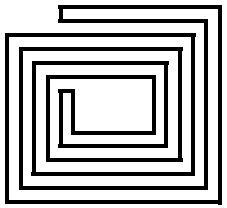
# Developing a Practice-Based Theory of Mathematical Knowledge for Teaching


**Deborah Loewenberg Ball, Heather Hill, Hyman Bass**
**Laurie Sleep, Imani Masters Goffney, Mark Thames**
**University of Michigan**
**Learning Mathematics for Teaching/Study of Instructional Improvement**

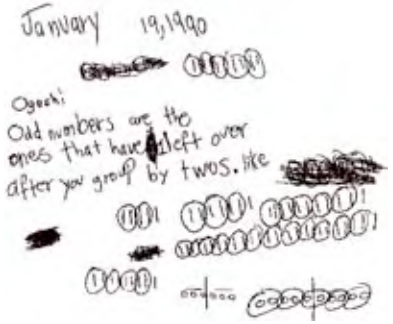

**National Council of Teachers of Mathematics**
**April 5, 2005**
**Anaheim, California**

**Length of side of main square**

| Possible sized squares | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 10 | 50 |
|---|---|---|---|---|---|---|---|---|---|
| 1x1 | 1 | 4 | 9 | 16 | 25 | 36 | 49 | | |
| 2x2 | | 1 | 4 | 9 | 16 | 25 | 36 | | |
| 3x3 | | | 1 | 4 | 9 | 16 | 25 | | |
| 4x4 | | | | 1 | 4 | 9 | 16 | | |
| 5x5 | | | | | 1 | 4 | 9 | | |
| 6x6 | | | | | | 1 | 4 | | |
| 7x7 | | | | | | | 1 | | |
| 8x8 | | | | | | | | | |
| 9x9 | | | | | | | | | |
| 10x10 | | | | | | | | | |
| Total | 1 | 5 | 14 | 30 | 55 | 91 | 140 | | |








(B) 1 2
35
x 25
255
80
1055



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# Overview

1. What is "mathematical knowledge for teaching"?
2. Efforts to "measure" teachers' mathematical knowledge: Purposes, history
3. Our measures development approach
4. Examples of measures use
5. Writing items

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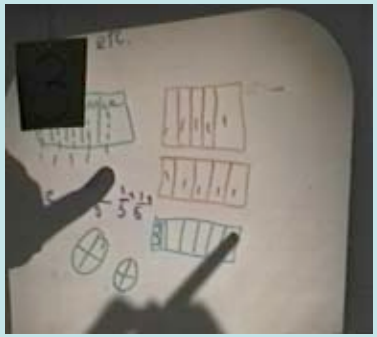
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# 1. What Do We Mean By "Mathematical Knowledge for Teaching"?



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# Common Framing of the Problem → Failure to Solve the Problem

- Teachers do not know mathematics well enough (evidence and causes vary)
- Concentration of under-qualified teachers in urban and high poverty schools

- Increase number and rigor of mathematics requirements for teacher certification
- Recruit mathematically-trained people into teaching



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# What *is* "Mathematical Knowledge for Teaching"? An Example from Multiplication of Decimals

**Multiply:**

$$\begin{array}{r} 3.5 \\ \times\ 2.5 \\ \hline \end{array}$$

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# Analyzing Incorrect Answers for $\begin{array}{r} 3.5 \\ \times\ 2.5 \\ \hline \end{array}$

**Why** do you multiply before you add?

(a)
$$\begin{array}{r} 3.5 \\ \times\ 2.5 \\ \hline 255 \\ 80 \\ \hline 10.55 \end{array}$$

(b)
$$\begin{array}{r} 3.5 \\ \times\ 2.5 \\ \hline 62.5 \end{array}$$

**Why** do you have as many decimal places in the answer as the total number of decimal places in the problem?



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# Analyzing Correct Answers for

$$\begin{array}{r} 3.5 \\ \times\ 2.5 \\ \hline \end{array}$$

(a)

$$\begin{array}{r} 3.5 \\ \times\ 2.5 \\ \hline .25 \\ 1.5 \\ 1 \\ 6 \\ \hline 8.75 \end{array}$$

(b)

$$\begin{array}{r} 3.5 \\ \times\ 2.5 \\ \hline 1.25 \\ 7.5 \\ \hline 8.75 \end{array}$$

