

With an Eye on the Mathematical Horizon: Knowing Mathematics for Teaching to Learners' Mathematical Futures



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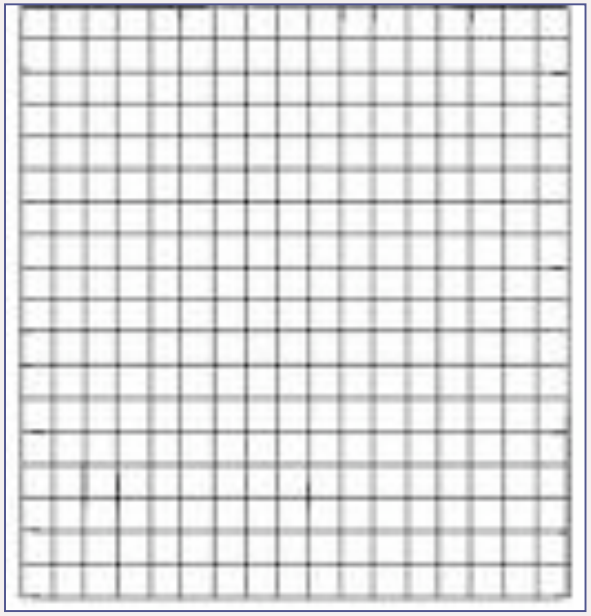
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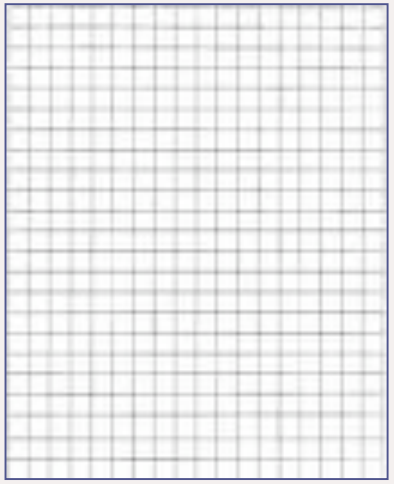
Three brief snapshots of mathematics learning

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Measuring area in first grade







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Making and proving a conjecture in third grade

An odd number plus an odd number equals an even number.

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ho•ri•zon |həˈrīzən|

noun

1 [usu. in sing.] the line at which the earth's surface and the sky appear to meet : *the sun rose above the horizon.*

• (also **apparent** or **visible horizon**) the circular boundary of the part of the earth's surface visible from a particular point, ignoring irregularities and obstructions.

• (also **celestial horizon**) Astronomy a great circle of the celestial sphere, the plane of which passes through the center of the earth and is parallel to that of the apparent horizon of a place.

