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Developing A Mathematically Proficient American Public:

What Are The Problems, What Do We Know About Them, and What Would it Take to Solve Them?

Deborah Loewenberg Ball, Ph.D.
Arthur F. Thurnau Professor of Mathematics Education and Teacher Education
University of Michigan

This is an important moment for mathematics education in the United States. For the first time, widespread agreement exists that mathematical proficiency matters on a wide scale. Most people agree that American adults will require substantial mathematical proficiency to participate fully in society and the economy in the twenty-first century. No longer can mathematics be only for some; knowing and being able to use mathematics is increasingly seen as an essential form of literacy. In addition, some occupations will continue to require even higher levels of mathematical skill.

But we have a problem. For all the rhetoric about the centrality of mathematics to the life and progress of this new century, the United States did not manage to achieve high levels of mathematical proficiency among most American adults by the end of the last. In this paper, I examine the nature of our problem, consider what is known about it, and about what it would take to solve it.

What Is The Problem?

Consider how you would answer the following questions. Why does it work to "add a zero" (i.e., place a zero to the right) when multiplying by ten, or two zeroes when multiplying by a hundred? And then why, when the number includes a decimal, do we move the decimal point over instead? Is 0 a number? If it is a number, is it even or odd? What does it mean to divide by one-half? What is an "irrational number"? What is the probability that in a class of 25, two people will share a birthday? As you respond to these questions, consider, too, the nature of your own mathematical understanding, and how it developed as it did.

Our problem in the U.S.? Most well-educated adults cannot answer the above questions comfortably, nor make judgments about orders of magnitude, estimate the likelihood of particular events, or reason effectively about quantitative relationships. Take the following everyday example: The television weather report forecasts 50 percent chance of rain on Saturday and 50 percent chance of rain on Sunday. A man listening to this is disappointed, muttering that there is therefore a 100 percent chance of rain that weekend and that his golf plans are ruined. Is he right? His companion disagrees with him, and tells him that he is not right—that if there is a 50 percent chance of rain on Saturday and a 50 percent chance on Sunday, there cannot be 100 percent chance of rain on the weekend. In reasoning about this, she explains that there cannot possibly be a greater chance of rain for the entire weekend than there is on either of the individual days. He nods, following what she is saying.

Unfortunately, although her conclusion is correct—there is not a 100 percent chance of rain on the weekend—her reasoning is not. Like so much of what people have learned in school mathematics, they know things to be true, but do not know the reasons. And without

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those reasons, their mathematical knowledge often misleads and falls short. What they think they know lets them down when they need to use it to figure out something new.

In this instance, counterintuitive though it may be, the probability that it will rain on the weekend is more than the probability that it will rain on either individual day. If there is a 50 percent chance of rain on Saturday and a 50 percent chance of rain on Sunday, there is actually a 75 percent chance of rain on the weekend. The following table, which displays all the possibilities, helps to reveal why.

Saturday	Sunday	Rain on weekend?
rain	rain	yes
rain	no rain	yes
no rain	rain	yes
no rain	no rain	no

The issue is realizing what "raining on the weekend" means. If someone were to ask, "Did it rain over the weekend?", you would answer affirmatively if it rained on either or both of the two days. Neither the man's original reasoning nor his companion's explanation was based on formal knowledge of probability; each relied instead on intuitive reasoning. However, some people do remember learning some probability in school and might attempt to use it to figure this out. Some might be tempted to use the fact that the joint probability of two independent events is given by the product of the two probabilities (1/2 on each day). However, this calculation ($1/2 \times 1/2 = 1/4$) would yield the probability that it rains on both days. It would also give the probability that it rains on neither day. In each case, the probability is 25 percent, which can be seen in either the first row (rain on both days) or last row (rain on neither day) of the chart above. However, the question asks about the chance of rain on one day or the other, and many people who do remember the formula are still uncertain of when and how to use it.

What is the point of this example? It is to illustrate the problem concretely by showing what is lacking for too many American adults. Many cannot remember what they learned in their mathematics classes, and when they do, all too often they cannot make appropriate use of what they learned. When average Americans are asked how they use mathematics in everyday life, the most frequent answers are either that they use mathematics to balance their checkbooks, or that they do not use mathematics at all. Distaste of mathematics is both widespread and socially acceptable. Well-educated people feel little compunction in announcing that they were never good at math. Imagine if those same people said similar things about reading or writing: "I never could read." In an age where everyone agrees that the workplace and life demands for mathematics will be greater than ever before, our educational system does not reliably prepare people to be mathematically proficient: skillful, able to reason about and solve problems using mathematical tools and ideas, confident in their own abilities and interested in mathematical questions.

