Learning mathematical knowledge for teaching:
Opportunities grounded in practice

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Session overview

1. Background on mathematical knowledge for teaching (MKT)
2. Designing ways for teachers to develop MKT in and for practice: Our approach
   - Overview of mod4 materials
   - Sample activities from our materials
   - Design of tasks to develop MKT
3. Previewing and piloting our materials

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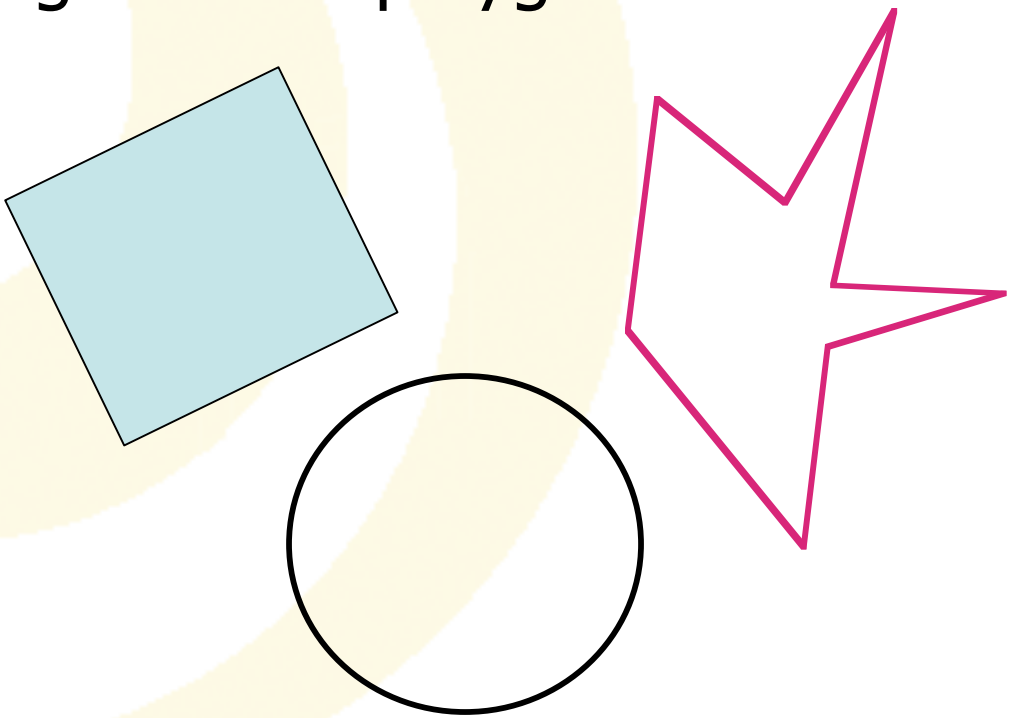
Background on mathematical knowledge for teaching and the mod4 project

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Knowing polygons

Which of these figures is a polygon?

