Exploring the Use of Mathematical Language in Teaching: What Do Teachers Need to Know?

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Overview of Session

1. Introduction: Mathematical knowledge for teaching, and why focus on mathematical language?
2. Investigate the use of mathematical language in practice: Examples from two classrooms
3. Implications for teacher education
4. Conclusions

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1. Introduction: Mathematical Knowledge for Teaching and Why Focus on Mathematical Language?

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Toward A Practice-Based Theory of Mathematical Knowledge for Teaching

1. **Study instruction**, and identify the mathematical work of teaching
2. Analyze what **mathematical knowledge is needed to do that work** effectively, and how it must be understood to be useful for the work
3. Develop and evaluate approaches to **helping teachers learn mathematical knowledge for teaching**
4. Develop, test, and refine **measures of mathematical knowledge for teaching** using multiple methods as a means to evaluate professional education, investigate effects on students' learning, and improve theory

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Mathematical Knowledge for Teaching

- **What do we mean when we use this term, "mathematical knowledge for teaching"?**
  - Mathematical knowledge, skill, habits of mind that are entailed by the work of teaching
- **What do we mean by the "work of teaching"?**
  - The tasks in which teachers engage, and the responsibilities they have, to teach mathematics, both inside and outside of the classroom

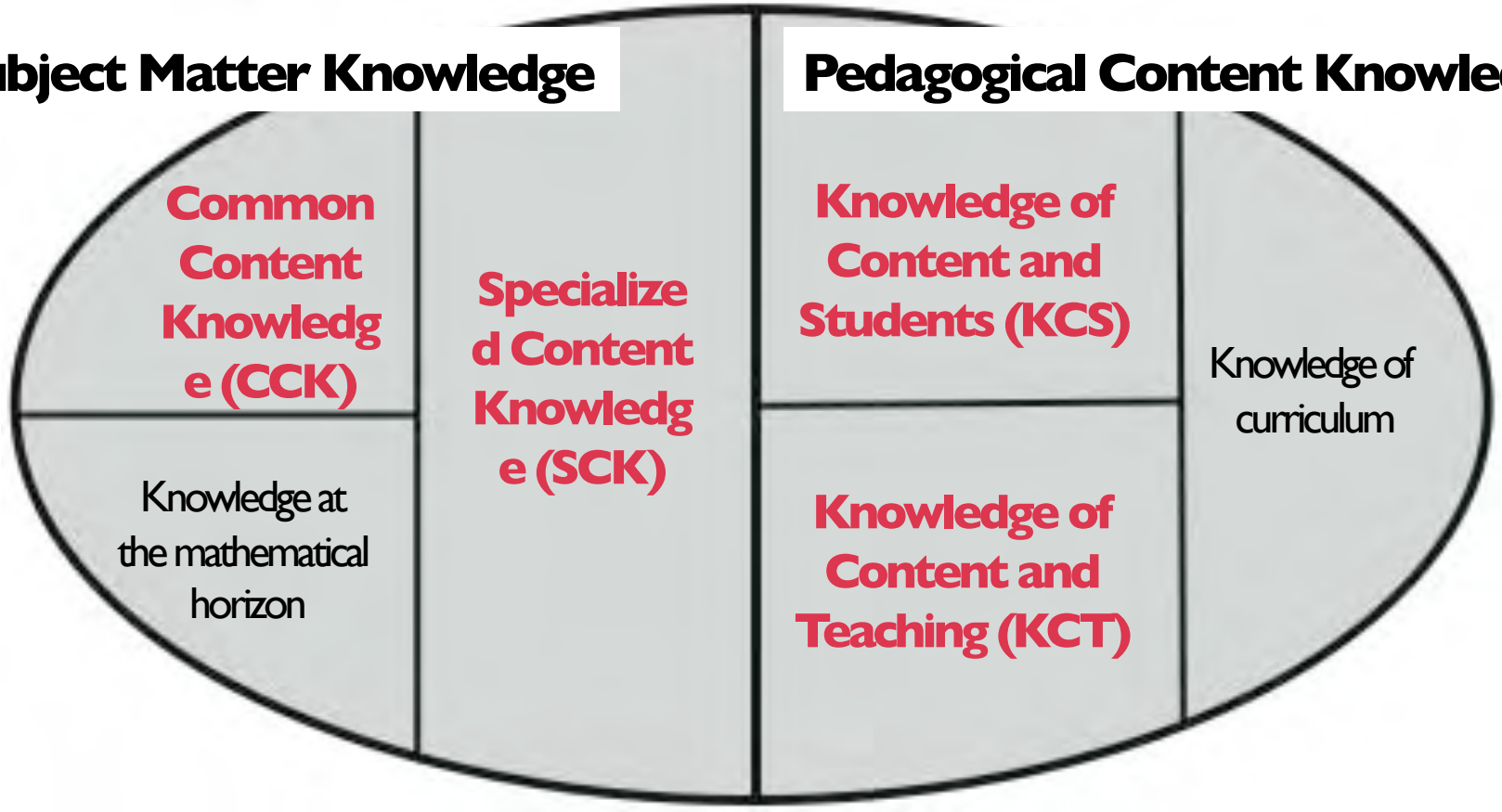
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Examples of Work of Teaching