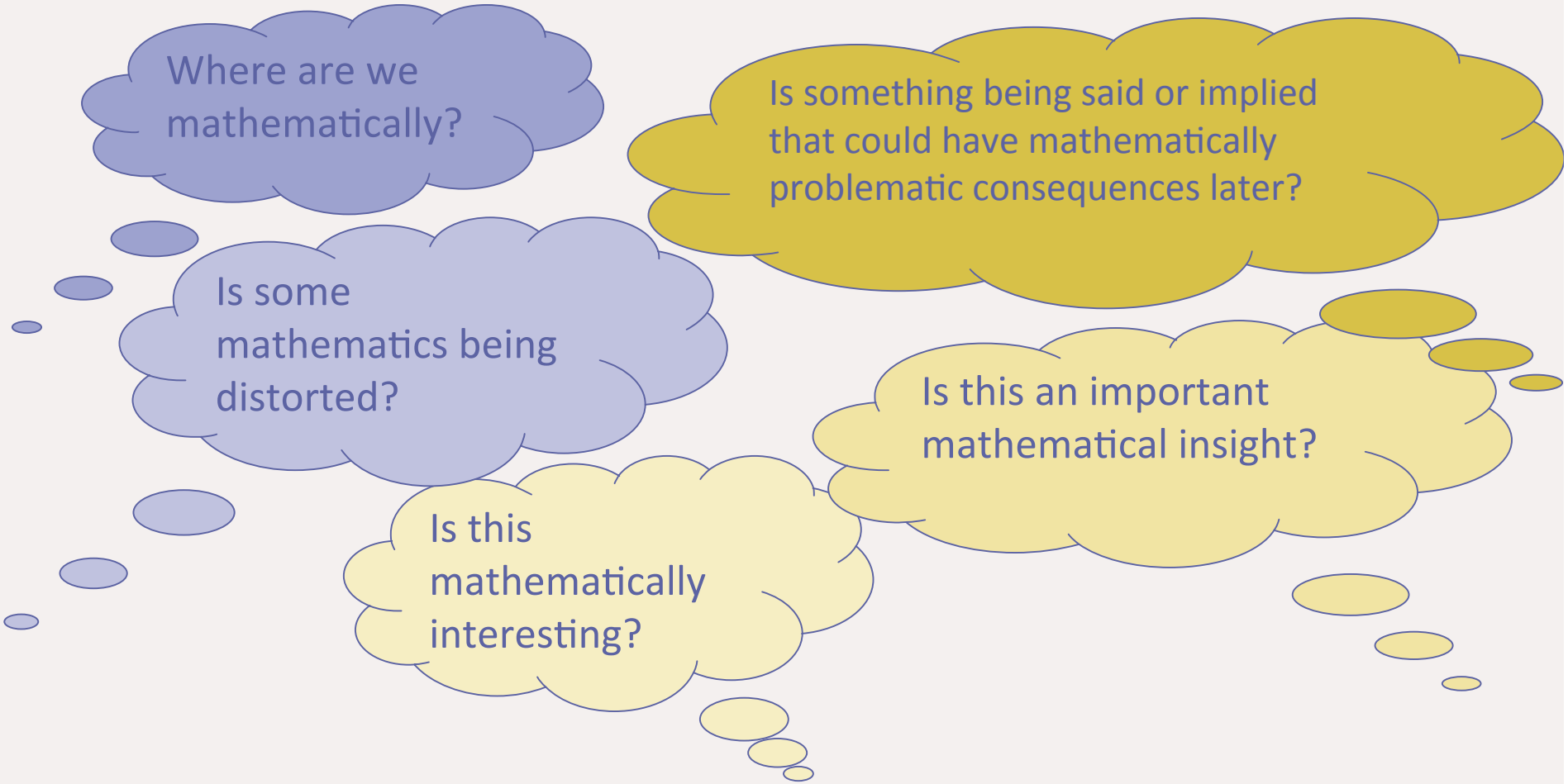
2 (often **horizons**) the limit of a person's mental perception, experience, or interest : *she wanted to leave home and broaden her horizons.*

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Teaching with an eye on the mathematical horizon



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Twin imperatives of teaching

1. To be intellectually honest
2. To be respectful of students and obligated to the learning of every student

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Teaching that is both responsible and responsive

- Hears the mathematics in students' representations and talk
- Notices misconceptions in students' representations and talk
- Balances mathematical rigor with generosity toward emerging ideas
- Manages mathematical opportunities wisely

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What does being mathematically responsible entail?

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What is mathematical knowledge for teaching (MKT)?

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Elements of our "practice-based" approach

1. Study instruction and identify the mathematical work of teaching
2. Analyze what mathematical knowledge is entailed by the work (MKT)
3. Test the working hypotheses based on these analyses by developing measures of MKT, validating teacher scores against practice and against student achievement gains
4. Develop and evaluate approaches to helping teachers learn mathematical knowledge for teaching

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Knowing multiplication

Multiply:

$$\begin{array}{r} 49 \\ \times\ 25 \\ \hline \end{array}$$

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Knowing multiplication for teaching

What mathematical steps produced each of these answers?

(a)

$$\begin{array}{r} 49 \\ \times\ 25 \\ \hline 405 \\ 108 \\ \hline 1485 \end{array}$$

(b)

$$\begin{array}{r} 49 \\ \times\ 25 \\ \hline 225 \\ 100 \\ \hline 325 \end{array}$$

(c)

$$\begin{array}{r} 49 \\ \times\ 25 \\ \hline 1250 \\ 25 \\ \hline 1275 \end{array}$$