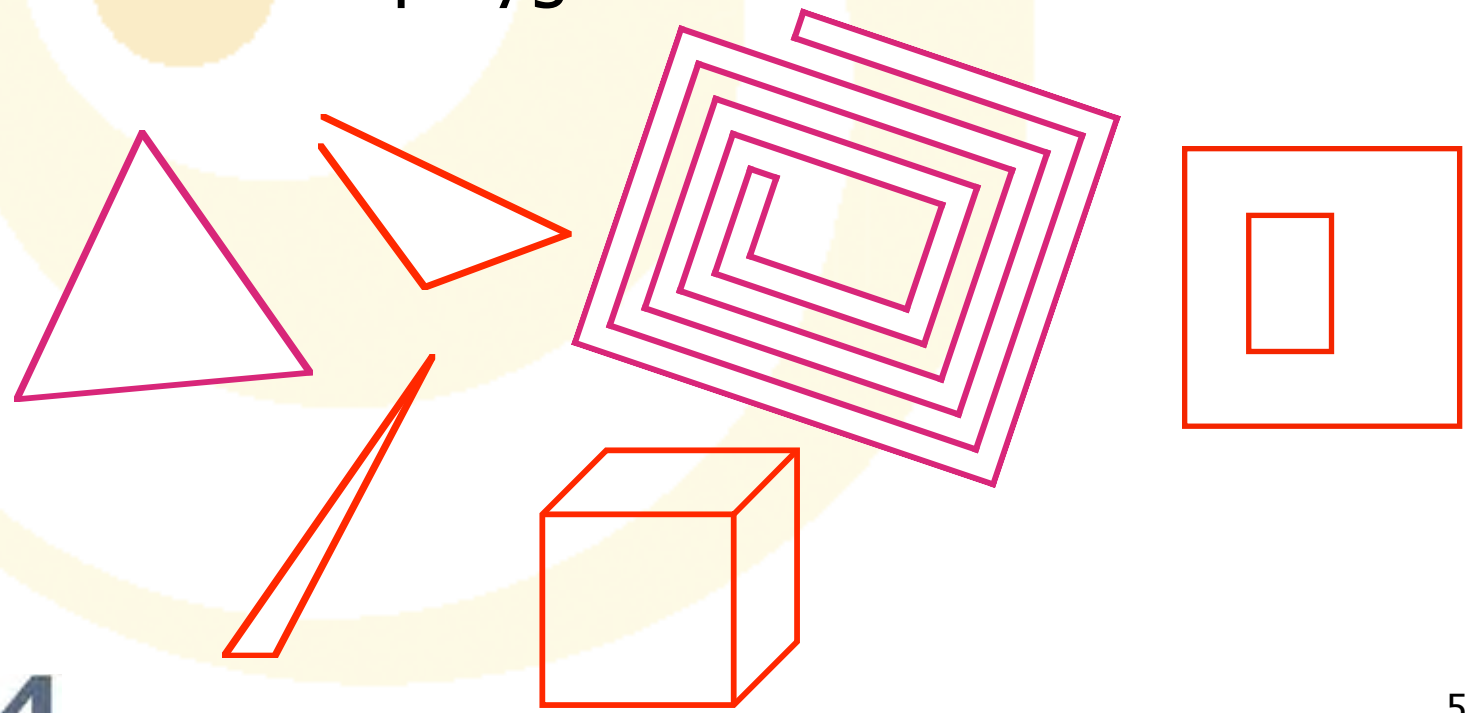


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Knowing polygons for teaching

In what ways could each of these shapes be used to precipitate a discussion of the definition of polygon?



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Mathematical knowledge for teaching (MKT)

- What do we mean when we use this term, "mathematical knowledge for teaching"?
  - Mathematical knowledge, skill, habits of mind that are entailed by the work of teaching
- What do we mean by the "work of teaching"?
  - The tasks in which teachers engage, and the responsibilities they have, to teach mathematics, both inside and outside of the classroom

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Knowing division of fractions

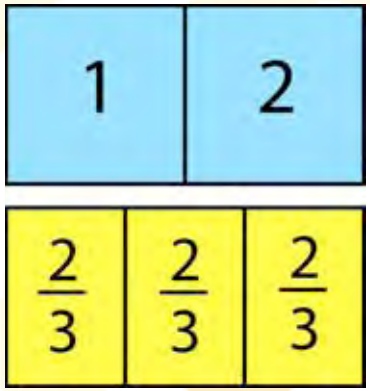
$$2 \div \frac{2}{3}$$

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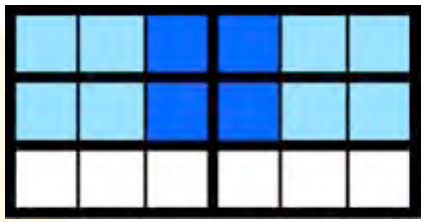
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Knowing division of fractions for teaching