That is not the worst of it. Significant inequities exist in mathematical proficiency. Mathematical failure is disproportionately associated with race, poverty, and gender. The mathematical underachievement of girls and students from particular socio-economic, ethnic, cultural and linguistic backgrounds remains a pervasive problem in the United States. The changing demographics of the population suggest that this problem could worsen in the future without a serious effort to understand and address the source of mathematical inequalities.

That so many Americans leave their formal experience of learning mathematics uninterested in and unskillful with the subject has long been an object of concern. How can we mobilize the resources to do something about it?

What Do We Know?

When confronted with the disappointing outcomes of U.S. mathematics education, some

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blame the poor quality of teachers. Others blame students, as less able, less well-prepared; or their families and out-of-school activities. Still others place the blame on curriculum materials. In fact, the core problems are with instruction—that is, what teachers and students do together with mathematics, in the real environments of schools. Our problems lie with teaching and learning, not with teachers and learners as individuals. Replacing the teachers and learners in our schools would leave those new people with the same problems. It is teaching and learning in the U.S. that is not as effective as it needs to be at the threshold of the twenty-first century.

Fortunately, some significant resources exist for the problem's solution: Much effort has already been invested in developing approaches, materials, and knowledge that can support the improvement of mathematics teaching and learning.

Within the last year, three documents have been released, each one the work of committees of unusually diverse membership: The National Council of Teachers of Mathematics' new *Principles and Standards for School Mathematics*; *Before It's Too Late*; the report of the Glenn Commission on Mathematics and Science Teaching for the Twenty-First Century; and the National Research Council's *Adding It Up: Helping Children Learn Mathematics*. Each offers significant resources for the problems we are facing. Although the reports differ in their scope and purpose, they overlap in their emphasis on improving mathematics instruction and teacher education.

Humility demands that we bear in mind that these problems that we are facing are not new. The past forty years have seen several waves of mathematics reform, each entailing serious efforts to improve mathematics learning. Each has attempted to upgrade what counts as "mathematics" in school, to alter students' mathematical experience, and to improve their grasp of fundamental ideas and skills. And yet change has been difficult, and much has remained the same as it was in 1950, or even 1900. Students still practice pages of sums and products and are still asked to solve improbable story problems. Students are still told to "invert and multiply" to divide fractions and to use "My dear Aunt Sally" to remember to multiply and divide before adding and subtracting in an expression. Teachers still explain how to do procedures, offer rules of thumb, give tests on definitions and procedures, and provide applications. These practices in and of themselves are not necessarily unhelpful. However, the prevalence of instruction that consists only of such teaching techniques helps to explain why the number of students who leave school as proficient with mathematics as they are literate with English remains small.

Despite pervasive concerns about mathematics education over the past fifty years, why has there been so little improvement, and what do we know now that might make a difference in these patterns?

1. **Curriculum materials alone cannot determine instruction. Teaching—what teachers do with curriculum materials with their students—is what matters.** The development, or adoption, of particular curriculum materials has been the main strategy in attempts to improve mathematics learning. It is not difficult to understand why. It seems, from the outside, like the central ingredient of classroom lessons and so improving curriculum seems like a direct input to improving learning. What this strategy overlooks, however, is that teachers exercise substantial discretion in their use of curriculum, making decisions about what to emphasize, augment, and omit. They make decisions about the order of topic presentation, and adapt the book's treatment of a topic in order to meet their students' needs. Moreover, teachers' own knowledge of mathematics influences their interpretation of the textbook authors' intentions and thus shapes their use of the material. Despite common assumptions, there is no such thing as completely "following the text."

   Moreover, students also interpret textbooks, and their interpretations shape their

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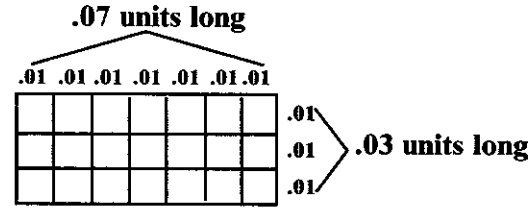
teachers' use of the curriculum. They may struggle with particular lessons and require re-teaching, or additional practice. They may already know the material and require extension. They may not understand the book's examples, and need alternative models. Effective instruction demands such adaptation between teachers and their students. Hence, even in districts where "curriculum pacing" is the instructional policy, how teachers and students interpret and use the textbook lessons results in variability from class to class.

2 **In order to teach mathematics well, teachers must know and be able to use mathematical knowledge flexibly to help students learn.** Teachers' mathematical knowledge is often a topic of interest. Many people bemoan the weak knowledge of U.S. teachers, and leap to remedying this weakness. The two most frequent remedies are recruitment—hire better-prepared teachers; and add coursework—increase the numbers of courses needed for professional certification. Neither is a promising strategy, however, for neither results in teachers who have and can use the mathematical knowledge needed for effective teaching.