Is there a method?
Does it work for
all decimal numbers?
**How do you know?**

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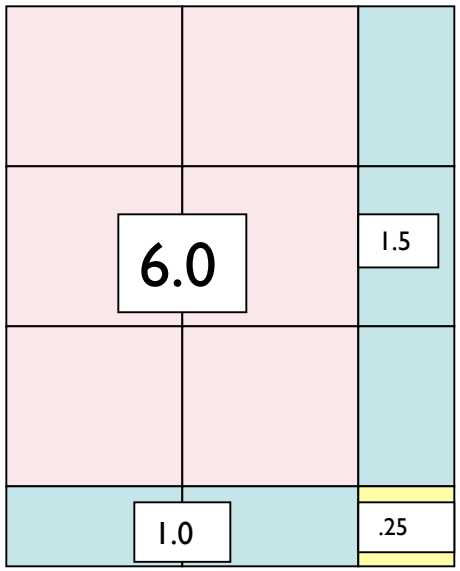
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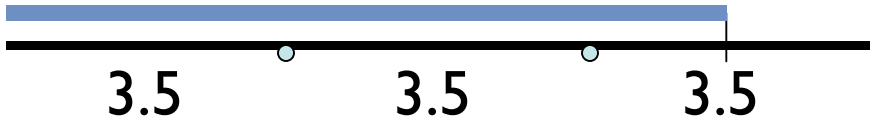
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# Using Representations for $3.5 \times 2.5$

(a)




(b)




**Explain:**
Where is the 3.5?
Where is the 2.5?
Where is 8.75?

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# Mathematical Knowledge for Teaching

What do we mean when we use this term, "**mathematical knowledge for teaching**"?

- Mathematical knowledge, skill, habits of mind that are entailed by the work of teaching

What do we mean by the "**work of teaching**"?

- The tasks in which teachers engage, and the responsibilities they have, to teach mathematics, both inside and outside of the classroom



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# Hypotheses About Knowledge of Mathematics for Teaching

## PREVALENT HYPOTHESES

1. Teachers need knowledge of the curriculum, plus N levels more knowledge
2. Pedagogical content knowledge, curricular knowledge

## OUR CURRENT HYPOTHESES

- **Common** content knowledge
- **Specialized** content knowledge
- Knowledge of content and **students**
- Knowledge of content and **teaching and curriculum**