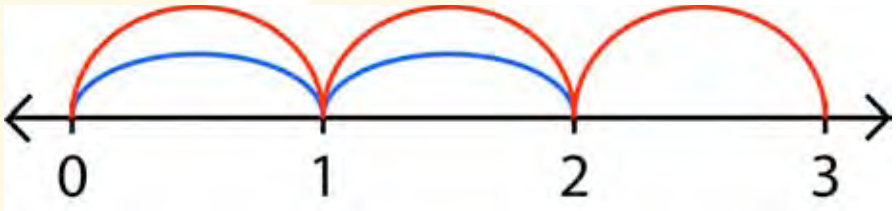
(a)



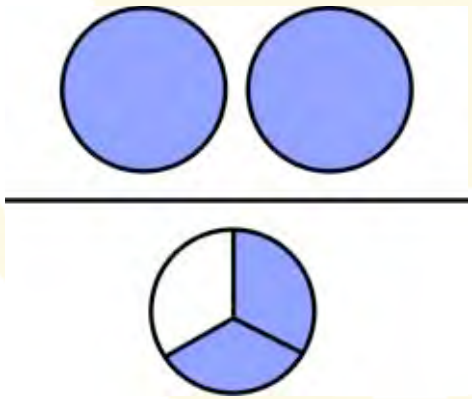
(b)



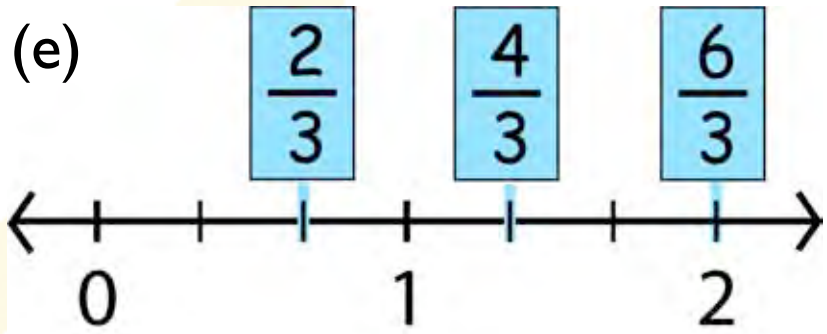
(c)



(d)



(e)



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Other tasks of teaching mathematics

- Unpacking and decomposing mathematical ideas
- Sequencing ideas
- Choosing and using representations and examples
- Explaining and guiding explanation
- Using mathematical language and notation
- Analyzing errors
- Interpreting and evaluating alternative solutions and thinking
- Analyzing mathematical treatments in textbooks
- Making mathematical practices explicit
- Attending to issues of equity (e.g., language, contexts, mathematical practices)

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Elements of our "practice-based" approach

1. Study instruction, and identify the mathematical work of teaching
2. Analyze what mathematical knowledge is needed to do that work effectively, and how it must be understood to be useful for the work
3. Develop, test, and refine measures of MKT using multiple methods as a means to evaluate professional education, investigate effects on students' learning, and improve theory
4. Develop and evaluate approaches to helping teachers learn mathematical knowledge for teaching

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An important problem: Insufficient opportunities for teachers to develop MKT

- Many of teachers' opportunities to learn mathematics are not aimed at developing the capacity to know and use mathematics in teaching
- Some teachers learn MKT from experience, but many do not
- Lack of materials for teaching mathematics to teachers in ways that enable them to know and use mathematics in practice (despite many current projects)

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Our approach to the problem

- Find/develop tasks that create opportunities for learning mathematical knowledge for teaching
- Situate teachers' opportunities to learn in the contexts of use
- Provide teachers with opportunities to practice the kinds of mathematical thinking, reasoning, and communicating used in teaching
- Enact tasks in ways that maintain the focus on developing MKT and the ability to use it in teaching

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Orientation to the mod4 materials

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Components of the mod4 materials

