Knowing Your Subject Well Enough to Teach It: What More Does It Take?

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Knowing Multiplication

Multiply:

$$\begin{array}{r} 49 \\ \times\ 25 \\ \hline \end{array}$$

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Knowing multiplication for teaching

(a)

$$\begin{array}{r} 49 \\ \times\ 25 \\ \hline 405 \\ 108 \\ \hline 1485 \end{array}$$

(b)

$$\begin{array}{r} 49 \\ \times\ 25 \\ \hline 225 \\ 100 \\ \hline 325 \end{array}$$

(c)

$$\begin{array}{r} 49 \\ \times\ 25 \\ \hline 1250 \\ 25 \\ \hline 1275 \end{array}$$

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Knowing radical expressions

Simplify: $\sqrt{150}$

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Knowing radical expressions for teaching

Which of the following is best for setting up a discussion about different solution paths for simplifying radical expressions?

(a) $\sqrt{54}$

(b) $\sqrt{156}$

(c) $\sqrt{128}$

(d) These examples all work equally well.

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Overview

- Defining "mathematical knowledge for teaching"
- How well do YOU know mathematics for teaching?
- Measuring mathematical knowledge for teaching: Results

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What is the "problem"?

The quality of mathematics teaching and learning

Teachers' knowledge of mathematics and their ability to use it in practice

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Some approaches to solving the problem, and the unresolved question

- Require more mathematics for certification
  - More mathematics courses
  - A major (or minor) in mathematics
  - Mathematics test
- Recruit mathematically trained people into teaching
  - Engineers, accountants, mathematicians, …
- Fund mathematically focused professional development

But what kind of mathematical knowledge, skill, and reasoning is needed in teaching, and how can it be developed?

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Toward a practice-based theory of mathematical knowledge for teaching

1. Study instruction, and identify the mathematical work of teaching
2. Analyze what mathematical knowledge is needed to do that work effectively, and how it must be understood to be useful for the work
3. Develop, test, and refine measures of MKT using multiple methods as a means to evaluate professional education, investigate effects on students' learning, and improve theory
4. Develop and evaluate approaches to helping teachers learn mathematical knowledge for teaching

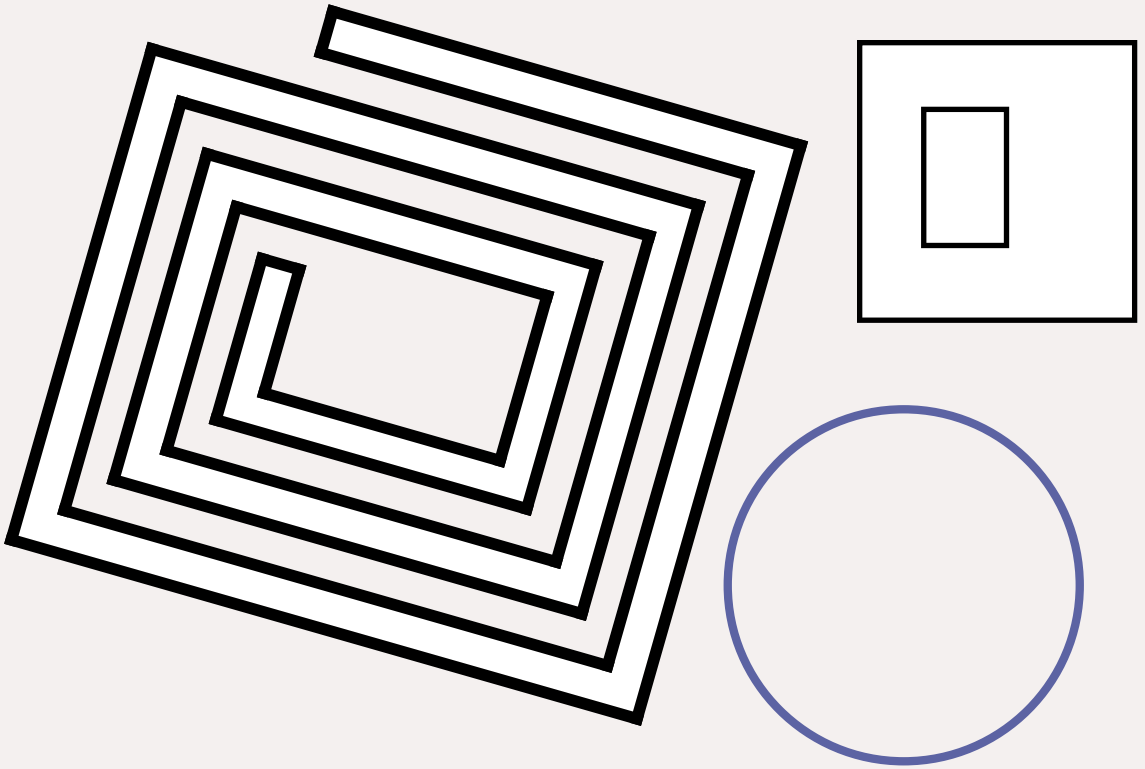
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Mathematical knowledge for teaching (MKT)

