



TEACHING ALL STUDENTS TO BE MATHEMATICALLY SKILLFUL:

Balancing the Equation

Deborah Loewenberg Ball

Scholastic Math Summit • Nashville, TN • February 20, 2013

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Today's session

1. What does does it mean to be mathematically skillful?
2. Skillful mathematics teaching: A balancing act
3. What are high-leverage teaching practices?
4. High-leverage practices for teaching students to be mathematically skillful

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What does it mean to be "mathematically skillful"?

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The 8's problem

Write as many 8's as you like. Insert plus signs (+) so that the value of the expression is equal to 1000.

Note: You can string 8's together, such as 888.

- How many different equations can you write following this rule?
- What is your approach?
- Can you find all of them? How do you know?

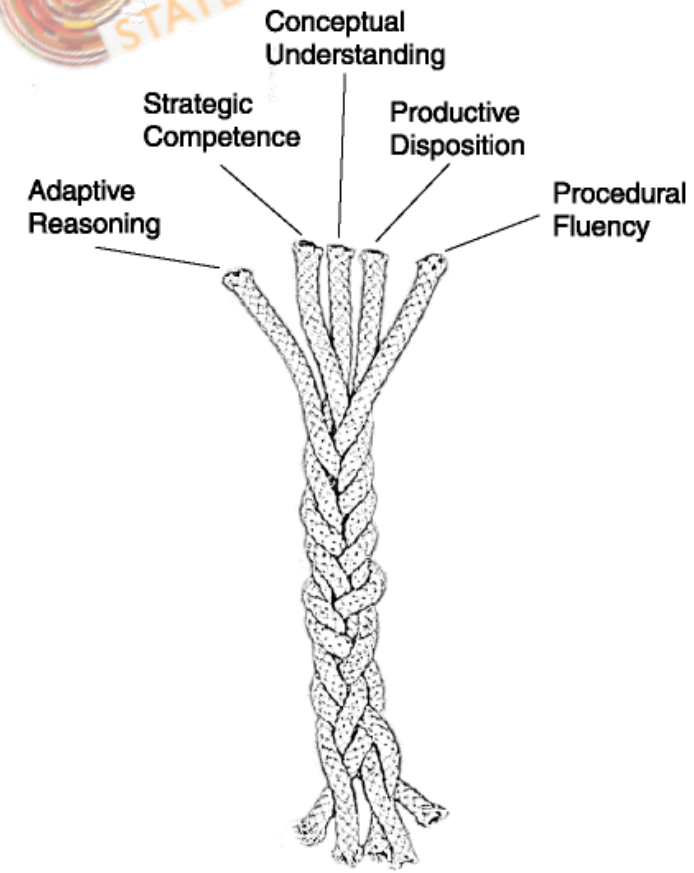
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Solutions to the 8's problem

number of 888's	number of 88's	number of 8's	number of terms
0	0	125	125
1	0	14	15
0	1	114	115
1	1	3	5
	2	103	105
	3	92	95
	4	81	85
	5	70	75
	6	59	65
	7	48	55
	8	37	45
	9	26	35
	10	15	25
	11	4	15

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Strands of mathematical proficiency



- ***Conceptual understanding*** - comprehension of mathematical concepts, operations, and relations
- ***Procedural fluency*** - skill in carrying out procedures flexibly, accurately, efficiently, and appropriately
- ***Strategic competence*** - ability to formulate, represent, and solve mathematical problems
- ***Adaptive reasoning*** - capacity for logical thought, reflection, explanation, and justification
- ***Productive disposition*** - habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy

Kilpatrick, J., J. Swafford, and B. Findell. (2001). *Adding It Up: How Children Learn Mathematics*. Washington, DC: National Academy Press.

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THE COMMON CORE:
What is entailed in being mathematically skillful?