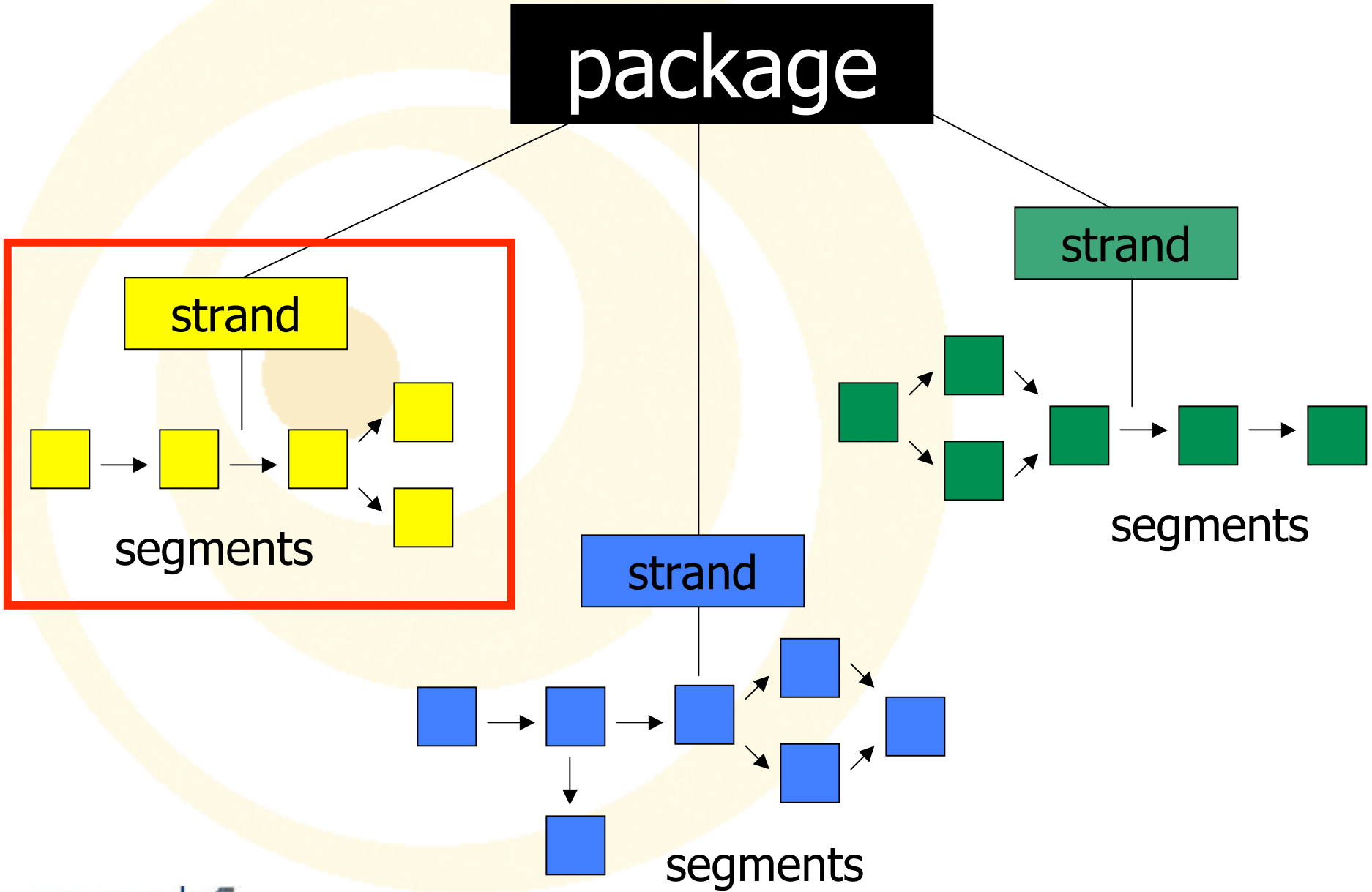
Package -- collection of instructional materials based on a particular set of records of practice

Strand -- sequence of work focused on developing a particular aspect of MKT

Segment -- instructional activities that make up a strand; each segment provides detailed descriptions of activities focusing on a particular part of the strand's theme