- What do we mean when we use this term, "mathematical knowledge for teaching"?
  - Mathematical knowledge, skill, habits of mind that are entailed by the work of teaching
- What do we mean by the "work of teaching"?
  - The tasks in which teachers engage, and the responsibilities they have, to teach mathematics, both inside and outside of the classroom

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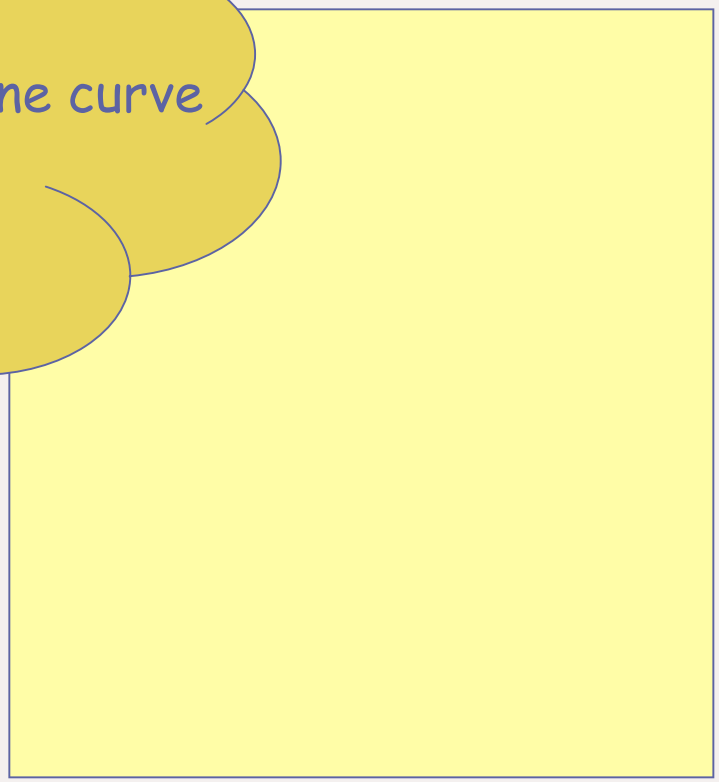
Knowing what a polygon is



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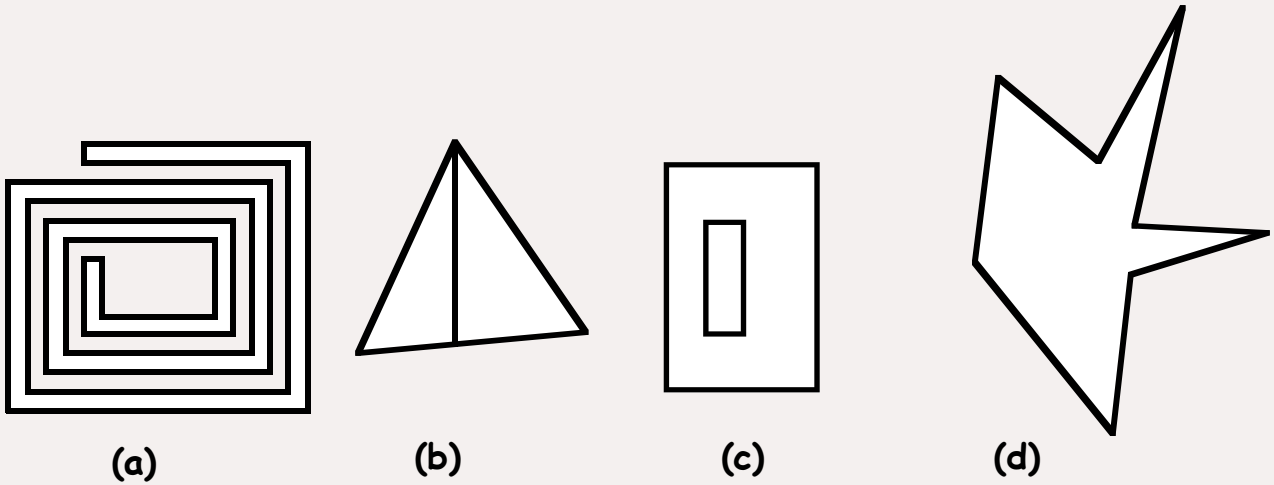
Developing a useful definition of "polygon"

