Designing and Using Problems to Teach Mathematical Knowledge for Teaching

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National Council of Teachers of Mathematics Research Presession
Salt Lake City, UT
April 8, 2008

mod4 @ M UNIVERSITY OF MICHIGAN

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Overview of session

1. Background on mathematical knowledge for teaching (MKT)
2. Framework for MKT task design
3. Examining sample MKT tasks together
4. Discussion: Tasks for developing MKT
5. Conjectures about "enactment" of MKT tasks
6. Current and future work

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Background on mathematical knowledge for teaching (MKT)

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The problem

How can we improve student learning?

Teachers' mathematical knowledge is a key factor shaping what students are able do.

What mathematical knowledge do teachers need for their work?

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Elements of our "practice-based" approach

1. Study instruction and identify the *mathematical work of teaching*
2. Analyze *what mathematical knowledge is needed* to do that work effectively, and how it must be understood to be useful for the work
3. Develop, test, and refine *measures* of MKT using multiple methods as a means to evaluate professional education, investigate effects on students' learning, and improve theory
4. Develop and evaluate approaches to *helping teachers learn* mathematical knowledge for teaching

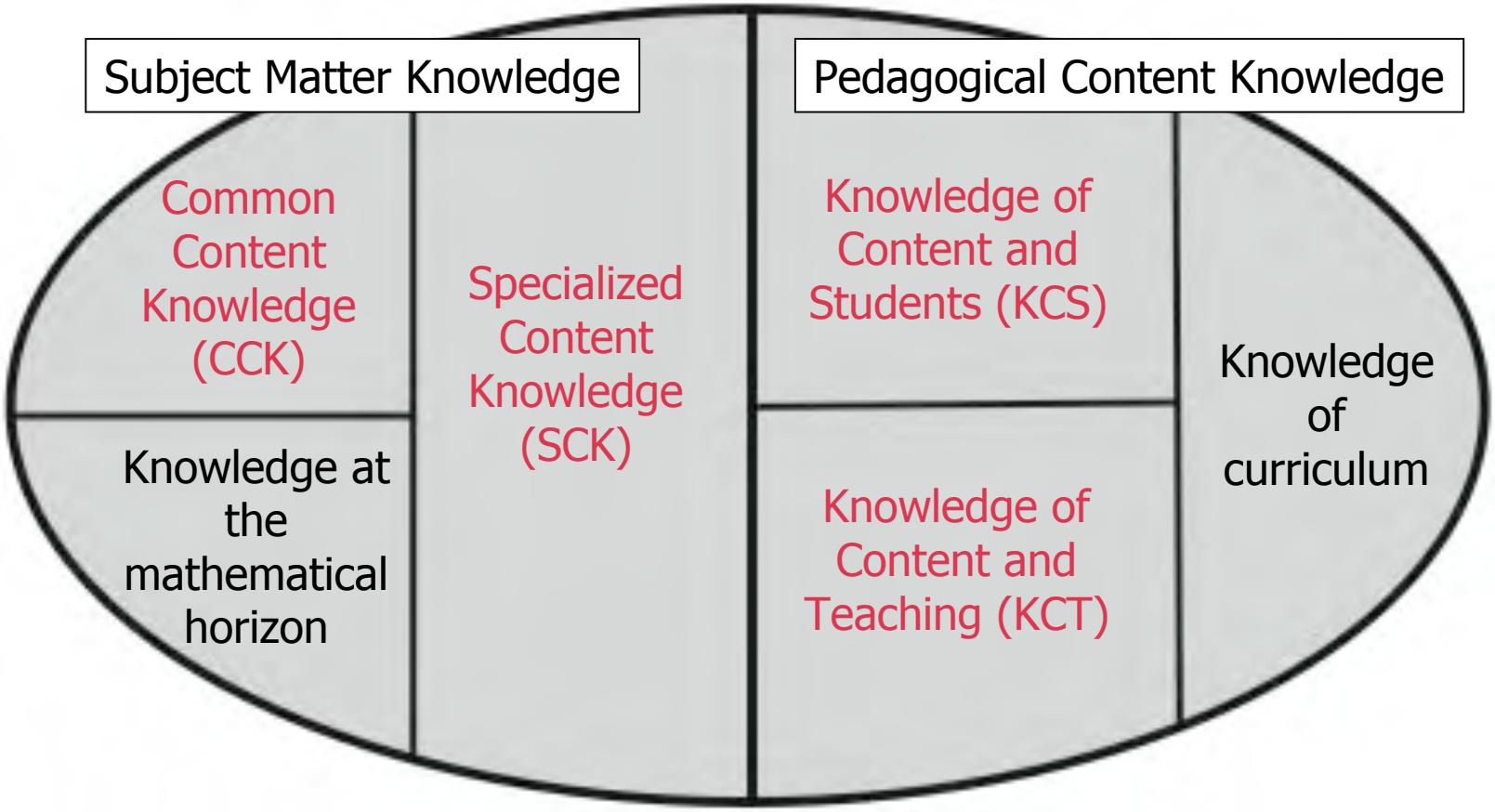
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Mathematical knowledge for teaching (MKT)