Knowing mathematics for teaching is different from knowing it for yourself. Take, for example, multiplication of decimals. Knowing how to multiply .3 x .7 requires being able to remember and follow the steps of the algorithm:

$$\begin{array}{r} .3 \\ \times \quad .7 \\ \hline .21 \end{array}$$

Most adults can do this. But being able to teach this requires much more. It may require answering the student who asks, "Why are there two decimal places in the answer when .3 and .7 each has only one decimal place?" Being able to say nothing more than, "That's the rule in multiplying decimals," leaves students without the reasoned knowledge that we examined in the first section of this paper. One way to explain the "rule" might be to represent this procedure pictorially to show how it works. One way to do this is to make a rectangle that is .3 on one side, and .7 on the other:



Each of the little squares inside the rectangle has an area of .01 units, and there are 21 of those.[1] This can set up an explanation of why multiplying tenths by tenths produces hundredths. Generalizing this to the multiplication of any decimal numbers is yet another step.

The point is that if we want to develop mathematical proficiency in students, then teachers need a kind of flexible knowledge of mathematics that permits them to "unpack" ideas and procedures to make their reasons available to students. Teachers need to use mathematics in ways that others do not. They need, for example, to be able to select an appropriate model for a particular mathematical idea, a model that illuminates the meaning of the idea, and that does not distort or obscure its essence. They need to be able to modify a problem to make it easier or more difficult. They need to ask the right mathematical question at precisely the crucial moment. And they need to be able to develop and deliver explanations that are comprehensible to learners.

Teaching is not a generic skill. Each of the tasks mentioned above is fundamentally mathematical work. Simply knowing how to ask a good question in general does not equip a teacher to ask a good question about a particular algebraic expression, or about a specific solution.

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Knowledge of mathematics needed in teaching is not easily developed in conventional mathematics courses that are designed for people intending to enter other kinds of careers—engineers, professionals in the mathematical sciences, physicists—who must use mathematics in ways significantly different from the uses to which teachers must deploy mathematical insight and knowledge.

3. **In order to teach mathematics well, teachers must be able to understand and work from where their students are.** Teachers cannot learn for students. No matter the format of instruction—whether lecture, small group activity, or individualized assignment—students make their own sense of what they are taught. Ideas do not flow directly from teachers' or peers' minds, or from well-designed worksheets or activities into learners' minds. Of course, instruction can be more or less well-designed to make it more likely that students will learn what teachers intend and that they will not be misled. Still, effective teaching requires teachers to be able to investigate skillfully what their students are taking from instruction, and to make adjustments along the way to ensure student learning. This requires teachers to mobilize their knowledge of mathematics as discussed above—to ask a good question to probe what students know, or to listen attentively to a stumbling or unexpected student explanation, or to design an additional assignment to provide appropriate help. Teachers must be able to do this with an ever-increasing variety of students, and to listen and work across wide gulfs of experience, language, and culture, to help each of their students learn.

4. **Good teaching is something to learn, not an inheritance.** Implied in all of this is that teachers are made, not born. Learning to teach effectively requires that teachers have sustained opportunities to learn mathematics, about students, and about ways to help students learn particular mathematical ideas and procedures, both before they begin teaching and as practicing professionals.

These opportunities to learn are most effective when they are connected to the work of teaching. Just as other professionals learn through clinical work, work in field settings, and professional practice, so too must teachers. They need opportunities to learn mathematics not like those appropriate for engineers, but in the ways that they must use it in teaching. They should learn about the primary resources of their practice—curriculum materials, for example—and about how to examine, modify, and use them effectively and with integrity. They must learn about common student difficulties with specific mathematical ideas and about ways to prevent and remediate those.

This sort of professional education is central to the educational systems in high-performing countries, where teachers' opportunities to develop professional knowledge and skill are part of their work. Not only do teachers in Japan and China, for example, have time to learn and to improve their practice as part of the regular work week, but what they work on is practice—curriculum, mathematical content, students' learning—together with other professionals. Their learning is ongoing, systemic, and systematically connected to the professional career. Just as the Japanese do not believe that mathematical accomplishment is the result of innate ability, neither do they leave good teaching to inheritance: Teaching is a complex practice that can be learned and continually improved.

5. **Most improvement efforts do not focus enough on teaching and learning, and are not designed adequately for what it takes to make them work in real contexts.**

If the goal is improved learning, then improvement efforts are most likely to work if they focus on teaching and learning. Indirect efforts are chancy, for they may or may not—and often do not—impact what teachers do with students, around particular mathematical content, in classrooms. For

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example, a new curriculum can be used in a wide variety of ways. But a new curriculum accompanied by ongoing opportunities for teachers to learn, to work with other teachers on the use of that curriculum, and to make wise accommodations of that material in their environments, with their particular students, would be much more likely to make a difference in the quality of instruction, and the effectiveness of learning.

