Conceptualizing Mathematical Knowledge for Teaching

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lmt Learning Mathematics for Teaching Project

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Overview of Session

Orientation

1. Developing a professionally-grounded theory and evidence about the knowledge demands of teaching
2. What is special about mathematical knowledge for teaching?
3. Is this a culturally-particular theory of mathematical knowledge for teaching?
4. Mathematicians' participation: Learning about multi-disciplinary work

Discussant's comments

Questions

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Who Knows Mathematics Well Enough to Teach Third Grade, and How Can We Decide?

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Common Framing of the Problem
⟶ Failure to Solve the Problem

- Teachers do not know mathematics well enough (evidence and causes vary)
- Concentration of under-qualified teachers in urban and high poverty schools

- Increase number and rigor of mathematics requirements for teacher certification
- Recruit mathematically trained people into teaching

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Two Currents in the Contemporary Environment



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Knowing Mathematics

Multiply:

$$\begin{array}{r} 3.5 \\ \times\ 2.5 \\ \hline \end{array}$$

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Knowing Mathematics for Teaching

(a)

$$\begin{array}{r} 3.5 \\ \times\ 2.5 \\ \hline 255 \\ 80 \\ \hline 10.55 \end{array}$$

(b)

$$\begin{array}{r} 3.5 \\ \times\ 2.5 \\ \hline 62.5 \end{array}$$

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Tackling the Challenge to Professional Jurisdiction

Develop rigorously and relevantly warranted professional knowledge

- credibility
- usefulness

– for education professionals (teachers, teacher educators, researchers)

– for other stakeholders (policymakers, public)

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Toward A Practice-Based Theory of Mathematical Knowledge for Teaching

1. Study instruction, and identify the mathematical work of teaching
2. Analyze what mathematical knowledge is needed to do that work effectively, and how it must be understood to be useful for the work
3. Develop, test, and refine measures of mathematical knowledge for teaching using multiple methods as a means to improve theory

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Opportunity: Study of Instructional Improvement

- Study of three Comprehensive School Reforms; teacher knowledge a key variable
- Instrument development goals:
  - Usable with 5000 teachers: survey, multiple choice
  - Measure content knowledge teachers *use* in teaching – not just what they *know*
  - Differentiate among teachers
  - Non-partisan

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Articulating Domains of Mathematical Knowledge for Teaching

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Hypotheses About Knowledge of Mathematics for Teaching

PREVALENT HYPOTHESES

1. Teachers need knowledge of the curriculum, plus N levels more knowledge
2. Pedagogical content knowledge

OUR CURRENT HYPOTHESES

- **Common** content knowledge
- **Specialized** content knowledge
- Knowledge of **content & students**
- Knowledge of **content & teaching**

What does this have to do with "pedagogical content knowledge"?

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Common Content Knowledge

The mathematical knowledge and skill expected of any well-educated adult

Teachers need to be able to:

- Recognize wrong answers
- Spot inaccurate definitions in textbooks
- Use notation correctly
- Doing the work assigned to students

Example:

$$\begin{array}{r} 307 \\ -\ 168 \\ \hline 139 \end{array}$$

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Specialized Content Knowledge

The mathematical knowledge and skill needed by teachers in their work and beyond that expected of any well-educated adult

Teachers need to be able to:

- Analyze errors and evaluate alternative ideas
- Give mathematical explanations and use mathematical representations
- Be explicit about mathematical language and practices

Example:

$$\begin{array}{r} 307 \\ -\ 168 \\ \hline 169 \end{array}$$

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Specialized Content Knowledge
(Sample Item)

Which of these students is using a method that could be used to multiply any two whole numbers?