- Examining the mathematical quality of a textbook treatment
- Using and defining terms
- Producing and evaluating mathematical explanations
- Using notation
- Interpreting and evaluating alternative solutions and thinking
- Choosing contexts with care for mathematical integrity, diversity, and transparency for learning
- Explaining goals and mathematical purposes to others
- Designing homework and quizzes; Selecting and modifying mathematics problems
- Choosing and using representations
- Building correspondences between a model and a concept or procedure
- Evaluating students' work and statements (often quickly)
- Posing questions
- Choosing examples

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Shulman's Original Category Scheme (1985) Compared with Ours



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mathematical language

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What Do We Mean By "Mathematical Language"?

- Three forms of mathematical language:
    1. Words
    2. Symbols
    3. Diagrams
- These are all forms of text; doing mathematics requires reading, writing, speaking, and interpreting these.









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What is Important About Language in Mathematics?

- Necessary for mathematical communication
- Foundational to mathematical reasoning

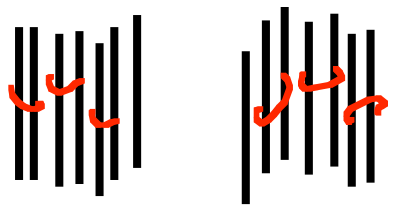
An odd number plus an odd number equals an even number.

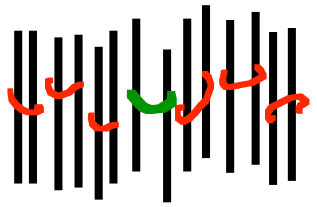
- Precision matters

What is an even number?

- Words, symbols, and diagrams must map onto one another.

$$(n2 + 1) + (j2 + 1) = (n + j + 1)2$$





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Words: More Than "Story Problems"

- Mathematics uses words also used in other settings. This can be an advantage, but also presents challenges.

  Examples:

  - Big, odd, even, prime, rational, radical, square
  - Right, similar, circle, regular, acute, obtuse
- The mathematical uses of these words are specialized.
- Definitions are used to specify technical uses of terms within mathematics.

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Symbols

- Symbols are a set of tools for encoding ideas.
- Symbols compress ideas in useful ways, making them conceptually and linguistically smaller to work with.
- Example:

It is easier and more precise to write:

$$x^2 - y^2 = (x + y)(x - y)$$

than:

The difference of the squares of any two numbers is the product of their sum with their difference.

8 + 4 = __ + 5

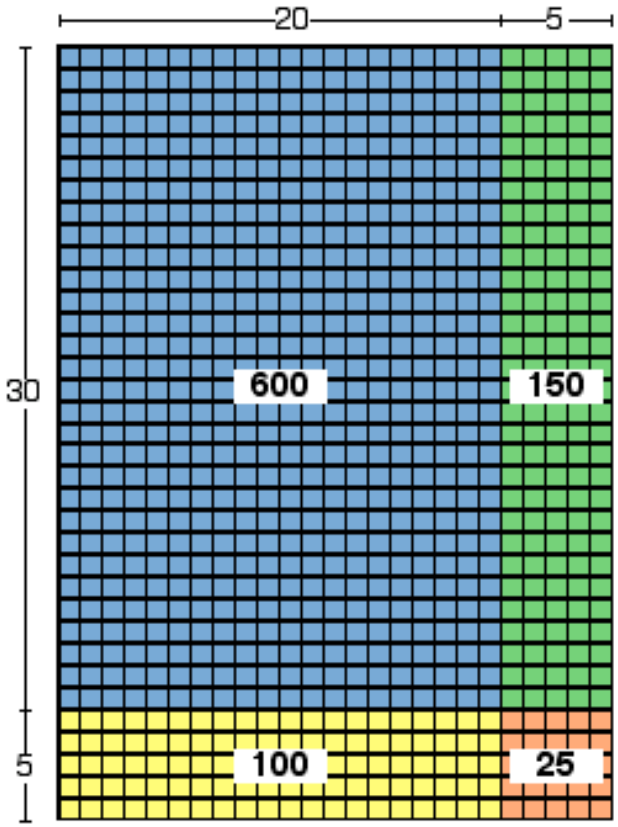
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Diagrams

