

Making Complex Mathematics Learnable By All

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Carnegie Learning Webinar

April 14, 2010

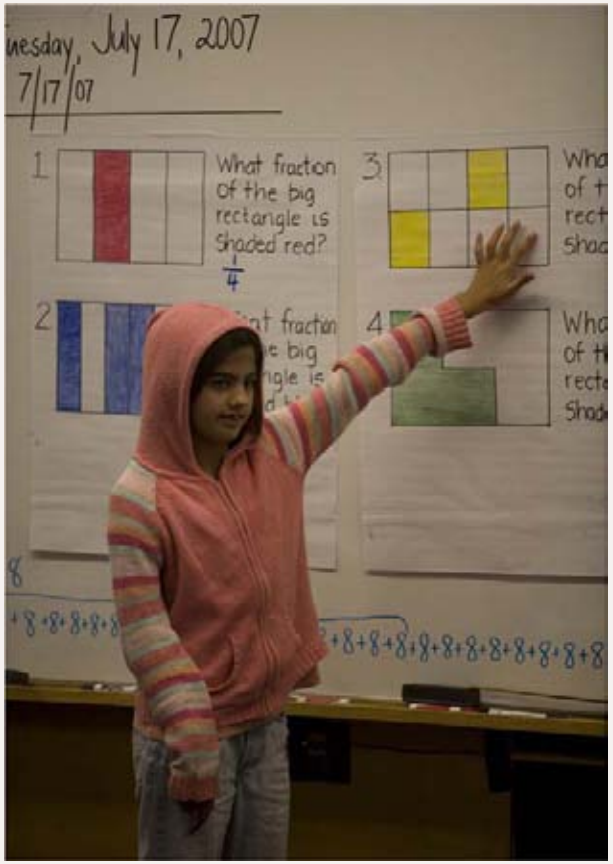
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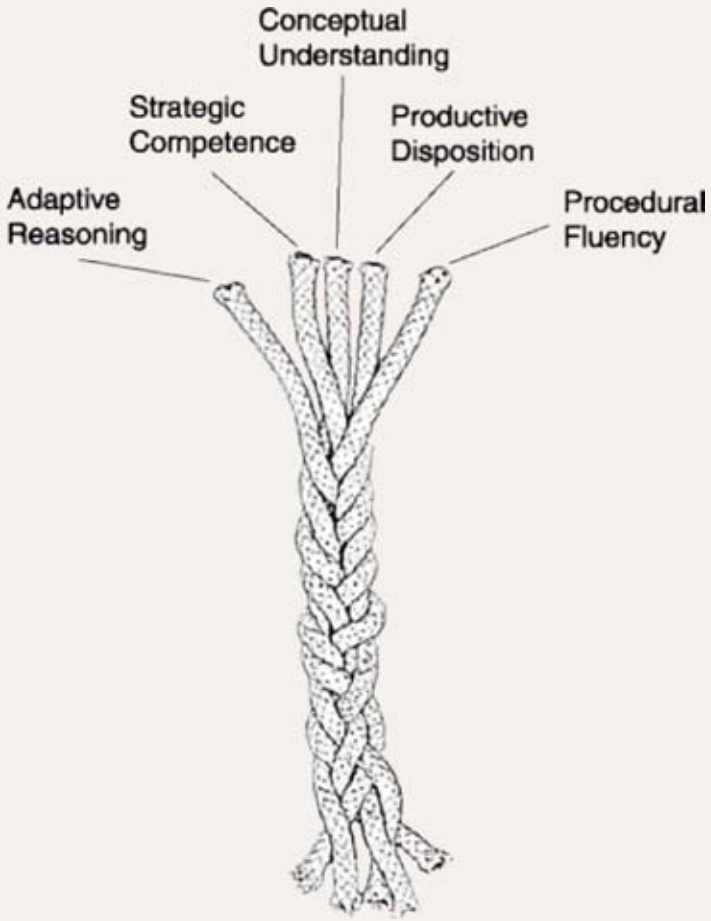
The urgency

1. Enormous gaps in learning opportunities and disparities in achievement (within U.S. and in international comparisons)
2. Rapidly changing school population
3. Higher, more complex academic goals
4. High expectations for all students



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Strands of mathematical proficiency



- ***Conceptual understanding*** - comprehension of mathematical concepts, operations, and relations
- ***Procedural fluency*** - skill in carrying out procedures flexibly, accurately, efficiently, and appropriately
- ***Strategic competence*** - ability to formulate, represent, and solve mathematical problems
- ***Adaptive reasoning*** - capacity for logical thought, reflection, explanation, and justification
- ***Productive disposition*** - habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy

Kilpatrick, J., J. Swafford, and B. Findell. (2001). *Adding It Up: How Children Learn Mathematics*. Washington, DC: National Academy Press.