Student A	Student B	Student C
35 x25 125 +75 875	35 x25 175 +700 875	35 x25 25 150 100 +600 875

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Knowledge of Content & Students

The knowledge that combines knowledge of content and students

Teachers need to be able to:

- Anticipate student errors and common misconceptions
- Interpret students' incomplete thinking
- Predict what students are likely to do with specific tasks and what they will find interesting or challenging

Example:

$$\begin{array}{r} 307 \\ -\ 168 \\ \hline 261 \end{array}$$

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Knowledge of Content & Students
(Sample Item)

Ms. Violeta was looking carefully at her students' papers, and she saw the following responses to the problem:

$$8 + 4 = ___ + 5$$

i) 12 ii) 17 iii) Can't do it iv) 1

Which of the following is the most likely explanation of the difficulty the students are having? (Mark ONE answer.)

a) They do not know their basic addition facts.
b) They cannot do multi-step problems.
c) They do not know that addition is commutative.
d) They do not understand the meaning of the equals sign.

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Knowledge of Content & Teaching

The knowledge that combines knowledge of content and teaching

Teachers need to be able to:

- Sequence content for instruction
- Recognize instructional pros and cons of different representations
- Size up mathematical issues in responding to students' novel approaches

Example:

$$\begin{array}{r} 307 \\ -\ 168 \\ \hline 261 \end{array}$$

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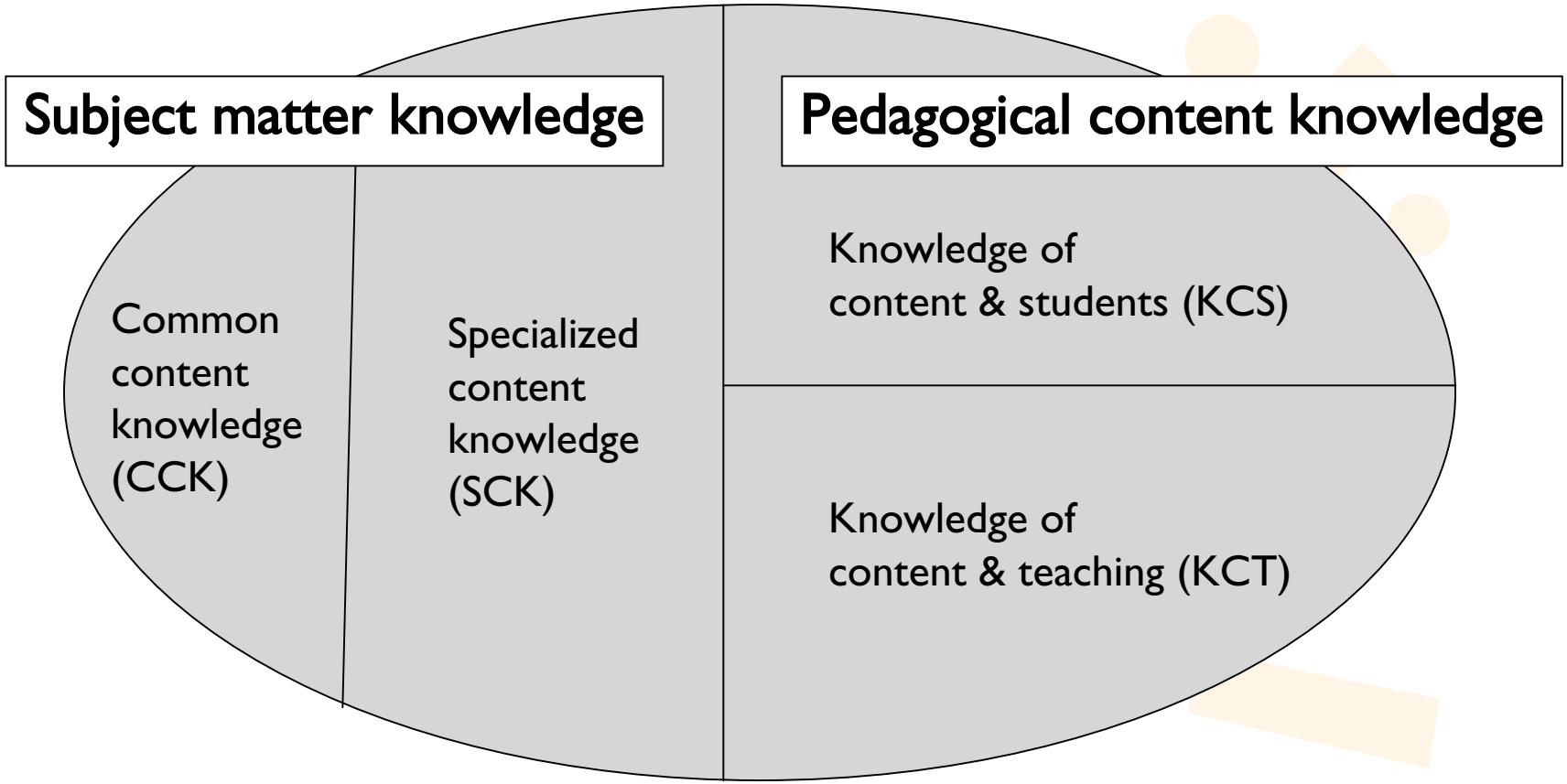
Knowledge of Content & Teaching
(Sample Item)