**What does this have to do with "pedagogical content knowledge"?**

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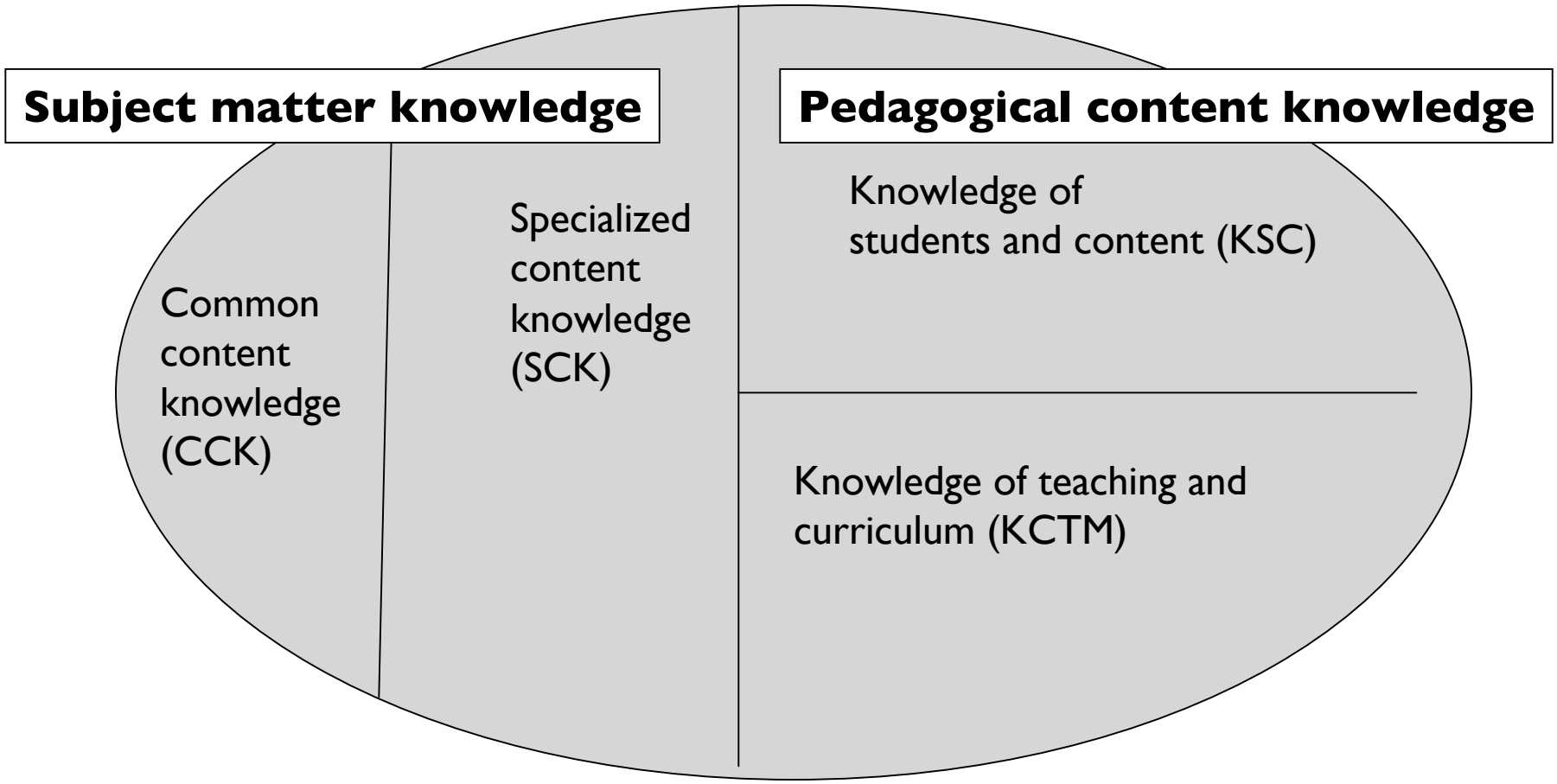
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# Shulman's Original Category Scheme (1985) Compared with Ours



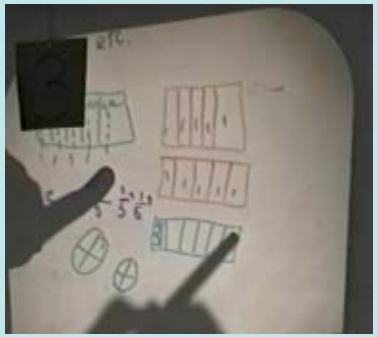

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# 2. Efforts to "Measure" Teachers' Mathematical Knowledge: Purposes and History



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# *Why* Would We Want to "Measure" Teachers' Content Knowledge for Teaching?

- To understand role of teachers' content knowledge in students' performance
- To study and compare outcomes of professional development and teacher education
- To inform design of teachers' opportunities to learn content knowledge



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# Study of Instructional Improvement

- Study of three Comprehensive School Reforms; teacher knowledge a key variable
- Instrument development goals:
  - Develop measures of content knowledge teachers *use* in teaching – not just *what* they teach
  - Develop measures that discriminate among teachers (not criterion referenced)
  - Non-partisan

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# Problems As We Began This Work

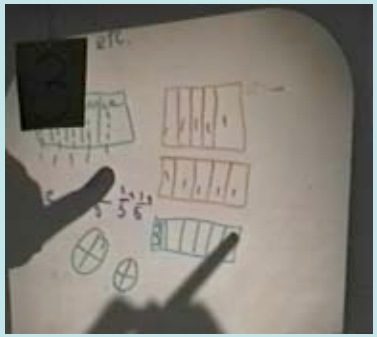
- No way to measure teachers' content knowledge for teaching on a large scale
  - Small number of items, many written by Post, Ball, others appeared on every instrument
  - Nothing known about the statistical qualities of those items (difficulty, reliability)
  - Studies relying on single items -- single items unlikely valid or reliable measures of teacher knowledge



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# 3. Our Measures Development Approach



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# Early Decisions and Activity

- Survey-based measure of content knowledge for teaching mathematics
  - 3000 teachers participating in SII
  - Multiple choice
- Specified domain map
- 5 people + 5 lbs cheese + 5 weeks = 150 items (May 2001)
- Large-scale piloting, summer 2001