A polygon is a simple closed plane curve composed of finitely many straight line segments.



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Possible examples: What is the reason for each?



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Other work of teaching

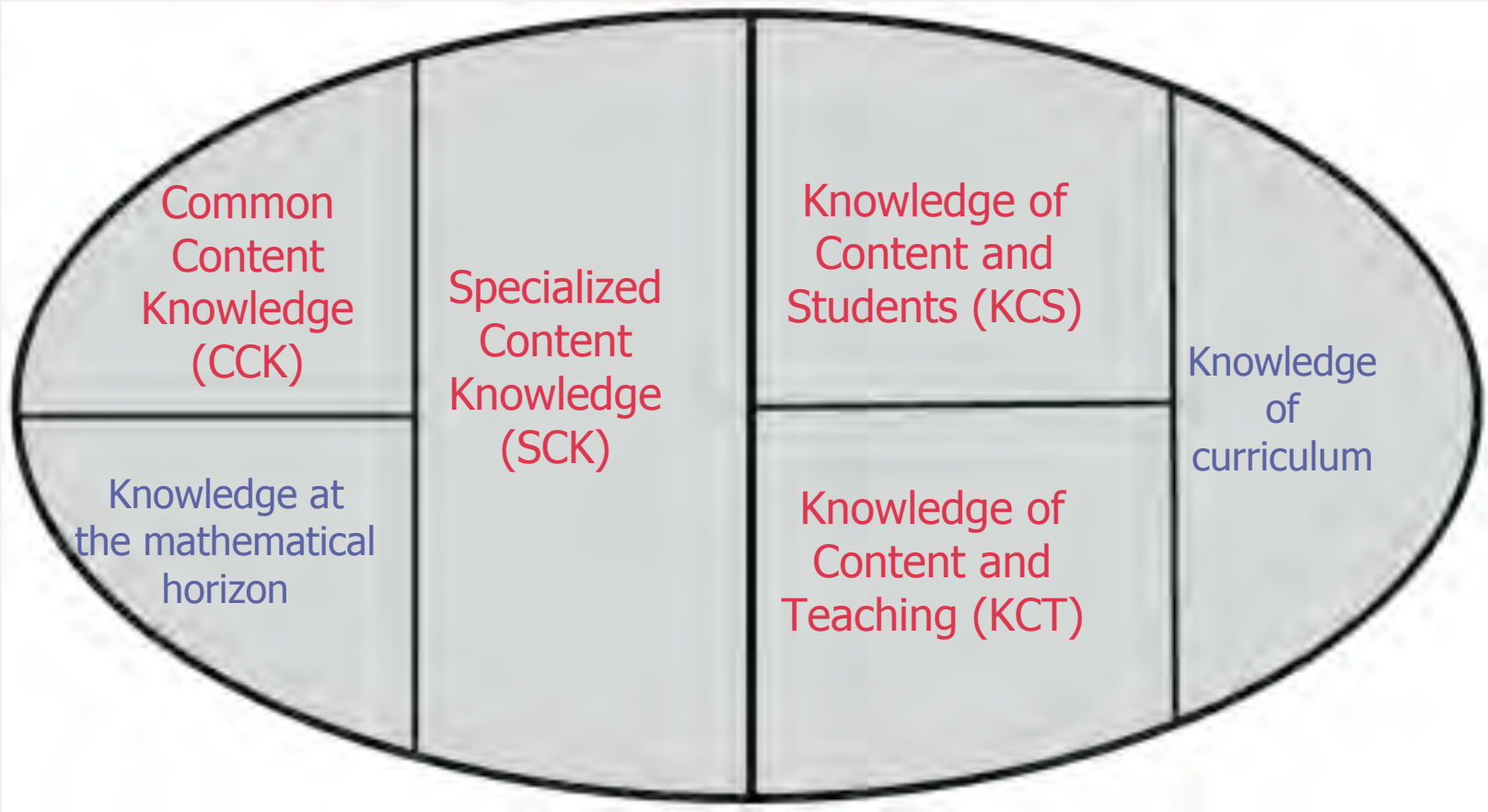
- Examining the mathematical quality of a textbook treatment of a topic
- Using and defining terms
- Supporting the linguistic demands for English language learners (ELLs)
- Producing and evaluating mathematical explanations
- Using notation
- Interpreting and evaluating alternative solutions and thinking
- Choosing contexts with care for mathematical integrity, diversity, and transparency for learning
- Explaining goals and mathematical purposes to others
- Designing homework and quizzes; Selecting and modifying mathematics problems
- Choosing and using representations
- Building correspondences between a model and a concept or procedure
- Evaluating students' work and statements (often quickly)
- Posing questions
- Choosing examples

