understandings of even and odd numbers? Finally is an issue of level. How are the issues faced by teachers affected by the age of their students? On one hand, it may seem that issues of distance and unfamiliarity are greatest in the case of very young children whose talk and manner is so different from adults.' On the other hand, older students, with greater self-consciousness and awareness of their vulnerability about performance, may mask their thinking and be less accessible.

Second is much-needed study of how different kinds of support affect teachers' learning to hear and interpret students. The approaches sketched in the last section of this chapter are plausible hypotheses, not substantiated methods. Examined logically, they hold promise because of their relationship to the work of knowing students in practice – particularly its interactive, situated, and uncertain features. But many questions remain: Do teachers learn subject matter from such resources and experiences? Does such situated learning of subject matter ideas, located pedagogically within student thinking, provide the kind of intellectual resources that can help teachers hear and read unexpected student responses? Does such learning affect teachers' capacities to hear students in mathematical areas different from those addressed in the cases, curriculum materials, or artifact investigations?

A related set of questions centre on design. That the approaches discussed here are hypotheses is also to say that they are under development. How can such materials and experiences be designed to support teachers' learning? How might they be designed so as to be accessible, engaging, and useful to a wide variety of teachers in facing the challenges of knowing students? How might they provide opportunities for teachers to develop the mathematical resources crucial to hearing students, as well as to have better anticipations of what students might say? What variations in design are worth trying? What related but distinct approaches are worth pursuing?

Another set of crucial questions has to do with what teachers can learn about students from such work. Do they develop expanded case knowledge? What is

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it like? And how does such knowledge affect the work of interpreting their own students amid the challenges of distance, context, and desire?

Finally, how does the experience of engaging in collective professional discourse about student thinking affect teachers' standards for student work and their skills at interpreting and assessing particular students' thinking? That teaching has been traditionally so isolated makes attractive the idea of building a professional discourse and sense of community around this core task of teaching. A reasonable hypothesis is that such collegial work can build and extend individual teachers' resources – what they know about and how they look at students, as well as their flexibility and imagination in reasoning. It can, in Lord's (1994) terms, yield resources developed in the context of professional communities in ways that individual teachers cannot do alone. However, we need to track the work of such teacher groups and examine closely how norms of discourse evolve and how standards for knowing develop. We need to document what is discussed and what participants take from their engagement in such collective work.

The task of hearing students is complex; interpreting what they may be learning still harder. This chapter analyses aspects of the challenge and explores possible approaches to helping teachers prepare for the work. I close by returning to the children, who seem often much more comfortable with the uncertainties of their own understandings and the ambiguities of something called knowledge.

After Jeannie announced her sudden realization, arrived at by arguing with Sheena, that they could not prove for all numbers that even + even = even because 'numbers go on forever,' several children took notice. Some argued that the conjecture was true, and they reported how many examples they had tried. Then Mei challenged her on the basis of what she thought Jeannie already believed about other, previously agreed-upon, 'theorems':

Mei:	... we haven't tried every number that there was, so can't really say those *(motions to conjectures posted on the wall)* are true if you're saying that you want to try every number there ever was.
Jeannie:	I never *said* they were true all the time.
Mei:	Then why didn't you disagree when Lucy and everybody agreed with those conjectures?

Jeannie paused thoughtfully. With a matter-of-factness about changing her thinking, she answered simply, 'Because I wasn't thinking about it.' And Sheena piped in, 'She *just* thought about it. That's why we were trying to think about it.' Jeannie's thinking had changed, she had told Sheena about it, and they were both in the process of remodelling their assumptions and ideas.

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Learning to figure out what students think is in many ways like learning to think mathematically. Conjectures improve from concentrated attention and exploration; interpretation and analysis become more complicated, and new problems emerge. Knowing students is a domain worthy of such development, and promising for increased understanding and consequent new challenges.