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Key practices of teaching to enable mathematics achievement

1. Expecting and enabling complex mathematical work of each student
2. Coordinating mathematical, school, and students' language
3. Using contexts with sensitivity to students' funds of knowledge and to the integrity of the mathematics
4. Creating a respectful mathematically-focused learning environment

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Enacting high expectations in mathematics instruction

1. Broaden what it means to be successful in math class
2. Make mathematical practices explicit
3. Support students' mathematical work both publicly and privately
   - Listen carefully to students' talk
   - Notice and improve ambiguous talk
   - Ask mathematical questions
4. Teaching students to be "people who study mathematics"

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What is the work of teaching?

1. SCAFFOLDING STUDENTS' EXPECTATIONS AND STANDARDS
2. CHOOSING MATHEMATICAL TASKS
3. ENACTING HIGH EXPECTATIONS DURING CLASS

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1. Scaffolding students' expectations and standards

What is the work of teaching?

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Student contract

I understand that my ability to succeed in the Elementary Mathematics Laboratory class depends on my commitments and effort this summer. The following are the commitments asked of me to participate in the special class:

1. I will attend class every day.

2. I will keep neat detailed records of my mathematical thinking in my math notebook.

3. I will complete my homework carefully ea ch night.

4. I will seek help or advice from my teacher or parents/caregivers if I have questions or concerns.

5. I will give my best effort to all the work I complete for this class.

I have read, understand and will follow through with the statements in this contract to the best of my ability.

Print student's full name: _____

Student signature:_____
Date:_____

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Teacher contract with

Student's name

I understand that your ability to succeed in the Elementary Mathematics Laboratory class depends on my commitments and effort this summer. The following are the commitments I make to you as your teacher in this special class:

1. I will attend our class every day.
2. I will come to class prepared to teach interesting and appropriate material to you.
3. I will expect you to do your very best work all the time and I will help you learn to do so.
4. I will make sure that you understand the mathematics and problems we work on.
5. I will answer any questions or concerns that you may have about this class or our work.
6. I will keep careful records of my mathematical thinking and ideas.

I have read, understand and will follow through with the statements in this contract to the best of my ability.

Print teacher's full name: _____

Teacher's signature:_____ Date:_____

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Focus

Where are opportunities taken or missed to:

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2. Make mathematical practices explicit
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4. Teach students to be "people who study mathematics"

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2. Choosing mathematical tasks

What is the work of teaching?

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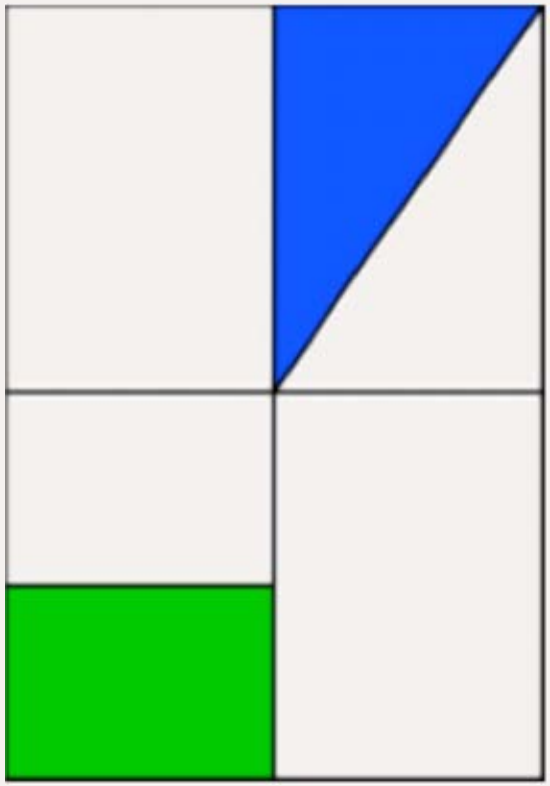
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Reasoning with representations



What fraction of the big rectangle is shaded blue?

What fraction of the big rectangle is shaded green?

What fraction of the big rectangle is shaded altogether?

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Analyzing others' solutions

A fourth grade student did the calculations below. For each one, decide whether the answer is correct or incorrect and explain how you know. If the answer is incorrect, try to explain what the error is.

$$\begin{array}{r} 48 \\ +\ 27 \\ \hline 615 \end{array}$$

$$29 + 37 + 18 = 84$$

Is the answer correct or incorrect?