Most reform efforts fail. They fail because they are too vague, too far from the core of instruction to make a difference, and because they overlook the learning needed to enact the improvement. Incentives alone cannot produce professional learning. Neither can sanctions. If effective instruction is something to learn, then reform efforts must be designed for learning.

What Would It Take to Solve Our Problems in Mathematics Education?

Across the various things we know, one big point stands out: Instructional practice must be the central focus of efforts to improve students' learning. Hence, investments in knowledge about effective practice, coupled with substantial opportunities to develop the resources for such practice, must be our main investment. What we know about our past efforts at improvement shows that if we are serious about developing a mathematically proficient American public for the twenty-first century, we will require a new orientation to our efforts and investments, an orientation that centers on teaching and learning, on teachers' learning, and on the continued development of knowledge about effective teaching.

We must build on what we know and use the resources that we have. We will need to change our assumptions, and the policies that flow from those. We will need to make the development of high-quality mathematics teaching and learning the central thrust of our efforts. This means:

1. *Providing sustained, systemic opportunities for teachers to learn to develop the effectiveness of their practice in ways connected to the complex tasks of their work.* Professional work and education must afford ongoing opportunities for teachers to develop useful and usable knowledge of mathematics, of students' learning of mathematics, of curriculum and its effective uses, and of teaching itself.
2. *Building a knowledge base for excellent practice.* Research is needed to build shared and usable knowledge about effective instruction. Such research must be focused on problems of practice and its inquiries and products made widely available in usable form. Such studies might examine successful teachers and schools to understand and articulate the elements that contribute to their effectiveness. Other studies might probe the most frequent challenges faced by teachers as they work to develop mathematical proficiency in particular environments. Studies might also design interventions aimed at improvement and then examine closely whether and how they work, including successes achieved and obstacles met. Studies should track students over time, through different instructional approaches, in different environments, so that we understand the long-term trajectories of students' mathematical development. And we need, perhaps most of all, reliable and careful studies of what helps teachers learn—the substance, materials, approaches, and organizational structures that enable teachers to develop the sort of professional knowledge and skill fundamental to effective mathematics instruction and its continuous improvement.
3. *Mobilize interdisciplinary expertise to work on the improvement of mathematics instruction.* Teaching and learning, and improving instruction in the complex environments of American schools, present policymakers and practitioners with compound challenges. These challenges require multiple kinds of expertise of many different kinds, including that of experienced expert practitioners, scholars of teaching and learning, and of

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school improvement, research mathematicians, and psychologists and sociologists, along with many others. At the core, the problems of mathematics education are problems of practice, and to be useful, these different sorts of expertise must be mobilized to focus on practice.

4. *Attend to and coordinate conflicting signals and initiatives in the environments in which instruction takes place.* If teachers and others in schools are to focus intensively on the improvement of mathematics teaching and learning, these efforts depend on better coordination of the multiple messages and policies. For example, high stakes testing often works against the need for teachers to attend closely to what students actually understand and whether they can use what they are learning. Such high stakes testing creates incentives for rapid coverage, and supports teaching that will help students remember for the test. Such testing also takes time away from instruction as teachers spend increasing time equipping students with the skills needed to succeed on tests. Assessment of learning is unquestionably fundamental to good instruction, and accountability for results matters. However, assessments should be designed to provide useful information to teachers connected more intimately to students' progress. Another area in need of coordination can be seen in the pervasive commitment to local educational decision making which results in students who move frequently—most often students from the poorest environments—needlessly re-starting their mathematics learning over and over as they shift from one curriculum and assessment environment to another. And there are many other needs for coordination in the very fragmented environments in which mathematics teaching and learning, and schools, exist. The challenge is clearly before us. It is time to move past the worries about how we compare internationally. It is time to move past thinking that education can be most readily improved through the recruitment of better players—teachers and learners. And it is time to stop seeing good teaching as little more than the simple exercise of common sense, and to support its improvement as a complex professional practice that can be learned. It is time to take the investments and learning of the past century to make this the century when mathematical proficiency in the United States is as common as competent reading and writing. If we are serious, then we must move wisely and strategically, setting aside ideology and belief, and working across communities to develop a mathematically proficient American public.

Endnotes

1 To see that this is true, imagine a square of side length 1. If you cut that square up into ten equal strips, then the area of each is one-tenth of the original square. If you further cut each strip into ten squares, then the original square has been cut into 100 little squares. The length of the side of each of these little squares is .1, and their area is .01.

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