ENDNOTES

1. My work on this chapter has profited greatly from the critical comments and helpful suggestions of Suzanne Wilson, Hyman Bass, Hilda Borko, and Deborah Schifter, as well as the editors of this volume. I wish also to thank Kara Suzuka for her assistance.
2. This work was supported in part by a post-doctoral fellowship from the National Academy of Education and the Spencer Foundation, and a grant from the National Science Foundation (TPE 92-55242). The opinions herein are mine and do not reflect the position, policy, or endorsement of either of the grantors.
3. David Cohen (in preparation) has analysed and written about issues of endemic uncertainty and how teachers manage to constrain or open their access to students' thinking in what he calls 'teachers' acquaintance with students' knowledge.'
4. See Lampert (in press) for a helpful examination and analysis of what it might mean to conduct research on teaching *from the perspective of practice.*
5. Teachers' content knowledge was an early focus of researchers on teaching who sought to prove what seemed an obvious relationship between what teachers knew and what the students learned. When given the ways in which they conceived 'subject matter knowledge' and 'student learning,' the expected correclations did not appear, this line of investigation was shelved for over a decade (see Ball, 1991).
6. A more complete discussion of this episode and an analysis of it in light of issues of representation and discourse can be found in Ball (1993b).
7. Kara Suzuka, Margery Osborne, Sylvia Rundquist (my classroom teacher-collaborator), Jim Reineke, and Erin Eberz.
8. Other researcher-teachers who conduct research on teaching from the perspective of practice include Lampert (e.g., 1986, 1990a, 1992, in press); Chazan (1992); Hammer (1995); Heaton (1994); Lensmire (1993, 1994a, 1994b); Osborne (1993); Roth (1992); Wilson (1995, in press); and Wong (1995). See also Ball (in press) for a discussion of first-person research.
9. I argue elsewhere (see Ball, 1995) that collective engagement in such abstraction can serve as common ground for students in ways preferable to applications and connections to the everyday world.
10. See Osborne (1993) for more on children's multiple expressions, including art, movement, body in a first-grade science class.
11. My thinking about Ofala's mathematical understanding has been greatly enhanced by studying Hyman Bass's written commentary on an earlier version of this analysis.
12. See Ball (1993a) for other instances of my trying to help Cassandra find a spot for presenting her ideas to this class.
13. It was not clear why claiming that 69 was odd was so obvious to Cassandra.
14. Again, it is not clear why 71 was so clearly odd to Cassandra.
15. Magdalene Lampert and I have been engaged over the past several years in exploring 'investigation' as a central task in teaching and teacher education and drawing parallels between this way of thinking about practice and professional education on one hand and

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children's learning of mathematics on the other (see, for example, Lampert & Ball, 1995). Dewey also makes this connection (1916/1944, p. 172).

REFERENCES

Anderson, J., Reder, L., & Simon, H. (1996). Situated learning and education. *Educational Researcher, 25*(4), 5-11.

Ball, D. L. (1991). Research on teaching mathematics: Making subject matter part of the equation. In J. Brophy (Ed.), *Advances in research on teaching* (Vol. 2, pp. 1-48). Greenwich, CT: JAI Press.

Ball, D. L. (1993a). Moral and intellectual, personal and professional: Restitching practice. In M. Buchmann & R. E. Floden (Eds.), *Detachment and concern: Topics in the philosophy of teaching and teacher education* (pp. 193-204). New York: Teachers College Press.

Ball, D. L. (1993b). With an eye on the mathematical horizon: Dilemmas of teaching elementary school mathematics. *Elementary School Journal, 93*, 373-397.

Ball, D. L. (1995). Transforming pedagogy – Classrooms as mathematical communities: A response to Lensmire and Pryor. *Harvard Educational Review, 65*(4), 670-677.

Ball, D. L. (in press). Working on the inside: Designing to use one's own practice as a site for studying mathematics teaching and learning. In R. Lesh (Ed.), *Designing research for reform: In mathematics and science education.*

Ball, D. & Wilson, S. (1996). Integrity in teaching: Recognizing the fusion of the moral and intellectual. *American Educational Research Journal, 33*(1), 155-192.

Barnett, C. (1991). Building a case-based curriculum to enhance the pedagogical content knowledge of mathematics teachers. *Journal of Teacher Education, 42*(4), 263-273.