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Mathematical knowledge for teaching

Subject Matter Knowledge

Pedagogical Content Knowledge



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How well do you know mathematics for teaching?

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Overview: Trying your hand at using mathematics in teaching

1. Noticing and identifying the mathematics in students' productions
2. Posing strategic mathematical questions to students
3. Composing usable and precise mathematical definitions
4. Writing a quiz and choosing examples

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Case 1:
Noticing and identifying the mathematics in students' productions

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Classroom Context

- Third graders (ages 8 - 10)
- Mathematics: even and odd numbers, definitions, showing why something is true
- Multi-lingual and multi-cultural class

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What do you hear and notice mathematically?

- What is going on mathematically?
- What definitions of even and odd do you hear (implicit) in children's talk?

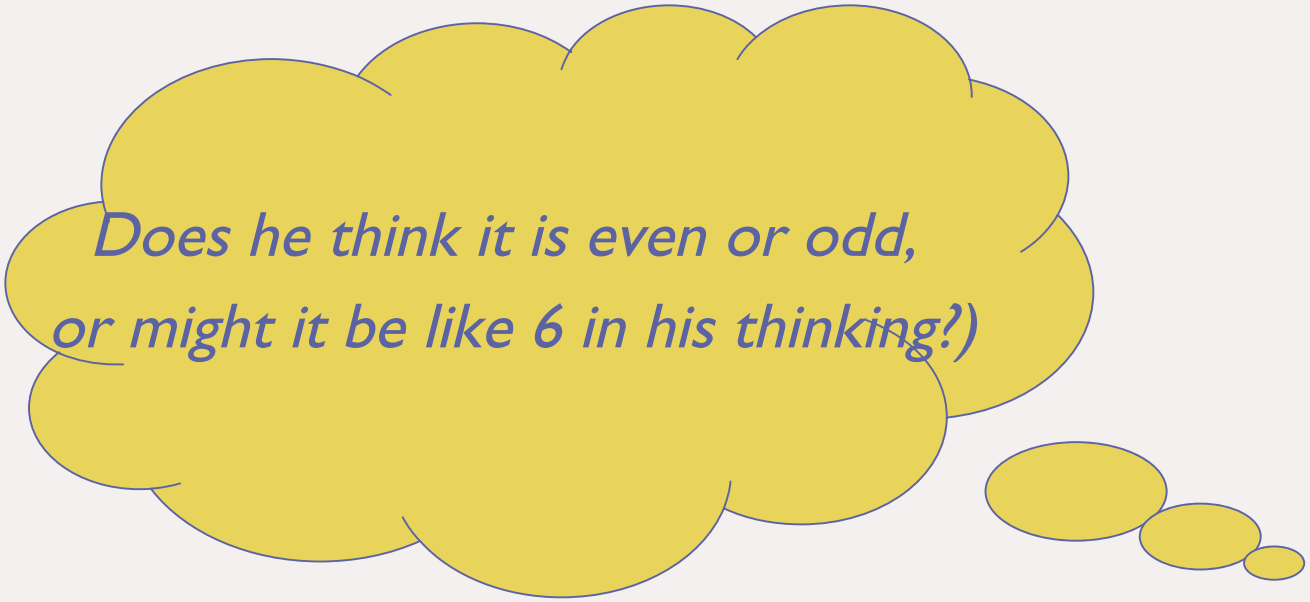
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Case 2: Posing mathematical questions to students

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What would be good to ask Shea?