To introduce the idea of grouping by tens and ones with young learners, which of the following materials or tools would be most appropriate? (Choose ONE.)

a. A number line
b. Plastic counting chips
c. Pennies and dimes
d. Straws and rubber bands
e. Any of these would be equally appropriate for introducing the idea of grouping by tens and ones.

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Shulman's Original Category Scheme (1985) Compared with Ours



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Why Are New Categories Needed?
(Why not just use PCK?)

To understand better :

- Do some aspects of teachers' mathematical knowledge predict student achievement better than others?
- How can we design with more precision teacher education, professional development, and support materials for teachers?
- How do different approaches to teacher education shape particular aspects of teachers' mathematical knowledge?

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Is Mathematical Knowledge for Teaching a General or Culturally-Particular Construct?

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Teachers' Knowledge of Mathematics

- Comparative studies of students
- Comparing teaching and teachers
- *Knowing and Teaching Elementary Mathematics* (Liping Ma, 1999)
- Growing interest in teachers' knowledge
- Irish interest in teachers' knowledge

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"Mathematical Knowledge for Teaching"

- Theory being developed by the Learning Mathematics for Teaching (LMT) Research Group
- Based on studying the work of teaching mathematics
- Multiple choice items developed to measure Mathematical Knowledge for Teaching (MKT)

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Teaching is a Cultural Activity

- LMT items informed by US mathematics teaching
- *The Teaching Gap* (Stigler and Hiebert, 1999)
- Mathematics teaching differs across countries
- If teaching is culturally-bounded, does the mathematical knowledge for teaching differ across countries?

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Research Questions

- Is mathematical knowledge for teaching culturally-particular or general?
  - Specifically, is it similar or different in Ireland and the US?
- How might the items need to be adapted to be suitable for Ireland?

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Adapting the Items

- Items translated by me
- Checked by four Irish teachers
- Checked by one Irish mathematician
- Changes were documented

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Types of Possible Changes

- Contextual
- Deletions
- Mathematical change
- Change in layout
- Change in structure

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Adaptations: Findings

- Some items were observed to be sensitive to cultures and required changes
- Mathematical knowledge for teaching was robust in most cases and did not need to be changed

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A Typical Item

Which of the following story problems could be used to illustrate $1\frac{1}{4}$ divided by $\frac{1}{2}$? (Mark YES, NO, or I'M NOT SURE for each possibility.)

	Yes	No	I'm not sure
a) You want to split $1\frac{1}{4}$ pies evenly between two families. How much should each family get?	1	2	3
b) You have $1.25 and may soon double your money. How much money would you end up with?	1	2	3
c) You are making some homemade taffy and the recipe calls for $1\frac{1}{4}$ cups of butter. How many sticks of butter (each stick = $\frac{1}{2}$ cup) will you need?	1	2	3

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Context Change (Units)

Which of the following story problems could be used to illustrate $1\frac{1}{4}$ divided by $\frac{1}{2}$? (Mark YES, NO, or I'M NOT SURE for each possibility.)