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# Original Sampling Frame

| | Types of knowledge | | |
|---|---|---|---|
| Mathematical content | | Common and specialized content knowledge | Using knowledge of students and content |
| | Number | | |
| | Operations | | |
| | Patterns, functions, and algebra | | |

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# Sample Items
# K-5 Number and Operations

- **Common knowledge**

  **What number is halfway between 1.1 and 1.11?**

- **Specialized knowledge**

  1. Representing mathematical ideas and operations
  2. Providing explanations for mathematical ideas and procedures
  3. Evaluating mathematical methods, claims, or solutions



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# Representing Number Concepts

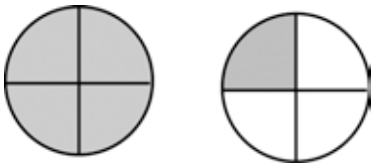
Mrs. Johnson thinks it is important to vary the whole when she teaches fractions. For example, she might use five dollars to be the whole, or ten students, or a single rectangle. On one particular day, she uses as the whole a picture of two pizzas. What fraction of the two pizzas is she illustrating below? (Mark ONE answer.)

a) 5/4

b) 5/3

c) 5/8

d) 1/4





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# Providing Mathematical Explanations

Ms. Harris was working with her class on divisibility rules. She told her class that a number is divisible by 4 if and only if the last two digits of the number are divisible by 4. One of her students asked her why the rule for 4 worked. She asked the other students if they could come up with a reason, and several possible reasons were proposed. Which of the following statements comes closest to explaining the reason for the divisibility rule for 4? (Mark ONE answer.)

a) Four is an even number, and odd numbers are not divisible by even numbers.

b) The number 100 is divisible by 4 (and also 1000, 10,000, etc.).

c) Every other even number is divisible by 4, for example, 24 and 28 but not 26.

d) It only works when the sum of the last two digits is an even number.



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# Evaluating Alternative Solution Approaches

| Student A | Student B | Student C |
|---|---|---|
| 35<br>x25<br>125<br>+75<br>875 | 35<br>x25<br>175<br>+700<br>875 | 35<br>x25<br>25<br>150<br>100<br>+600<br>875 |

**Which of these students is using a method that could be used to multiply any two whole numbers?**



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# Using Data to Test and Improve Theory

- Factor analyses
- Analyses of validity
- Uses of measures
  - To predict student achievement
  - To evaluate professional development



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# Factor Analysis

- Enables analyses of the number of underlying factors in a data set
- Can link items to constructs
- Ask similar question of our data:
  - How many factors? What are they named?
  - How do items group?



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# Overarching Findings: Factor Analyses

- Multidimensionality of mathematical knowledge for teaching
  - Number and and operations mathematical knowledge for teaching
    - Specialized
    - Common
  - Patterns, functions & algebra content knowledge for teaching
  - Geometry mathematical knowledge for teaching
  - Knowledge of students and content
- Tentative conclusion: presence of professional knowledge for teaching

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# Validating Our Measures

How do we interpret teachers' performance on our questions?

1. Their score reflects their mathematical thinking
   - *Cognitive interviews*
2. Higher scores mean higher-quality mathematics instruction
   - *Videotape validation study*
3. Scores reflect common and specialized knowledge of content
   - *Mathematician and non-teacher interviews*
4. Higher scores related to improved student learning
   - *Study of Instructional Improvement student gains analysis*

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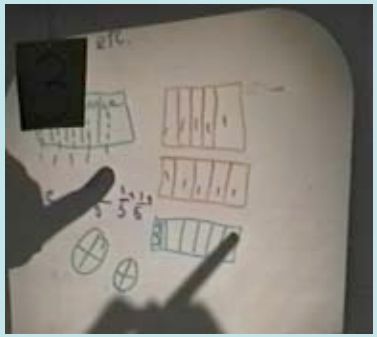
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# 4. Examples of Measures Use



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# (a) First Example of Use: Predicting Student Achievement



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# Links to Study of Instructional Improvement Student Achievement Analysis

- Question: Do students learn more when teachers perform better on our measures?
  - SII data and analysis – measured CKT on questionnaire
  - SII measures highly reliable, from test construction perspective
- Model: Student Terra Nova gains predicted by:
  - Student descriptors (family SES, absence rate)
  - Teacher characteristics (math methods/content, content knowledge)
- Teacher content knowledge significant
  - Small effect (< 1/10 standard deviation)
  - But student SES is also on same order of magnitude