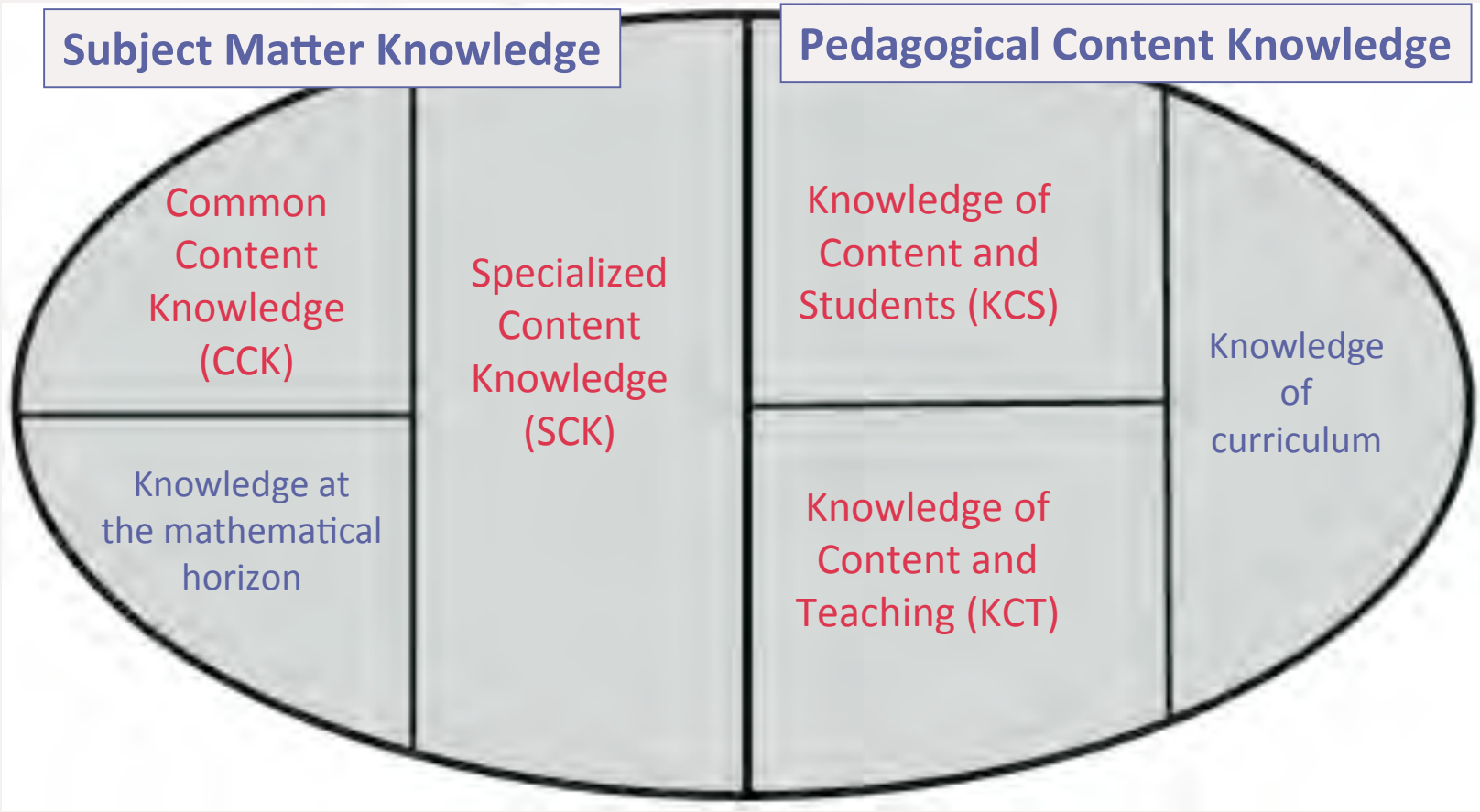
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Examples of work of teaching

1. Selecting/designing tasks
2. Identifying and working toward the mathematical goal of the lesson
3. Listening to and interpreting students' responses
4. Analyzing student work
5. Teaching students what counts as "mathematics" and mathematical practice
6. Making error a fruitful site for mathematical work
7. Attending to ambiguity of specific words
8. Deciding what to clarify, what to make more precise, what to leave in student's own language