		Yes	No	I'm not sure
a) You want to split $1\frac{1}{4}$ pies evenly between two families. How much should each family get?		1	2	3
b) You have €1.25 and may soon double your money. How much money would you end up with?		1	2	3
c) You are making some homemade taffy and the recipe calls for $1\frac{1}{4}$ cups of butter. How many sticks of butter (each stick = $\frac{1}{2}$ cup) will you need?		1	2	3

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Context Change (Cultural Activity)

Which of the following story problems could be used to illustrate $1\frac{1}{4}$ divided by $\frac{1}{2}$? (Mark YES, NO, or I'M NOT SURE for each possibility.)

		Yes	No	I'm not sure
a) You want to split $1\frac{1}{4}$ pies evenly between two families. How much should each family get?		1	2	3
b) You have €1.25 and may soon double your money. How much money would you end up with?		1	2	3
You are making Rice Krispie™ buns and the recipe calls for 1¼ cups of chocolate. How many bars of chocolate (each bar = ½ cup) will you need?		1	2	3

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Inappropriate Mathematical Changes (Local Curriculum, Clarification)

Which of the following story problems could be used to illustrate 3 divided by $\frac{1}{2}$? (Mark YES, NO, or I'M NOT SURE for each possibility.)

	Yes	No	I'm not sure
a) You want to split 3 pies evenly between two families. How much should each family get so that the pies are equally shared?	1	2	3
b) You have €3 and may soon double your money. How much money would you end up with?	1	2	3
You are making Rice Krispie™ buns and the recipe calls for 3 cups of chocolate. How many bars of chocolate (each bar = ½ cup) will you need?	1	2	3

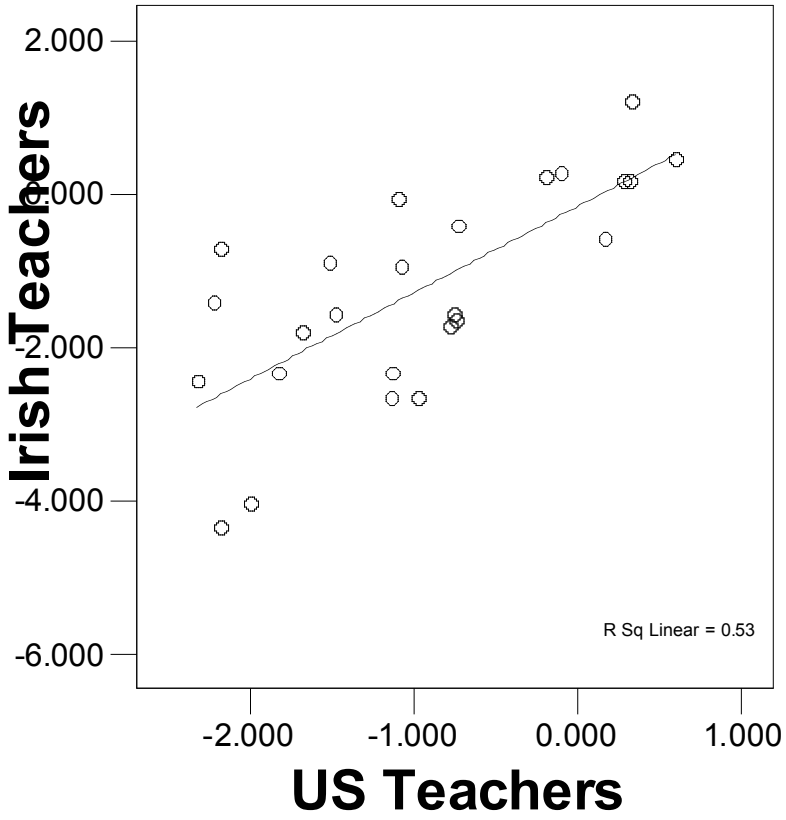
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Is Mathematical Knowledge for Teaching Culturally-Particular or General?