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# (b) Second Example of Use: Evaluating Teacher Professional Development



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# Example: Tracking Teacher Growth

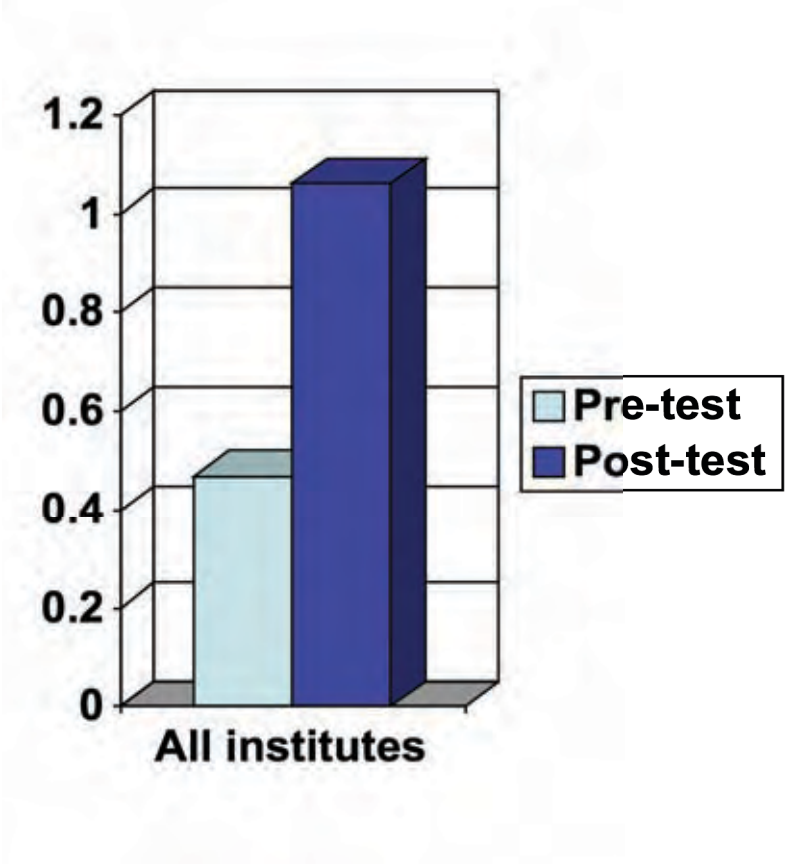
- Items piloted in California's Mathematics Professional Development Institutes (MPDI)
  - Instructors: Mathematicians and mathematics educators
  - 40-120 hours of professional development
  - Focus is squarely on mathematics content
  - Summer 2001
  - Pre/post assessment format (parallel forms)



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# MPDI Teacher Growth

- For all institutes for which we have data, teachers gained .48 logits, or roughly ½ standard deviation
- Translates to 2-3 item increase on assessment
- Considered substantial gain