- Visual representations enhance mathematical exposition and support problem solving
- Obvious in geometry; less obvious in arithmetic, algebra
- Shows the connectivity and unity of mathematics

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Representing Multiplication as Area



Student A	Student B	Student C
35 x25 125 +75 875	35 x25 175 +700 875	35 x25 25 150 100 +600 875

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Diagrams Can Mislead

A rectangle is a plane figure with four straight sides and four right angles.

Typical examples shown:

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False Generalization from Examples

- Squares are not rectangles.

- Parallelograms are "tilted rectangles"

so this is not a rectangle:

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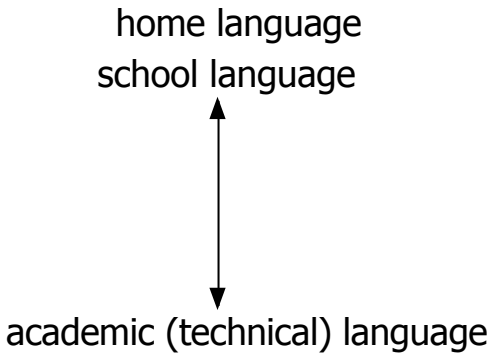
Importance of Mathematical Language

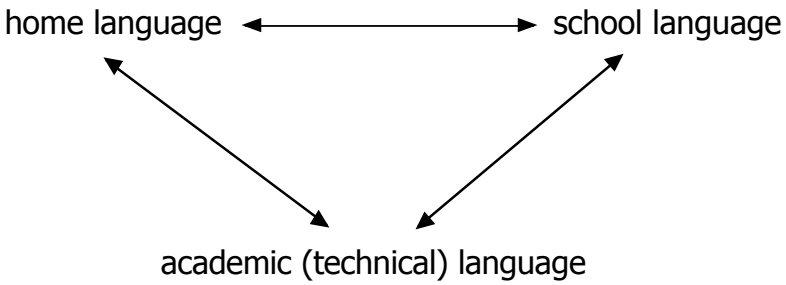
- Mathematical knowledge and reasoning depends on and is supported by mathematical language.
- Teaching and learning mathematics depends on and is supported by language.
- Mathematical language is both mathematical content to be learned and medium for learning mathematical content.

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Issues with Mathematical Language: Mathematical vs. Everyday Language

- Mathematics often uses and specializes everyday language rather than coining a separate technical vocabulary -- both enabling and complicating entry to its register.
- Equity issue: Additional dimension for students who must navigate between home, school, and mathematical languages





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Goals for Session

- Uncover other mathematical language issues that arise in practice
- Consider the mathematical knowledge demands of using mathematical language in teaching
- Consider implications for teacher education

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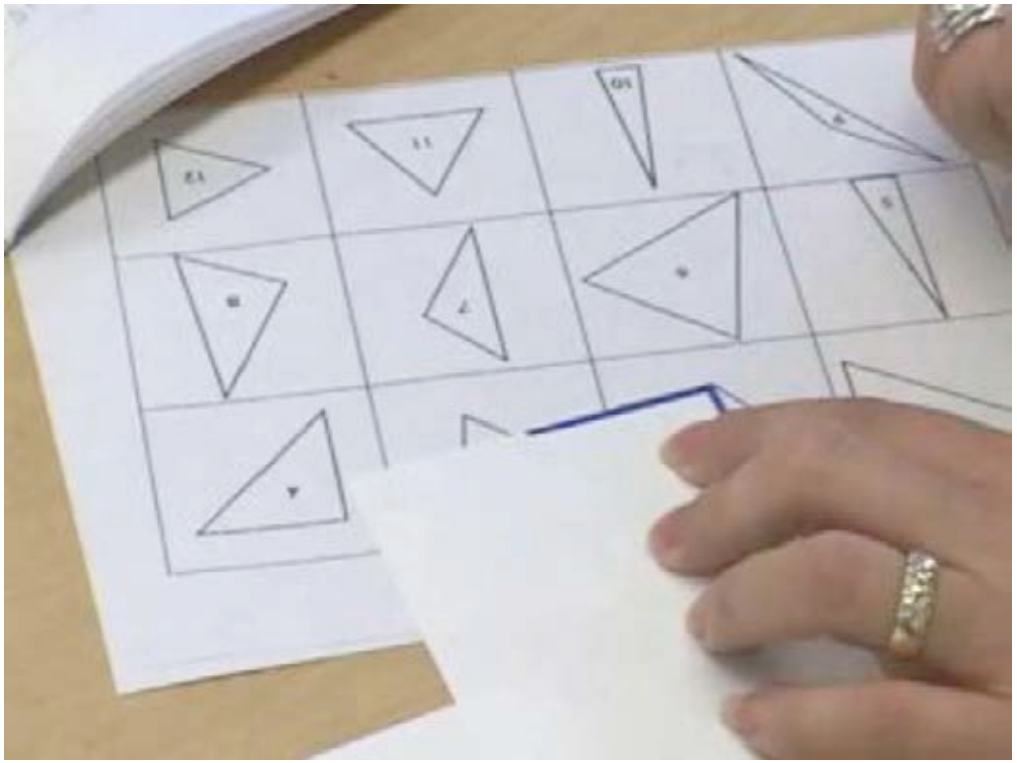
2. Investigating the Use of Mathematical Language in Practice: Examples from Two Classrooms