- Administered items to 100 teachers; followed up with interviews of 5 teachers
- Teachers thought the items reflected the type of mathematics that teachers in Ireland need to do their work
- Item difficulty measured in standard deviations, average teacher ability = 0
- Number items content knowledge, correlation of difficulty of items 0.73

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Item Difficulties (Irish/US Teachers)



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Conclusion

- Evidence that mathematical knowledge for teaching in Ireland has common features with in the US
- Suggests that mathematical knowledge for teaching is not just a culturally-particular construct
- Categories of changes identified.
- Some changes necessary for cultural sensitivity
- Some changes may compromise tool's ability to measure and to compare mathematical knowledge for teaching

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Future Work

- Continue the Irish study with a larger sample of Irish teachers
- Measure mathematical knowledge for teaching in a culturally-similar country where English is not the language of teaching
- Measure mathematical knowledge for teaching in a different culture
- Compare mathematical knowledge for teaching across cultures when issues like administration are standardized

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What's in Collaborative Work?

Mathematicians and Educators Developing Measures of Mathematical Knowledge for Teaching

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The Call for Multidisciplinary Collaboration in Education

- Strong theoretical grounds for collaborative work (e.g., Schwab; Hutchins)
- In its absence, contested jurisdiction
- How to make it work is unspecified, and complex

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Our Research Question

For the work of developing measures of mathematical knowledge for teaching, what productive roles can research mathematicians play, based on their disciplinary expertise and culture?

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Our Data

- Observation (direct and video) of interactions in "item camps"
- Analysis of mathematician-generated draft items
- Analysis of mathematician commentaries (oral and written) on draft items of others

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The Multidisciplinary Work of Developing Items to Measure Mathematical Knowledge for Teaching

- Existing theory of the mathematical knowledge to be measured
- Educators (scholars and practitioners), mathematicians, psychometricians, statistical modelers
- "Item camps"

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Claim #1

Mathematicians generally did not write items usable for measuring the mathematical knowledge for teaching.

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A Mathematician's Draft Item

Charmine and Alison had cut out a large copy of each digit. They were playing with the 1, the 2, the 3 and the 4. Charmine said, "The largest number I can make is 4321." Alison said, "And the smallest number is just the opposite: 1234!" "Wow," said Charmine, "that's a big difference from rearranging them – over 3000! I wonder what the smallest difference is between any two of the numbers that we can make?"

Which answer is correct?

(a) 1
(b) 9
(c) 11
(d) none of the above

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Claim #2

Mathematicians were often highly skilled in the review of draft items:

- –Mathematical integrity of the content
- –Precision with language

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A Good Item

(authored by a teacher, revised by a mathematician)

Mr. Hosko was wondering what it meant to say that division by 0 was *undefined*. He asked his colleague, Mrs. King, what she thought. **Which of the following BEST explains this?** (Mark ONE answer.)

(a) Division by 0 is undefined because you cannot do it.

(b) Division by 0 is undefined because you cannot make 0 groups of something.

(c) Division by 0 is undefined in school curricula because college-level mathematics is needed to do this calculation.

(d) Division by 0 is undefined because there is no answer you can get that you can multiply by the divisor 0 to get the original number.

(e) Division by 0 is undefined because every number divided by 0 equals 0.

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Correction to Item

Replace the "correct" answer

(d) Division by 0 is undefined because there is no answer you can get that you can multiply by the divisor 0 to get the original number.

by

(d') Division by 0 is undefined because there is no **single** answer you can get that you can multiply by the divisor 0 to get the original number.

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Analysis of Correction

(d) Division by 0 is undefined because there is no answer you can get that you can multiply by the divisor 0 to get the original number.

Suppose

$a \div 0 = b$

But there is no number b that can be multiplied by 0 to equal a.

So this does not fit the definition of division which requires that

If $a \div d = b$, then $b \cdot d = a$

BUT. . . .

Consider the case where a = 0.

Then with

$a \div 0 = b$, you have $0 \div 0 = b$

In this case, any value for b works, because $b \cdot 0 = 0$, no matter what b is.

Therefore the correct response should be edited to accommodate the case of $0 \div 0$:

(d') Division by 0 is undefined because there is no single answer you can get that you can multiply by the divisor 0 to get the original number.