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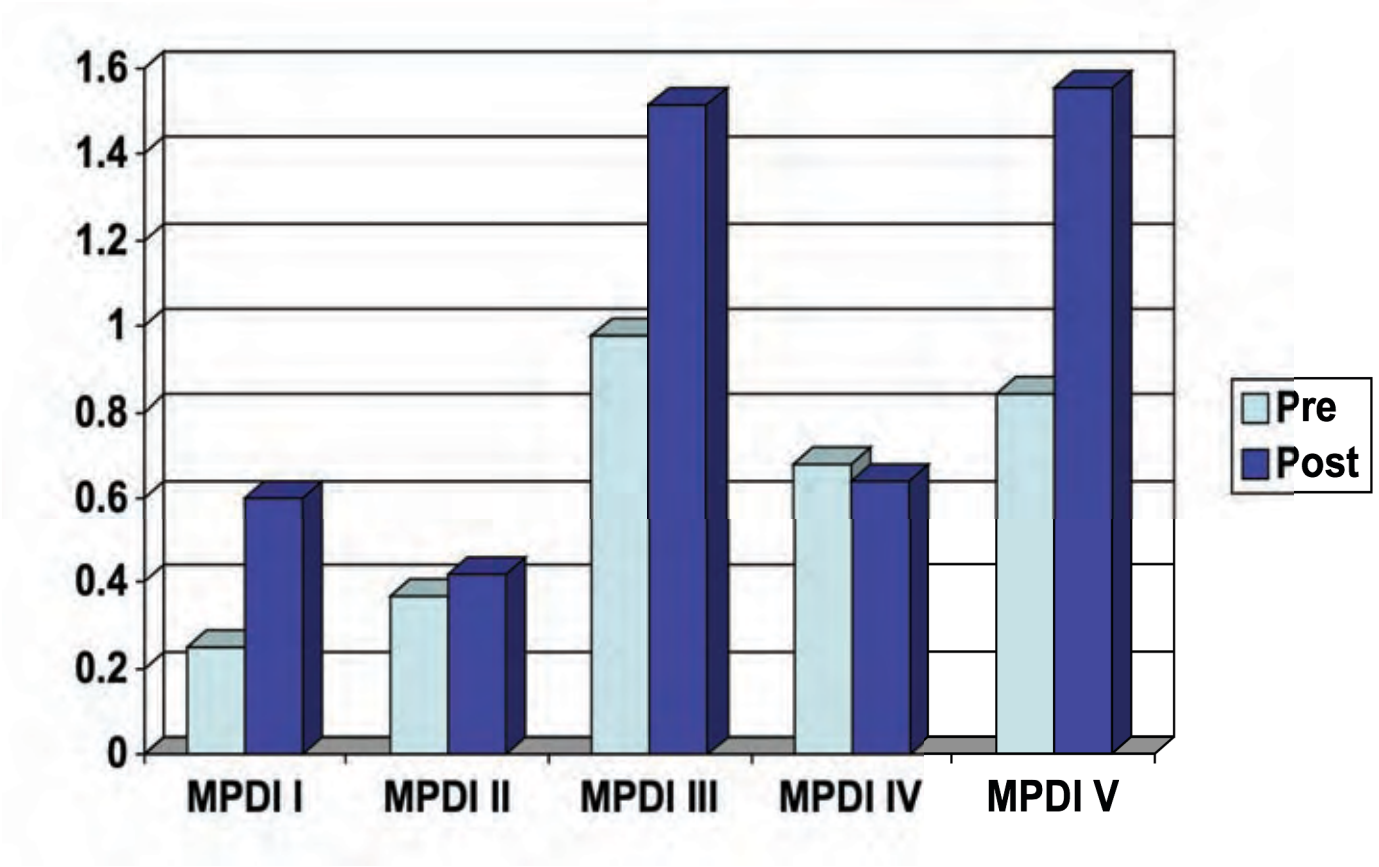
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# Results from Sample Institutes




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# MPDI Evaluation: Other Findings

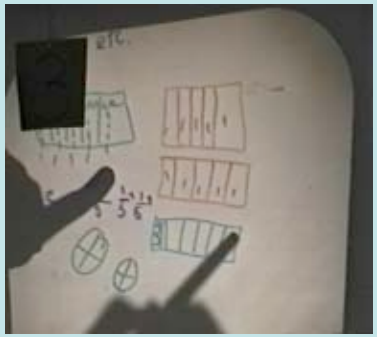
- Length of institute predicts teacher gains
  - 120-hour institutes most effective, on average
  - But some 40-hour institutes very effective
- Focus on mathematical analysis, proof, and communication leads to higher gains
- Many questions remain
  - Effects of content (e.g., mathematics vs. student thinking)
  - Treatment of content: "packed"/"unpacked"
  - Effects of teacher motivation



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# 5. Trying Your Hand at Writing Items



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# Criteria for Writing Items

- Items must be unambiguously right or wrong
- Think of clever distractors—what would someone with little knowledge think?
- Write easy, medium, and hard items
- Ignore (or subvert) ideologies about teaching mathematics
- Avoid excess verbiage



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# Criteria for Writing Items

- Write items that reflect how mathematics is <u>used in teaching</u>
  - Use your knowledge of classrooms and teachers – what issues arise? What do new teachers struggle with?



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# Conclusions

- It is possible to write questions that reliably and validly measure mathematical knowledge for teaching.
- Such questions are more likely to seem credible to teachers.
- The work of producing usable items requires multiple kinds of expertise, a good conceptual frame, and many cycles of revision and improvement.

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