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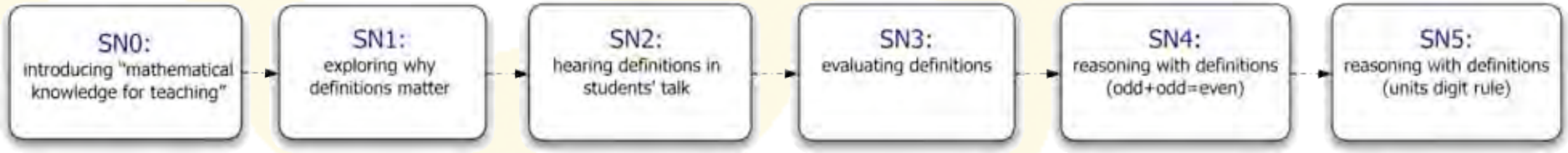
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Prototype strand:

Using definitions in mathematics learning and teaching

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Sampling activities from the materials

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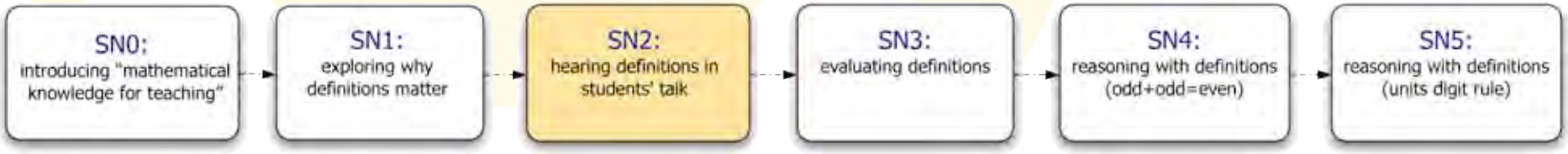
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Activity 1

From segment #2:

Hearing definitions in students' talk

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In the following student exchange,
what definitions of even and odd numbers do you hear in the way they talk about these concepts?

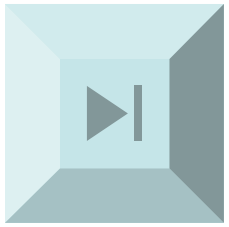
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Shea's thoughts about "six"

"I was just thinking about six, that it's a... I'm just thinking. I'm just thinking it can be an odd number, too, 'cause there could be two, four, six, and… three twos, that'd make six…"

Christina disagrees with Shea



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Christina and Shea on "six"



video

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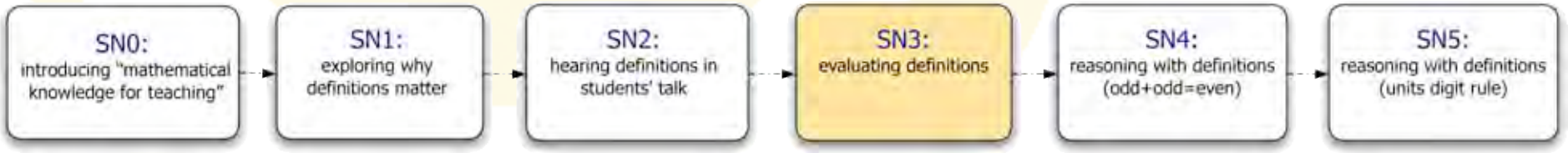
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Activity 2

From segment #3:

Evaluating definitions

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What makes a "good" definition?

- Mathematically precise — correctly identifies the kind of object, process, property
- Usable by user community — based on already-defined and understood terms

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Write a mathematically precise definition for **even number** that is also usable by third graders.

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Sample responses

a) An even number is a number that can be divided into two equal parts.

b) An even number is any multiple of 2.

c) An even number is any number whose unit digit is 0, 2, 4, 6, or 8.

d) An even number is a whole number with zero remainder when divided by 2.

e) A whole number is even if it is the sum of a whole number with itself.