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Qualities of Good Items

- Grounded in the "work of teaching"
- Mathematically sound
  - Valid
  - Precise
  - Measure mathematical proficiency for teaching
  - Unpacked, not compressed

Mathematicians and educators diverged in their understanding of these dimensions.

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Unpacking Compressed Knowledge

- Mr. Squire is designing a lesson on simplifying radical expressions. He wants to pick one example from the previous day's homework problems to review at the beginning of today's class. His goal is to select a problem that will lead to a good discussion about different solution strategies for simplifying radicals.
- **Which of the following is <u>best</u> for setting up a good discussion about strategies for simplifying radical expressions?** (Mark ONE answer.)

  (a) $\sqrt{48}$

  (b) $\sqrt{63}$

  (c) $\sqrt{99}$

  (d) $\sqrt{150}$

  (e) All four would work equally well.

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What Was Difficult for Mathematicians to See?

The cases all seemed useful:

$63 = 3^2 \cdot 7$ $99 = 3^2 \cdot 11$ $150 = 5^2 \cdot 2 \cdot 3$ $48 = 4^2 \cdot 3$

(3 primes)

They did not notice how 48 uniquely responded to the stem:

$48 = 4^2 \cdot 3$, but also $= 2^2 \cdot 2^2 \cdot 3$,

so there are different pathways to the simplification to try and discuss.

Can extract 4 or, successively, 2 twice

Compressed knowledge of radicals made it difficult to discriminate among these specific numerical examples as offering different learning opportunities (a mathematical perspective needed professionally by teachers).

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Conclusions

- The epistemology of disciplinary mathematics seems to differ from the epistemology of mathematics for teaching.
  - The nature and setting of problems
  - What is involved in "hard" problem solving
  - What precision and clarity involve
- Mathematicians' contributions were valuable for critiquing and polishing items; but less so for generating good items.

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Using Data to Test and Improve Theory

- Factor analyses
- Analyses of validity
- Uses of measures
  - To predict student achievement
  - To evaluate professional development

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Overarching Findings: Factor Analyses

- Multidimensionality of mathematical knowledge for teaching
  - Knowledge of students and content different from content knowledge
  - Patterns, functions & algebra content knowledge for teaching
  - Geometry mathematical knowledge for teaching
  - Number and and operations mathematical knowledge for teaching
    - Specialized
    - Common
- Tentative conclusion: presence of professional subject matter knowledge for teaching

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Validating Our Measures

How do we want to be able to interpret teachers' performance on our questions?

1. Scores capture teachers' mathematical knowledge
   - Cognitive interviews
2. Higher scores are related to higher-quality mathematics instruction
   - Videotape validation study
3. Higher scores are related to improved student learning
   - Study of Instructional Improvement student gains analysis
4. Scores reflect different dimensions of "content knowledge for teaching mathematics"
   - Mathematician and non-teacher interviews
   - Item response theory and factor analysis

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Linking Teacher Knowledge and Student Achievement

- Questionnaire consisting of 30 items (scale reliability .88)
- Model: Student Terra Nova gains predicted by:
  - Student descriptors (family SES, absence rate)
  - Teacher characteristics (math methods/content, content knowledge)
- Teacher content knowledge significant
  - Small effect (< 1/10 standard deviation): 2 - 3 weeks of instruction
  - But student SES is also about the same size effect on achievement

(Hill,Rowan, and Ball, AERJ, in press)

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Develop Rigorously and Relevantly Warranted Professional Knowledge

- Build a theoretical foundation in the work of teaching: identifying and studying problems of practice
- Test practice-based hypotheses using data and rigorous analytic methods
- Use expertise from other disciplines wisely

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Discussion

- Miriam Sherin, Northwestern University

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Mathematical Knowledge for Teaching

What do we mean when we use this term, "**mathematical knowledge for teaching**"?

- Mathematical knowledge, skill, habits of mind that are entailed by the work of teaching

What do we mean by the "**work of teaching**"?

- The tasks in which teachers engage, and the responsibilities they have, to teach mathematics, both inside and outside of the classroom

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