Barnett, C. & Sather, S. (1992, April). *Using case discussions to promote changes in beliefs among mathematics teachers.* Paper presented at the annual meeting of the American Educational Research Association, San Francisco.

Bruner, J. (1960). *The process of education.* Cambridge, MA: Harvard University Press.

Chazan, D. (1992). Implementing the 'Professional Standards for Teaching Mathematics.' *Mathematics Teacher, 85*(5), 371-375.

Cohen, D. K. (in preparation). *Teaching practice and its predicaments.* Unpublished manuscript, University of Michigan, Ann Arbor.

Cohen, D. (1995). Rewarding teachers for students' performance. In S. Fuhrman & J. O'Day (Eds.), *Rewards and reform* (pp. 60-112). San Francisco: Jossey Bass.

Darling-Hammond, L., Ancess, J., & Falk, B. (1995). *Authentic assessment in action: Studies of schools and students at work.* New York: Teachers College Press.

Delpit, L. D. (1988). The silenced dialogue: Power and pedagogy in educating other people's children. *Harvard Educational Review, 58*, 280-298.

Dewey, J. (1902/1990). *The child and the curriculum.* Chicago: University of Chicago.

Dewey, J. (1916/1944). The nature of method. In *Democracy and education* (pp. 164-179). New York: Macmillan.

Duckworth, E. (1987). *The having of wonderful ideas and other essays.* New York: Teachers College Press.

Edwards, D. (1993). What do children really think? Discourse analysis and conceptual content in children's talk. *Cognition and Instruction, 11*, 207-225.

Edwards, D. & Mercer, N. (1989). Reconstructing context: The conventionalization of classroom knowledge. *Discourse Processes, 12*, 91-104.

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission

Elbow, P. (1986). *Embracing contraries: Explorations in learning and teaching*. New York: Oxford University Press.

Erlwanger, S. (1975). Benny's conceptions of rules and answers in IPI mathematics. *Journal of Children's Mathematical Behavior, 1*, 157-283.

Hammer, D. (1995). *Epistemological considerations in teaching introductory physics*. Newton, MA: Center for the Development of Teaching, Education Development Center.

Hawkins, D. (1974). Nature, man, and mathematics. In *The informed vision: Essays on learning and human nature* (pp. 109-131). New York: Agathon. (Original work published in 1972.)

Heaton, R. (1994). *Creating and studying a practice of teaching elementary mathematics for understanding*. Unpublished doctoral dissertation, Michigan State University, East Lansing.

Hiebert, J., Carpenter, T., Fennema, E., Fuson, K., Human, P., Murray, H., Olivier, A., & Wearne, D. (1996). Problem solving as a basis for reform in curriculum and instruction: The case of mathematics. *Educational Researcher, 25*(4), 12-21.

Jackson, P. (1968). *Life in classrooms*. New York: Holt, Rinehart, and Winston.

Jackson, P. (1986). *The practice of teaching*. New York: Teachers College Press.

Kitcher, P. (1984). *The nature of mathematical knowledge*. New York: Oxford University Press.

Kline, M. (1970). Logic versus pedagogy. *American Mathematical Monthly, 77*, 264-282.

Kuhn, T. (1962). *The structure of scientific revolutions*. Chicago: University of Chicago Press.

Lampert, M. (1986). Knowing, doing, and teaching multiplication. *Cognition and Instruction, 3*(4), 305-342.

Lampert, M. (1990a). Connecting conventions with inventions. In L. Steffe & T. Wood (Eds.), *Transforming children's mathematics education* (pp. 253-265). Hillsdale, NJ: Erlbaum.

Lampert, M. (1990b). When the problem is not the question and the answer is not the solution: Mathematical knowing and teaching. *American Educational Research Journal, 27*, 29-64.

Lampert, M. (1992). Practices and problems in teaching authentic mathematics. In F. Oser, A. Dick, & J. L. Patry (Eds.), *Effective and responsible teaching: The new synthesis* (pp. 295-314). San Francisco: Jossey-Bass.