- Frame: knowledge "entailed by the work of teaching"
  - Knowledge used or needed in practice.
- What do we mean by "knowledge"?
  - Mathematical knowledge, skill, habits of mind
- What do we mean by the "work of teaching"?
  - The activities in which teachers engage, and the responsibilities they have, to teach mathematics, both inside and outside of the classroom

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Mathematical knowledge for teaching



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Common content knowledge (CCK)

Calculate:

$$\frac{5}{6} \div \frac{1}{3}$$

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Specialized content knowledge (SCK)

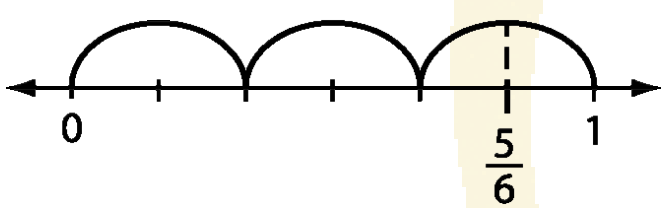
$$\frac{5}{6} \div \frac{1}{3} = \frac{10}{12} \div \frac{4}{12} = 10 \div 4 = 2\frac{1}{2}$$

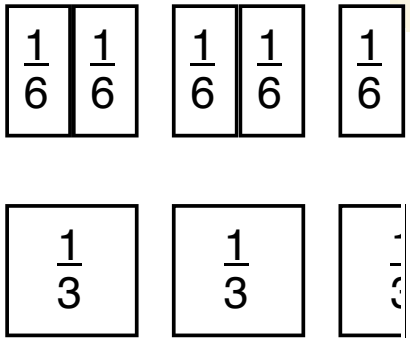
Is this a fluke?
Does it work in general?
If so, why does it work?

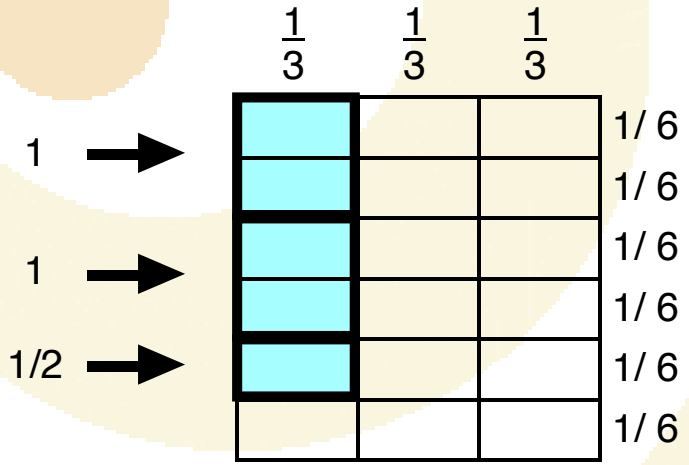
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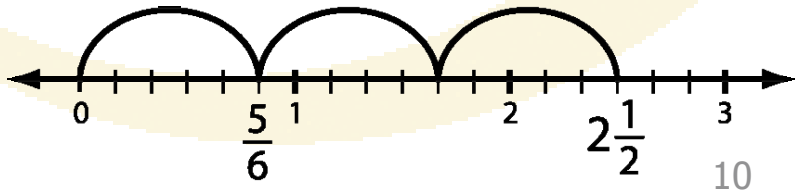
Specialized content knowledge (SCK)

Which of these can be used to represent $\frac{5}{6} \div \frac{1}{3}$?