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Inspecting definitions closely

a) An even number is a number that can be divided into two equal parts.

b) An even number is any multiple of 2.
All numbers, for example 7, 3/5, $\sqrt{2}$, π, are even!

c) An even number is any number whose unit digit is 0, 2, 4, 6, or 8.
In this case, 36.7 is an even number!

d) An even number is a whole number with zero remainder when divided by 2.
According to this, -6 is not even.

e) A whole number is even if it is the sum of a whole number with itself.
This is a correct definition of evenness for whole numbers, and is consistent with the general definition for integers that will arrive later.

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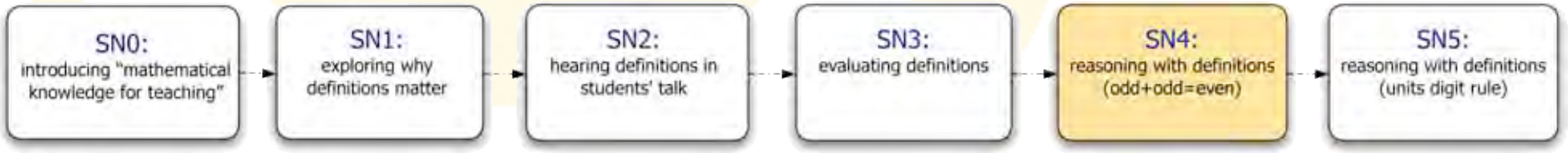
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Activity 3

From segment #4:

Reasoning with definitions

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A conjecture:

an odd number plus an odd number equals an even number

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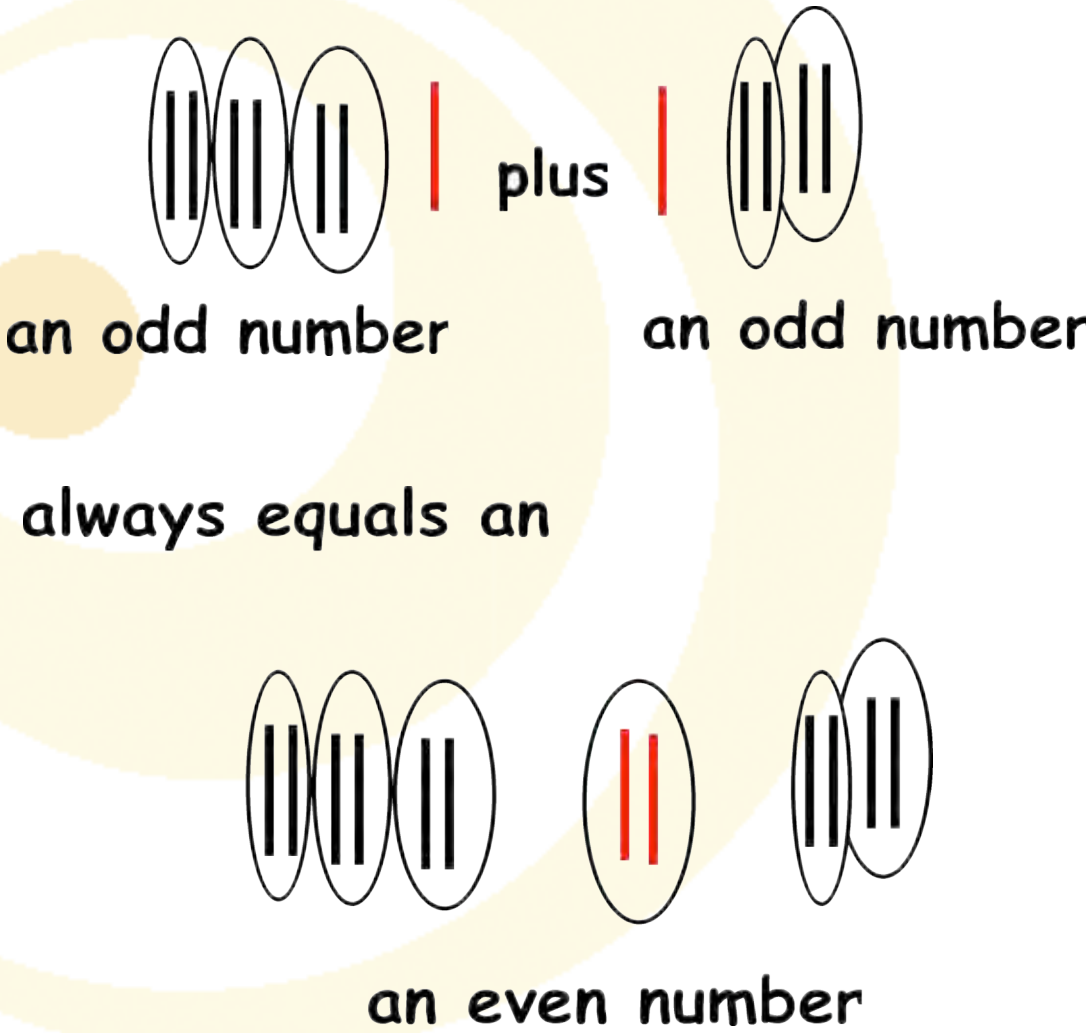
Try to "prove" the conjecture