Concepts and procedures

- Understanding of operations (+, x)
- Fluency with whole number computational skills (+, x)
- Multi-digit computation
- Understanding of mathematical structure

Mathematical practices

- Making sense of and interpreting problems
- Representing mathematical ideas in strategic ways
- Persevering with complex mathematical problems
- Constructing mathematical arguments
- Look for and use regularity in mathematical reasoning
- Look for and make use of mathematical structure

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What does it mean to teach for mathematical skillfulness?

- Set goals that combine strands of proficiency and that wrap the mathematical practices together with the mathematical "content"
- Develop students' learning opportunities through
  - The careful choice of mathematical tasks
  - The way that a mathematical task is framed and set up
  - How the mathematical task is used together

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Teaching as a complex balancing act

Teaching all students to become mathematically skillful entails balancing three traditional dualities:

1. Focus on mathematical concepts and procedures AND mathematical practices.
2. Teach in ways that are both teacher-directed AND student-centered.
3. Work to fill students' gaps AND to accelerate their capabilities with complex mathematical work.

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Responsible teaching is unnatural and intricate work

1. Teaching requires teachers to do things that are more than commonsense.
2. Teaching involves some moves or orientations that are unnatural in regular adult life.
3. Teaching practice is intricate, requiring a complex combination of knowledge, skill, timing, and relational work.

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High-leverage instructional practices

High-leverage practices (HLPs) are instructional tasks and activities that powerfully promote learning and are fundamental to skillful teaching.

Based on work done at the University of Michigan School of Education in the redesign of our teacher education program, and at TeachingWorks, a new organization whose mission it is to improve the quality of teaching and learning by transforming teachers' education.

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Five high-leverage teaching practices to develop students' skillfulness

1. Establish norms and routines for mathematical work.
2. Elicit students' mathematical thinking.
3. Respond to common patterns of students' mathematical thinking.
4. Modify mathematics tasks.
5. Lead group discussions about mathematics.

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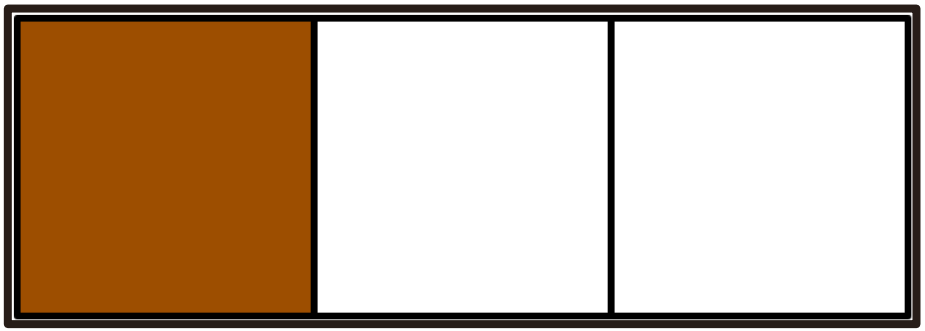
Teaching as a complex balancing act

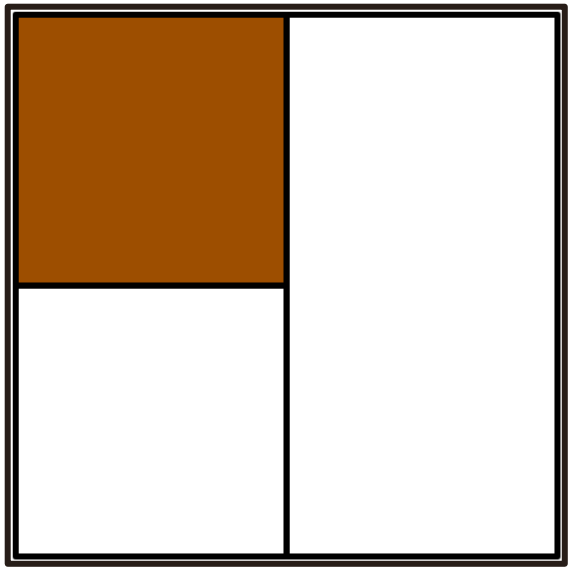
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How much of the rectangle is shaded brown?