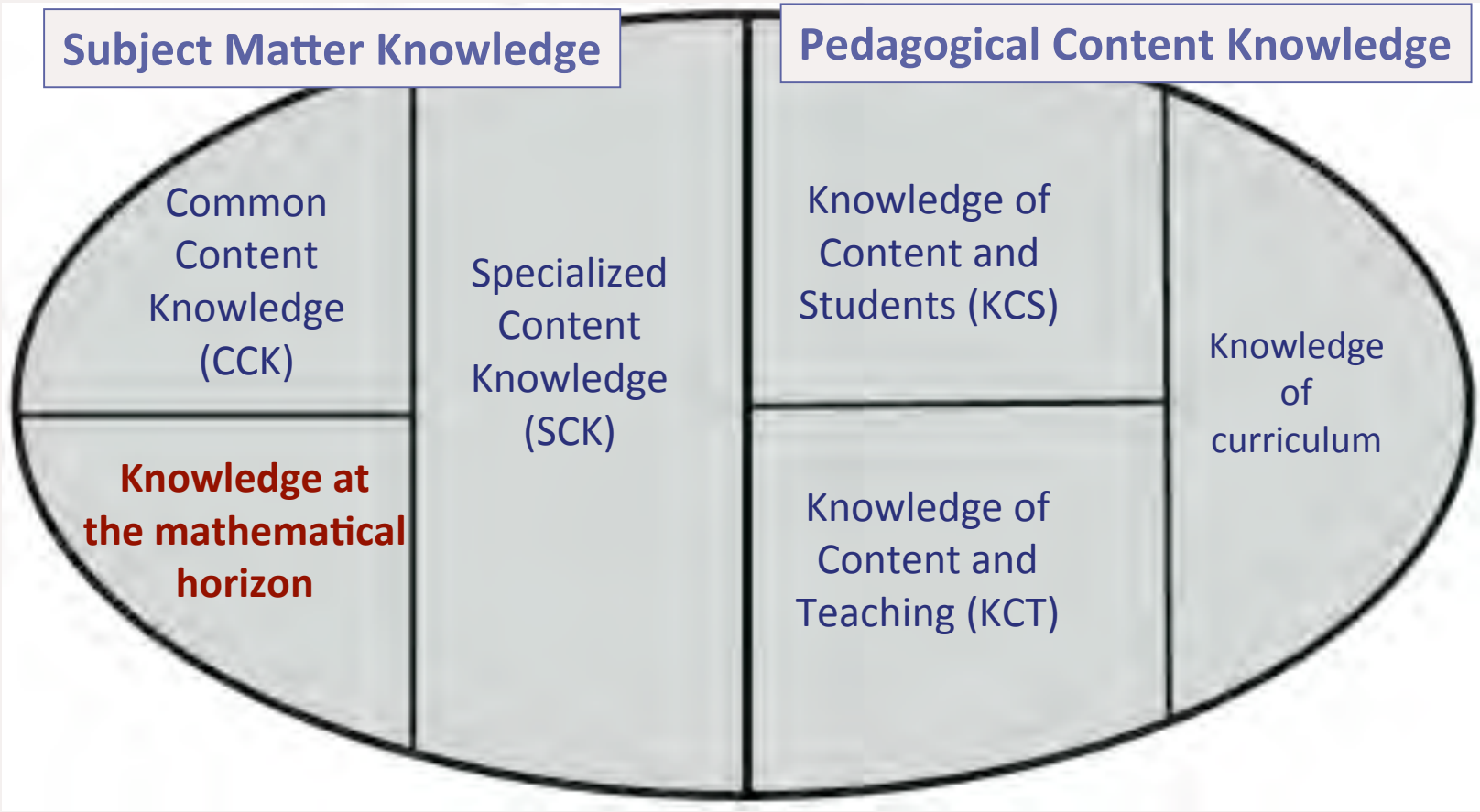
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Mathematical knowledge for teaching



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Mathematical knowledge for teaching



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Where is the mathematical horizon and how does it show up in the work of teaching?

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Video clip

- Grade 3
- Work on even and odd numbers
- Discussion of unexpected student idea
- Practices of mathematical argument and analysis
- Norms of respect

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Viewing focus

- What mathematics do you see in or implied in this segment?
- Where is the mathematical horizon?
- How might it affect the work of teaching?

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What mathematics do you see in this episode--

that the teacher—

- might need to be aware of
- might need to deal with
- might use to help navigate the situation

?

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What mathematics do you notice in this episode?

- Reasoning, and critical listening; imperative for mathematical justification
- Shea has a mathematical idea about 6; he appropriates (inappropriately) the name "even and odd"
- Three (implicit and unreconciled) definitions of "even number" in play: fair share, pair, alternating
- What would be mathematically precise definitions of even and odd usable by third graders?
- Lin clearly articulates Shea's idea, then generalizes it in order to convince him it is unreasonable
- What are these "Shea numbers?"

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Mathematical responsibilities in teaching

- –Make judgments about mathematical importance
- –Hear what students are saying
- –Highlight and underscore key points
- –Anticipate and make connections
- –Notice mathematical opportunities
- –Catch mathematical distortions or possible precursors to later mathematical confusion or misrepresentation

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Knowing mathematics for teaching to learners' mathematical futures

1. A sense of the mathematical environment surrounding current "location"
2. Major disciplinary ideas and structures
3. Key mathematical practices

1. Core mathematical values and aesthetics

1. Definitions, factorization, modular arithmetic
2. Number systems, field axioms, equations
3. Establishing equivalence, choosing representations, proving
4. Precision, consistency, efficiency

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Current thoughts about horizon knowledge as a domain of MKT

- Horizon knowledge is advanced knowledge that equips teachers with perspective for their work; it is not knowledge of the kind they need to understand in order to explain it to pupils
- Knowledge of the horizon does not create an imperative to act in any particular mathematical direction
- Some horizon knowledge is topics, some is practices, some is values
- We don't know several important things:
  - How far out or in what direction the useful horizon extends
  - How much detail is needed
  - How horizon knowledge can be acquired and developed
  - How horizon knowledge is related to student achievement

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