"What about the number ____?"



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Case 3: Composing usable and precise mathematical definitions

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What makes a "good" definition?

- Mathematically precise — correctly identifies the kind of object, process, property
- Usable by user community — based on already-defined and understood term
- Good for growth: Will expand easily as students advance

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What is a mathematically precise and usable definition of "even number" for third graders?

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Proposed definitions for even numbers

1. An even number is a number that can be divided into two equal parts.
2. An even number is any multiple of 2.
3. An even number is any integer multiple of 2.
4. An even number is any number whose unit digit is 0, 2, 4, 6, or 8.
5. An even number is a whole number with zero remainder when divided by 2.
6. A whole number is even if it is the sum of a whole number with itself.

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Consequences

a) An even number is a number that can be divided into two equal parts.

b) An even number is any multiple of 2.

All numbers, for example 7, 3/5, $\sqrt{2}$, π, are even!

c) An even number is any integer multiple of 2.

This is a correct definition of even number, but not usable by third graders.

d) An even number is any number whose unit digit is 0, 2, 4, 6, or 8.

In this case, 36.7 is an even number!

e) An even number is a whole number with zero remainder when divided by 2.

According to this, -6 is not even.

f) A whole number is even if it is the sum of a particular whole number with itself.

This is a correct definition of evenness for whole numbers, and is consistent with the general definition for integers that students will learn later.

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Case 4: Writing a quiz and choosing examples

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Designing a quiz on even and odd numbers for Shea and his classmates

- Choose a list of numbers that you would like Shea and his classmates to classify correctly so that you can learn whether they understand what makes a number even or odd
  - Does it matter what numbers you pose? Why or why not?

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Opportunity:
Study of Instructional Improvement

- Study of three Comprehensive School Reforms; teacher knowledge a key variable
- Instrument development goals:
  - Usable with 5000 teachers: survey, multiple choice
  - Measure content knowledge teachers *use* in teaching – not just what they *know*
  - Differentiate among teachers
  - Non-partisan

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Overarching findings: Factor analyses

- Multidimensionality of mathematical knowledge for teaching
  - Knowledge of students and content different from "pure" content knowledge
  - Knowledge differentiable by topic domain (e.g., algebra, geometry)
  - Number and and operations mathematical knowledge for teaching
    - Specialized
    - Common
- Tentative conclusion: existence of a kind of "professional" knowledge for teaching