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$$\frac{5}{6} \div \frac{1}{3} = 2\frac{1}{2}$$ (PCK)

Knowledge of students and content (KCS)

- What are common errors students make when dividing fractions?
- How do students' experiences with division of whole numbers support their understanding of division of fractions? How does it confuse them?
- What difficulties do students typically have interpreting the answer to a division of fractions problem?

Knowledge of teaching and content (KCT)

- Which representation would you use to introduce the meaning of division of fractions? Or to explain the 'invert and multiply' algorithm?
- What sequence of problems would you use to begin work on division of fractions?
- In a whole-class discussion, what solution methods would you want presented, and in what order?

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Other tasks of teaching mathematics

- Responding to students' "why" questions
- Unpacking and decomposing mathematical ideas
- Explaining and guiding explanation
- Using mathematical language and notation
- Generating examples
- Sequencing ideas
- Choosing and using representations
- Analyzing errors
- Interpreting and evaluating alternative solutions and thinking
- Analyzing mathematical treatments in textbooks
- Making mathematical practices explicit
- Attending to issues of equity (e.g., language, contexts, mathematical practices)

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Framework for MKT task design

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The challenge

How can MKT be developed? How can opportunities for learning MKT be provided?

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Our approach to the problem

- Find/develop tasks that create opportunities for learning mathematical knowledge for teaching
- Situate teachers' opportunities to learn in the contexts of use
- Provide opportunities to practice the kinds of mathematical thinking, reasoning, and communicating used in teaching
- Enact tasks in ways that maintain the focus on developing MKT and the ability to use it in teaching

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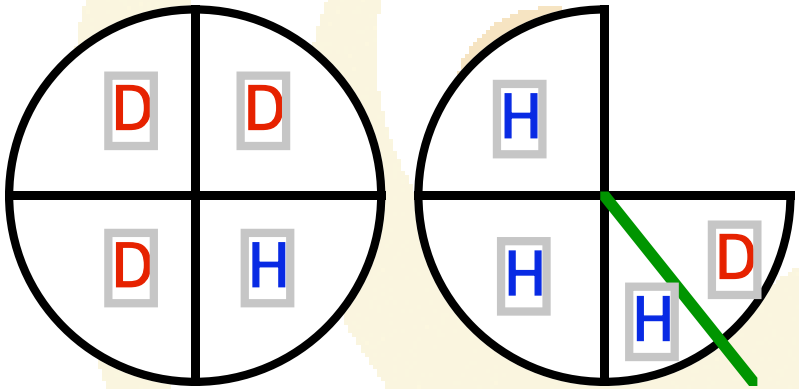
Division of fractions

$$1\frac{3}{4} \div \frac{1}{2}$$

1. Calculate the answer.
2. Write a story problem, or describe a situation, that corresponds to 1 3/4 ÷ 1/2.

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I have two pizzas. My friend eats one quarter of one of the pizzas. I have one and three quarters pizzas left. Then I split it evenly between two of my other friends. Each person gets three and a half pieces of pizza.



1. What is wrong with this?
2. Write a story problem that correctly represents the division.

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Important features of the task design

- Unpacks, makes explicit, and develops a flexible understanding of mathematical ideas that are central to the school curriculum
- Opens opportunities to build connections among mathematical ideas
- Provokes a stumble due to a superficial "understanding" of an idea
- Lends itself to alternative/multiple representations and solution methods
- Provides opportunities to engage in mathematical practices central to teaching (explaining, representing, using mathematical language, analyzing equivalences, proving, proof analysis, posing questions, writing on the board)

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The special problem of **specialized content knowledge**

As a field, we have relatively little experience creating opportunities for people to develop the specialized knowledge of mathematics (SCK) needed for teaching

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Designing tasks to develop SCK

What is the difference between
a good mathematics task
and one that is good for developing (specialized)
mathematical knowledge for teaching?

Why does this matter?