- What might you do to convince others?
- Can you show that this conjecture is true for all numbers?
- How does your "proof" make use of definitions of odd and even?
- Think broadly about "proof": this does not have to be a formal proof that you may have done in other math classes

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Would this constitute a proof?



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Design features of tasks

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Designing tasks to develop teachers' mathematical knowledge for teaching

What is the difference between

a good mathematics task

and one that is good for developing mathematical knowledge for teaching?

Why does this matter?

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Features of tasks designed to develop MKT

Learning MATHEMATICS for teaching:

- Unpacks, makes explicit, and develops a flexible understanding of mathematical ideas that are central to understanding the school curriculum
- Provokes a stumble due to a superficial "understanding" of an idea
- Lends itself to alternative/multiple representations and solution methods
- Provides opportunities to engage in mathematical practices central to teaching (explaining, representing, using mathematical language, analyzing equivalences, proving, proof analysis

Learning TEACHING in mathematics:

- Provides opportunities to engage in teaching practices that are central to mathematics teaching (interpreting others' thinking, posing questions, writing math on the board, talking)
- Benefits from collective work; different perspectives/solutions add to an understanding of the problem or concept

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How to get involved: Previewing the prototype and participating in our current pilot

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preview of winter$_{2007}$ prototype

process:

1. sign up (online)
   http://sitemaker.umich.edu/mod4/signup
2. UM friends account
   create a login for accessing content on the University of Michigan's servers
3. online preview
   download pdfs of strand overview and lesson sketches

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$winter_{2007}$ pilot

- **what:** small exploratory investigation into the use of our prototype strand
- **when:** now - may 15
- **who:** teacher educators and professional developers interested in using one or more of the segments in their work with teachers and providing us with feedback

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$winter_{2007}$ pilot

our purposes:

- learn about the **materials** we've developed so far
- learn about the types of **contexts** in which the materials will be used
- learn about the types of **supports** and **resources** needed to use the materials
- develop and refine effective **ways to learn** from those who are using our materials

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$winter_{2007}$ pilot

process:

1. sign up (online)
2. UM friends account
3. online preview

4. information exchange regarding materials use
   go over standard uses/responsibilities & adjust for your context
5. full set of pilot materials
   obtain online access to all the documents & a binder of materials (including the videos)
6. support, feedback, site visits, camps
   learn from each other

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Conclusions

1. Knowing more about content knowledge for teaching creates an imperative to develop approaches for teachers to develop such knowledge
2. Need to develop a clearer sense of what makes a task good for teaching mathematics for teaching
3. Tasks alone are not enough; teaching mathematics for teaching requires care with enactment to keep the focus on MKT
4. New need for teacher developers to learn MKT for teaching teachers (what is this?)

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slides will be available at

http://www-personal.umich.edu/~dball/
http://sitemaker.umich.edu/mod4/nctm2007

sign up for preview materials and pilot participation at

http://sitemaker.umich.edu/mod4/signup

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