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Video #1

- 3rd grade geometry lesson
- Students have been introduced to polygons and to different types of angles (*right, greater than, less than*)
- Introduction to lesson on measuring angles in triangles



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Focal Questions

- What mathematical language issues arise in practice?
- What are the mathematical knowledge demands of using mathematical language in teaching?

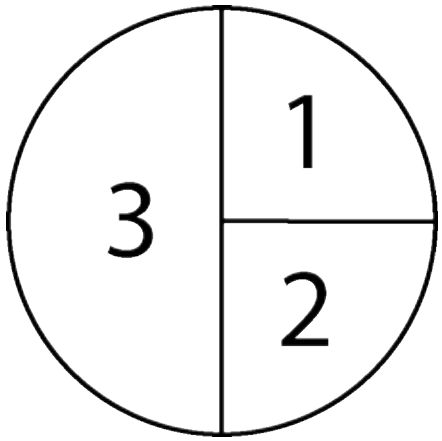
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Video #2

- 4th grade probability lesson
- Introduction to two spinner experiments:

Experiment #1: Spin the spinner once and record the outcome

Experiment #2: Spin the spinner twice and record the sum of the two spins



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Discussion Questions

- What mathematical language issues arise in practice?
- What are the mathematical knowledge demands of using mathematical language in teaching?
- What might be the implications for teacher education?

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3. Implications for Teacher Education

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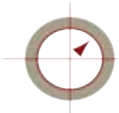
Examples of Teacher Education Tasks about Mathematical Language

- Talk and practice talking, critique one another's talk (also writing, listening, reading and interpreting)
- Keep a list of terms that have other uses, and learning to distinguish which ones require careful attention, and how
- Analyze and write mathematical definitions
- Analyze a textbook lesson for its language and modify it with mathematical care
- Writing worksheets and quizzes
- Analyze assessment items for their mathematical language demands
- Write letters to parents/others about student work









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What makes a "good" definition?

- Mathematically precise — correctly identifies the kind of object, process, property
- Usable by user community — based on already-defined and understood term









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What is a mathematically precise and usable definition of "even number" for third graders?

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Definitions: Even Numbers

a) An even number is a number that can be divided into two equal parts.

c) An even number is any multiple of 2.

e) An even number is any integer multiple of 2.

g) An even number is any number whose unit digit is 0, 2, 4, 6, or 8.

i) An even number is a whole number with zero remainder when divided by 2.

f) A whole number is even if it is the sum of a whole number with itself.

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Consequences

a) An even number is a number that can be divided into two equal parts.

c) An even number is any multiple of 2.

Of a) and b): All numbers, for example 7, 3/5, √2, p, are even!

c) An even number is any integer multiple of 2.

Of c): This is a correct definition of even number.

d) An even number is any number whose unit digit is 0, 2, 4, 6, or 8.

Of d): In this case, 36.7 is an even number!

e) An even number is a whole number with zero remainder when divided by 2.

Of e): According to this, -6 is not even.

f) A whole number is even if it is the sum of a whole number with itself.

Of f): This is a correct definition of evenness for whole numbers, and is consistent with the general definition for integers that will arrive later.

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Conclusions

1. Language is the medium of teaching and learning.
2. Language is foundational to mathematical proficiency.
3. Developing fluency and skills with mathematical language is a crucial part of mathematical knowledge for teaching.
4. Tasks designed to engage teachers in aspects of the work of teaching that involve language can help to develop that fluency.









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