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See *Brown Rectangle Video Sequence: Part 1*

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① Establish norms and routines for mathematical work

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Norms and routines for productive mathematical work

1. Teaching students to listen to and respond to others with interest and respect
2. Holding students accountable for reasoning
3. Using multiple calling on strategies, according to purpose
4. Explicitly supporting students to present at the board: audibly, clearly
5. Modeling and supporting the recording of one's own and others' mathematical ideas and reasoning
6. Encouraging writing, representing, drawing mathematical ideas
7. Analyzing errors as part of productive mathematical work

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② Elicit students' mathematical thinking

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③ Responding to common patterns of students' mathematical thinking

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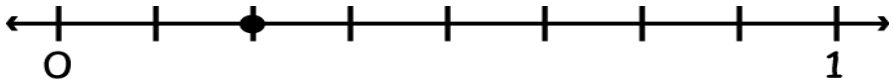
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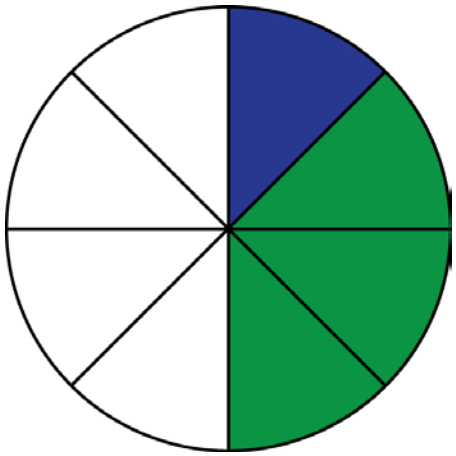
④ Modify mathematics tasks

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Notice "safe" tasks

What makes these tasks "safe"?



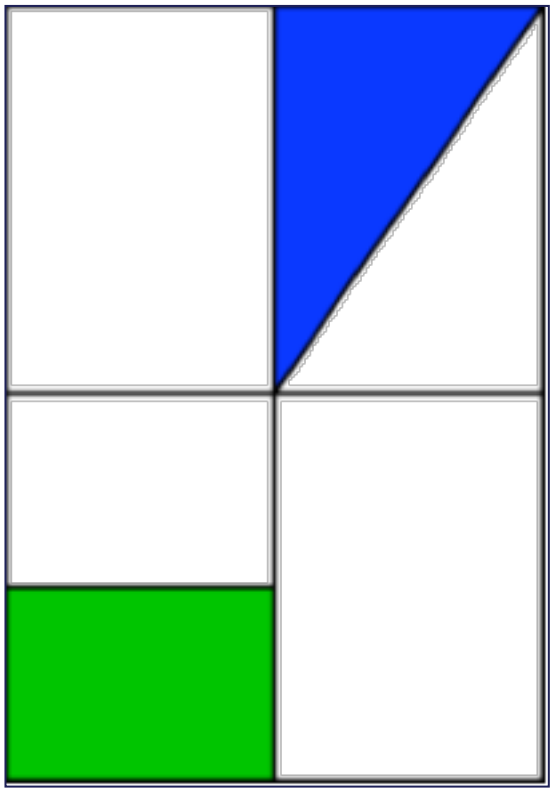


1. Much of the thinking is done <u>for</u> the children (e.g., equal partitioning, shading, labeling)
2. Protect from error — supports getting "correct" answer

THUS, THESE TASKS ARE ACTUALLY POTENTIALLY <u>UN</u>SAFE.