How do you know?

If the answer is incorrect, try to explain what the error is.

If the answer is incorrect, please do it correctly here:

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Reasoning with tools

Which Cuisenaire rod is three times as long as a red one?

Which rod is half as long as an orange one?

One rod is one-fourth as long as another rod. What colors might they be?

Which rod is one-third as long as the dark green one?

Which rod is five times as long as a red one?

The blue rod is how many times as big as the white Cuisenaire rod?

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Types of tasks in class and on homework: How are they similar and different?

SIMILAR

- Involve the use of mathematical skills as well as concepts
- Routinely require explanation and reasoning
- May have more than one solution or solution strategy

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Differences between in-class and homework tasks

HOMEWORK

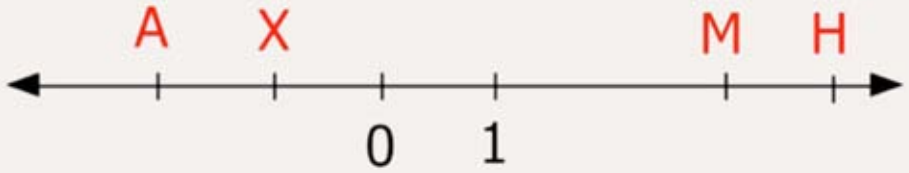
- Can be done independently
- As unambiguous as possible
- Structures the workspace more explicitly

IN-CLASS

- Profits from exchange with others
- Requires interpretation
- Requires more decisions about representation and recording
- Likely to yield multiple solution methods or solutions

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Homework



Is A less than or more than H? How do you know?

Is A less than or more than X? How do you know?

Label what numbers go at A, X, M, and H. Then, explain how you know what M is.

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In-class



What number can be written at the mark where the blue arrow is pointing?

Circle your answer:

$\frac{1}{2}$ $\frac{1}{4}$ $\frac{1}{3}$ 1

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Necessary for using complex mathematical tasks with "high expectations"

- Helping students to believe they can do – and do -- hard work
- Scaffolding complex work appropriately
- Making wise judgments about what to leave open and what needs to be made explicit
- Commenting on mathematics, and mathematical productions, not features of students
- Supporting error as a fruitful site for mathematical work, and teach students to use error productively

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3. Enacting high expectations during class

What is the work of teaching?

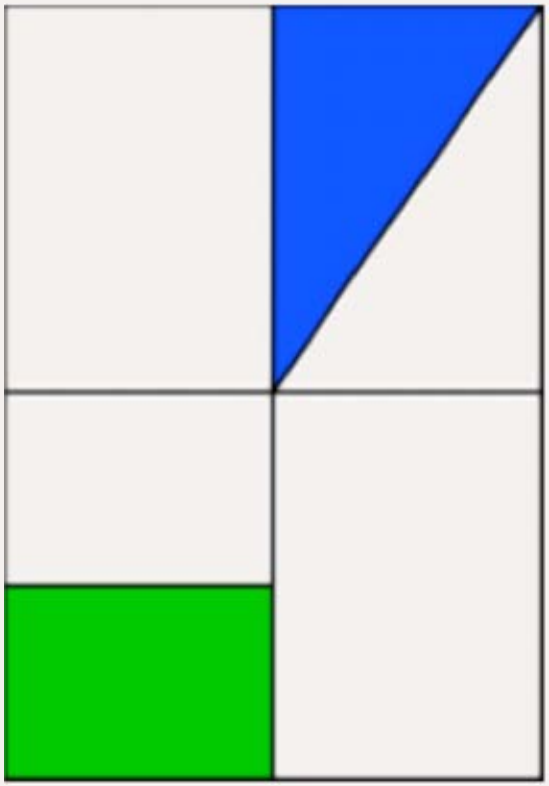
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Episode

- Day 7, discussion of warm up problem
- Focused attention on equal parts
- Developing working ideas about fractions
  1. Identify the whole
  2. Make equal parts
  3. Count how many equal parts out of the whole

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Setting up the problem



What fraction of the big rectangle is shaded blue?

What fraction of the big rectangle is shaded green?

What fraction of the big rectangle is shaded altogether?

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During these 5 minutes

00:00—Open the discussion: encourage participation by more students; establish expectation for explanation; use wait time; choose whom to call on; call on that child.

00:26—Mamadou gives answer of ½. Ask Mamadou to explain his reasoning; make sure other students can hear and are listening; interpret Mamadou's explanation; recognize relationship to key mathematical idea; determine how to respond (whether to take up, whether to clarify question, whether to call on different student).