Lampert, M. (in press). Studying teaching as a thinking practice. In J. Greeno & S. G. Goldman (Eds.), *Thinking practices*. Hillsdale, NJ: Lawrence Erlbaum.

Lampert, M. & Ball, D. L. (1995). *Aligning teacher education with contemporary K-12 reform visions*. Paper prepared for the National Commission on Teaching and America's Future, Teachers College, New York.

Lappan, G., Fey, J. T., Fitzgerald, W. M., Friel, S. N., & Phillips, E. D. (in press). *Connected mathematics project*. Palo Alto, CA: Dale Seymour Publications.

Lensmire, T. (1993). Following the child, socioanalysis, and threats to community: Teacher response to student texts. *Curriculum Inquiry, 23*, 265-299.

Lensmire, T. (1994a). *When children write: Critical revisions of the writing workshop*. New York: Teachers College Press.

Lensmire, T. (1994b). Writing workshop as carnival: Reflections on an alternative learning environment. *Harvard Educational Review, 64*(4), 371-391.

Lord, B. (1994). Teachers' professional development: Critical colleagueship and the role of professional communities. In N. Cobb (Ed.), *The future of education: Perspectives on national standards in America* (pp. 175-204). New York: The College Board.

Lortie, D. C. (1975). *Schoolteacher: A sociological study*. Chicago: The University of Chicago Press.

Osborne, M. (1993). *Teaching with and without mirrors: Examining science teaching in elementary school from the perspective of a teacher and learner*. Unpublished doctoral dissertation, Michigan State University, East Lansing.

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission

Paley, V. (1986). On listening to what the children say. *Harvard Educational Review, 56*(2), 122-131.

Paley, V. (1995). *Kwanzaa and me: A teacher's story.* Cambridge, MA: Harvard University Press.

Peterson. P. L., Carpenter, T., & Fennema, E. (1989). Teachers' knowledge of students' knowledge in mathematics problem solving: Correlational and case analyses. *Journal of Educational Psychology, 81*, 558-569.

Pimm, D. (1987). *Speaking mathematically: Communication in mathematics classrooms.* New York: Routledge & Kegan Paul.

Ramsey, P. (1991). Social dynamics of early childhood classrooms. In *Making friends in school: Promoting peer relationships in early childhood* (pp. 43-70). New York: Teachers College Press.

Roth, K. (1992). *The role of writing in creating a science learning community* (Elementary Subjects Center Series No. 62). East Lansing: Michigan State University, Elementary Subjects Center.

Russell, S. J. & Rubin, A. (1994). *Landmarks in the hundreds.* In *Investigations in number, data, and space.* Palo Alto, CA: Dale Seymour Publications.

Scheffler, I. (1965). *Conditions of knowledge: An introduction to epistemology and education.* Chicago: University of Chicago Press.

Schifter, D., Russell, S. J., & Bastable, V. (in press). Teaching to the big ideas. In M. Solomon (Ed.), *Reinventing the classroom.* New York: Teachers College Press.

Shulman, L. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4-14.

Schwille, J., Porter, A., Floden, R., Freeman, D., Knapp, L., Kuhs, T., & Schmidt, W. (1983). Teachers as policy brokers in the content of elementary school mathematics. In L. Shulman & G. Sykes (Eds.), *Handbook of teaching and policy* (pp. 370-391). New York: Longman.

Wilson, S. M. (1995). Not tension, but intention. A response to Wong's analysis of the researcher-teacher. *Educational Researcher, 24*( ),

Wilson, S. M. (in press). Mastodons, maps, and Michigan: Exploring uncharted territory while teaching elementary school social studies. *Elementary School Journal.*

Wilson, S. M., Shulman, L. S., & Richert, A. (1987). '150 different ways of knowing': Representations of knowledge in teaching. In J. Calderhead (Ed.), *Exploring teacher thinking* (pp. 104-124). Sussex: Holt, Rinehart, & Winston.

Wong, D. (1995). Challenges confronting the researcher/teacher: Conflicts of purpose and conduct. *Educational Researcher, 24*(3), 22-28.

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