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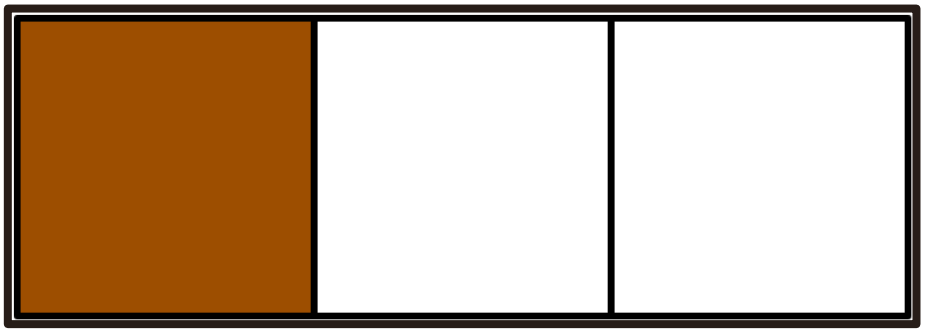
Build tasks that support students to do mathematical work

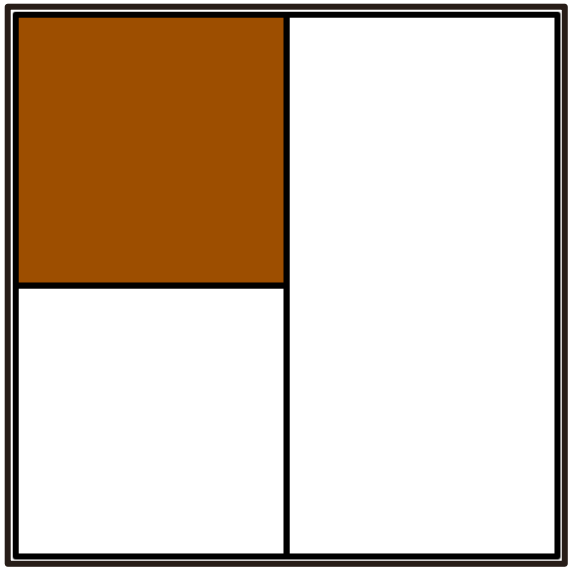


- What fraction of the big rectangle is shaded blue?
- What fraction of the big rectangle is shaded green?
- What fraction of the big rectangle is shaded altogether?

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Attend closely to wording

POSSIBLE PROBLEM:

I have pennies, nickels, and dimes in my pocket.

If I pull out two coins, what amounts of money might I have?

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Consider with care alternative wording

1. I have pennies, nickels, and dimes in my pocket. If I pull out two coins, what amount of money might I have?
2. I have pennies, nickels, and dimes in my pocket. If I pull out two coins, how many combinations are possible?
3. I have pennies, nickels, and dimes in my pocket. If I pull out two coins, how many different amounts of money are possible? Prove that you have found all the amounts that are possible.

What are the differences among these three formulations?

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Teaching students to be successful with complex tasks

- Help students to believe they can do hard work.
- Scaffold complex work appropriately.
- Make wise judgments about what to leave open and what needs to be made explicit.
- Comment on mathematics, and mathematical productions, not features of students.
- Support error as a fruitful site for mathematical work, and teach students to use error productively.

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⑤ Lead group discussions about mathematics

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Three types of math discussions

1. Practicing
2. Learning concepts or procedures together
3. Discussing solutions to a problem

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The 1, 2, 3 problem

- How many different three-digit numbers can you make using the digits 1, 2, and 3, and using each digit exactly once?
- Show all the three-digit numbers that you found.
- How do you know that you found them all?

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The 1, 2, 3 problem

Understanding combinations

Attending to language and precision

- How many different three-digit numbers can you make using the digits 1, 2, and 3, and using each digit exactly once?
- Show all the three-digit numbers that you found.
- How do you know that you found them all?

Constructing a mathematical argument

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Viewing focus

1. What are the purposes of the discussion?
2. What is the teacher doing to lead the discussion?
3. What roles are students playing?

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See *Warmup 123 Permutation Video*

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Conclusions

1. Teaching all students to be mathematically skillful requires teaching that is skillful.
2. Skillful teaching involves balancing traditional dualisms about content vs. process, teacher-direction vs. student-centeredness, and remediation vs. enrichment.
3. Supporting teachers to develop skill with high-leverage teaching practices is key to the development of students' mathematical skillfulness.

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