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Small group work: Three SCK tasks

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SCK task #1:
Writing division story problems

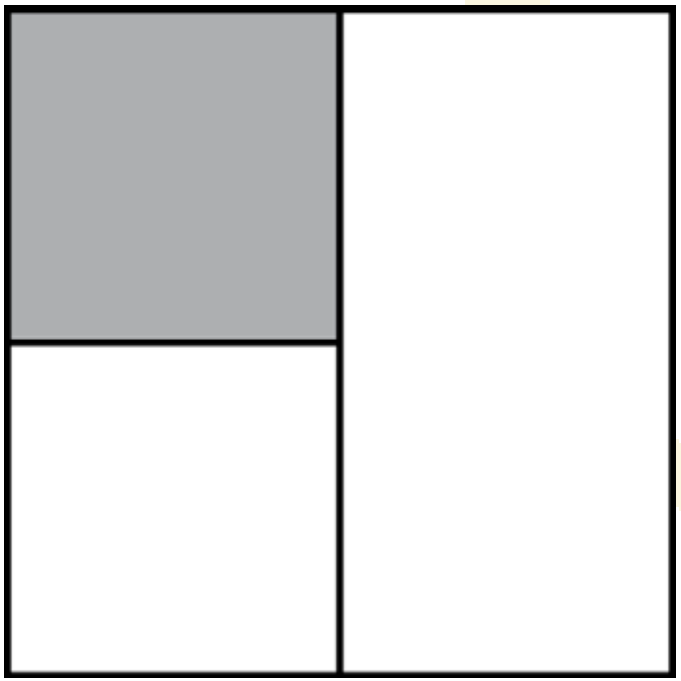
$$38 \div 4$$

Write as many different stories as you can that correspond to this division expression, representing different interpretations of the meaning of division, and with different possible numerical answers, depending on the context.

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SCK task #2: Analyzing solutions

What fraction is represented here?



What reasoning could produce each of these answers?

$\frac{1}{2}$ $\frac{1}{4}$ $\frac{1}{3}$ $1\frac{1}{2}$

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SCK task #3:
Analyzing errors

42	24	3.4	283
x 83	x 53	x 2.4	x 4
326	72	136	2062
	120	680	
	192	81.6	

For each problem:

- Do the problem correctly and explain the procedure you use and why it works.
- What mathematical steps likely produced each incorrect answer?

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Questions to consider when examining tasks

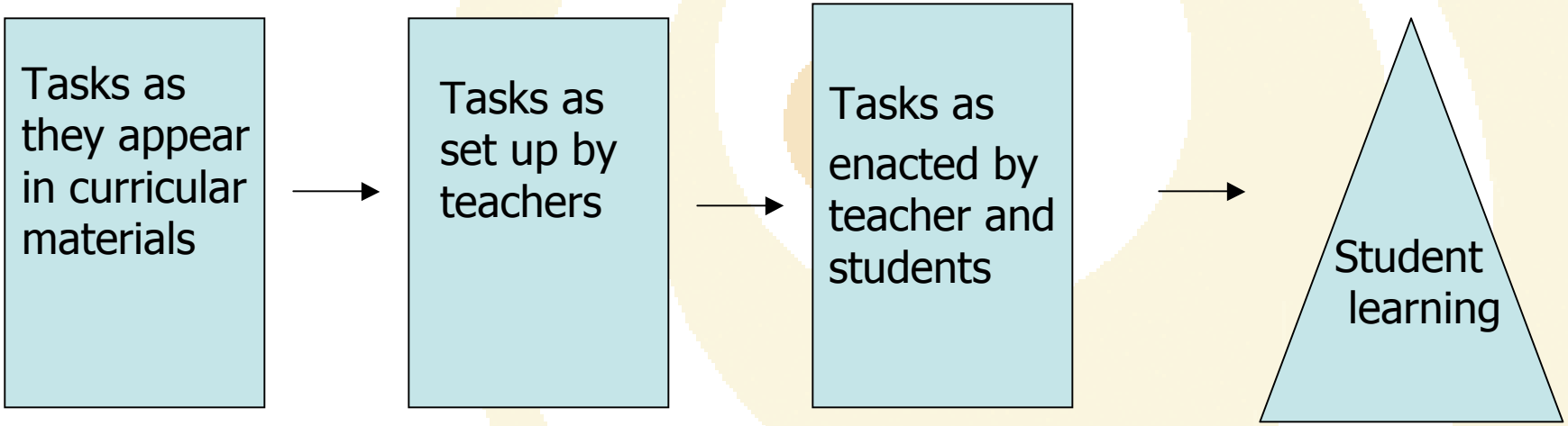
1. What opportunities to learn SCK (or MKT more generally) does each task provide?
2. Looking across the tasks: What are central features of these tasks that allow for work on SCK (or MKT)?
3. What are the challenges of enacting these types of "MKT tasks" with teachers?