00:47—Ask Mamadou to come to the board and explain using the diagram. Orient class toward Mamadou's explanation: get student to repeat what Mamadou said without explaining the error; comment about listening carefully; focus students on understanding reasoning.

01:37—Mamadou uses diagram to explain his answer. Invoke the working definition of fractions: ask Mamadou what he is calling the whole, how many equal parts, and how many are shaded.

02:26—Check class' understanding and manage risk of losing rest of class: call on student to explain Mamadou's solution; trace whole on diagram; establish correctness of Mamadou's answer given his selection of the whole.

03:00—Clarify the whole in the original question. Validate Mamadou's work, while establishing that the problem is asking something different. Ask student to read question and show on the diagram what is meant by "big rectangle"; ask Mamadou if he is watching; restate question and trace whole on a new copy of diagram.

04:06—Ask Mamadou to explain the difference between this question and the one he answered. Decide how to handle language "whole square" and "half the rectangle." Elicit answer to original question.

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Key practices of teaching to enable mathematics achievement

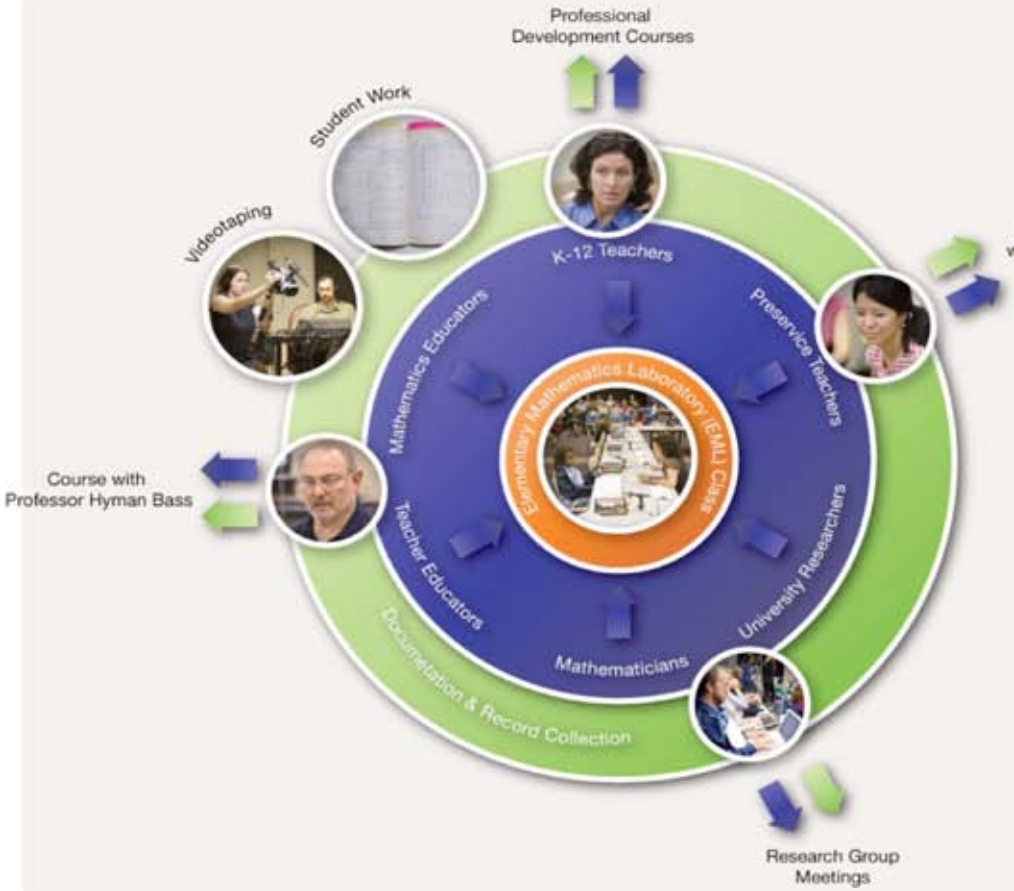
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Elementary Mathematics Laboratory



- July 26 – August 6, 2010
- Ann Arbor, Michigan
- Professional development for teachers, teacher developers, cooperating teachers, and teacher leaders
- For details: Please contact eml2010@umich.edu

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A closer look: The inner circle







Inner circle: The children's class

- An elementary summer school class — 27 local students entering fifth grade, wide range of mathematics achievement and dispositions
- 2.25 hours of instruction per day
- Mathematical topics: fractions, permutations, place value, number line, equivalence
- Mathematical practices: explaining, representing, proving, defining
- Practices of learning math: recording, summarizing, attending to language & precision, studying

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