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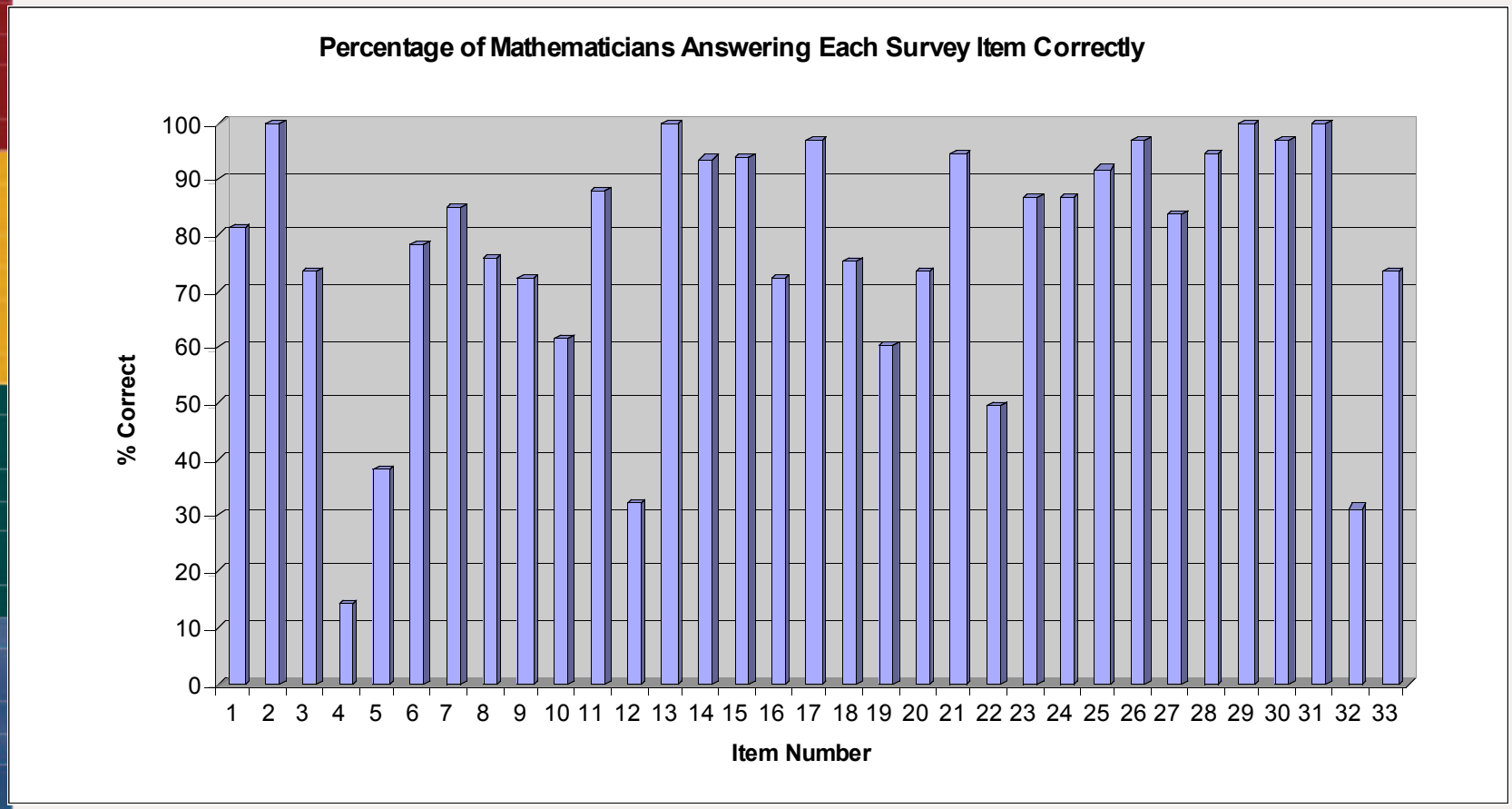
Validating our measures

How do we interpret teachers' performance on our questions?

1. Their score reflects their mathematical thinking
   - Cognitive interviews
2. Higher scores mean higher-quality mathematics instruction
   - Videotape validation study
3. Scores reflect common and specialized knowledge of content
   - Mathematician and non-teacher interviews
4. Higher scores related to improved student learning
   - Study of Instructional Improvement student gains analysis

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Percentage of mathematicians answering each survey item correctly



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Why did mathematicians get items wrong?

- Items had mathematical flaws.
- Items required knowledge of learners.
- Items demanded mathematical knowledge unique to the work of teaching:
  - Making sense of non-standard solutions or ideas
  - Choosing numerical examples
  - Choosing representations

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Which student is using a method that would work for any two whole numbers?

Student A	Student B	Student C
35 x25 125 +75 875	35 x25 175 +700 875	35 x25 25 150 100 +600 875

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Modeling multiplication

$$\begin{array}{r} .2 \\ \times\ .4 \\ \hline \end{array}$$

$$\begin{array}{r} .4 \\ \times\ .5 \\ \hline \end{array}$$

Model each problem using an area interpretation and a repeated addition interpretation of multiplication, matching each part of the problem to the drawing.

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Linking teacher knowledge and student achievement

- Questionnaire consisting of 30 items (scale reliability .88)
- Model: Student Terra Nova gains predicted by:
  - Student descriptors (family SES, absence rate)
  - Teacher characteristics (math methods/content, content knowledge)
- Teacher content knowledge significant
  - Small effect (< 1/10 standard deviation): 2 - 3 weeks of instruction
  - But student SES is also about the same size effect on achievement

(Hill,Rowan, and Ball, AERJ, 2005)

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Conclusions

1. Content knowledge is crucial to teaching effectiveness.
2. The mathematical knowledge for teaching needed is specialized and is more than common content knowledge known by any well-educated adult.
3. This kind of content knowledge in mathematics can be assessed.
4. This is a professionally-shaped way of knowing the subject. Who knows this content, and who can teach it to teachers?

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Slides will be available at

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