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Conjectures about "enactment" of MKT tasks

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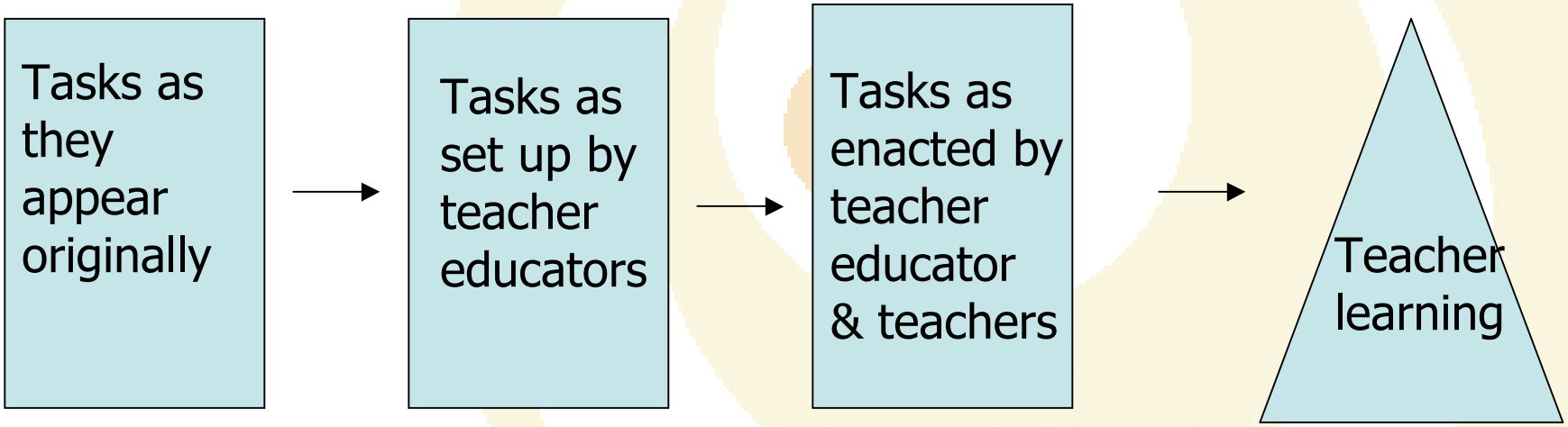
The Mathematical Task Framework (MTF)



Stein, Grover & Henningsen (1996)
Smith & Stein (1998)
Stein, Smith, Henningsen & Silver (2000)

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The Mathematical Task Framework adapted to teacher education



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Challenges of teaching MKT

- Staying focused on the *mathematics*, and not on how to *teach* the math
- Keeping the problems focused on MKT and not just "M"
- Unpacking the mathematics sufficiently and convincingly helping them see what there is to learn and do
- Making visible the connections to the kinds of mathematical thinking, judgment, reasoning one has to do in teaching

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Enactment:
What are key questions and moves that can be used to keep a task focused on developing MKT?

- Asking teachers to explain their solutions to the others
- Having teachers explain what is/was confusing them
- Asking teachers to figure out what might be confusing/difficult for someone else about the problem
- Provoking a common error
- Having teachers ask questions to become more clear about their colleagues' solutions
- Asking teachers to make correspondences between solutions and/or representations
- Asking teachers to explain someone else's thinking
- Providing opportunities to "talk mathematics" and write on the board
- Narrating how something a teacher does/says relates to or is a skill used in teaching

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Current and future work

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mod4: materials development

Developing materials that –

- Feature mathematical tasks designed for developing MKT
- Provide resources to support teacher educators and professional developers in the enactment of tasks
- Provide resources for facilitators and instructors to develop their own knowledge of mathematics for teaching

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Current pilot

- **what:** piloting the Using Definitions strand
- **when:** spring pilot and small summer pilot
- **who:** teacher educators and professional developers working with elementary (or possibly middle school) teachers who might be interested in using one or more of the sessions

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Other work

- Fractions strand (under construction)
- Future pilots (fall 2008)
- mod4camp (June 10-12; August 10-12)
  - Task design
  - Resource development
- Project news and information

  http://sitemaker.umich.edu/mod4

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slides will be available at

http://sitemaker.umich.edu/mod4/nctm2008

sign up for preview materials and pilot participation at –

http://sitemaker.umich.edu/mod4/signup

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