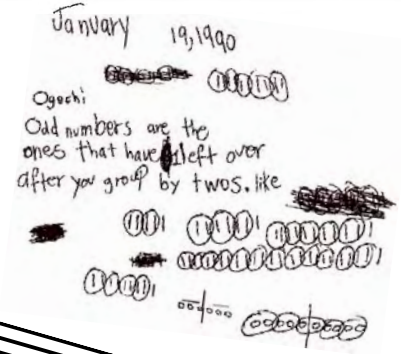
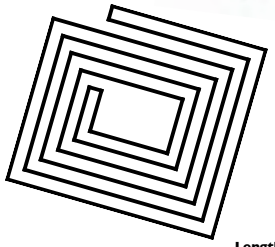
What is Special About Knowing Mathematics for Teaching and How Can It Be Developed?

Deborah Loewenberg Ball and Hyman Bass
University of Michigan

New York City Public Schools
Chancellor's Mathematics Advisory Panel
May 26, 2006

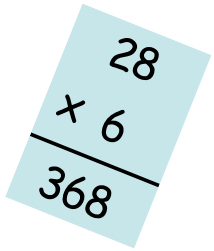




Length of side of main square

Possible sized squares	1	2	3	4	5	6	7	10	50
1x1	1	4	9	16	25	36	49		
2x2		1	4	9	16	25	36		
3x3			1	4	9	16	25		
4x4				1	4	9	16		
5x5					1	4	9		
6x6						1	4		
7x7							1		
8x8									
9x9									
10x10									
Total	1	5	14	30	55	91	140		







www-personal.umich.edu/~dball/

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Overview

1. Defining "mathematical knowledge for teaching"
2. How well do YOU know mathematics for teaching?
3. Measuring mathematical knowledge for teaching: Results
4. Implications for teachers' opportunities to learn MKT

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Clarifying the Problem

The quality of mathematics teaching and learning

Teachers' knowledge of mathematics

and their ability to use it in teaching

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But: U.S. Teachers Often Lack Sufficient Mathematical Knowledge

- What's the evidence?
  - Anecdotes
  - Research
- What does it tell us and what is not well understood?
  - Lack of a system
  - This is a societal problem, widespread

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Some Approaches to Solving the Problem

- Require more mathematics for certification
  - More mathematics courses
  - A major (or minor) in mathematics
  - Mathematics test
- Recruit mathematically trained people into teaching
  - Engineers, accountants, mathematicians, …
- Fund mathematically focused professional development

But -- these approaches assume we know what mathematical knowledge and skill is needed for high-quality teaching.

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Toward A Practice-Based Theory of Mathematical Knowledge for Teaching

1. **Study instruction**, and identify the mathematical work of teaching
2. Analyze what **mathematical knowledge is needed to do that work** effectively, and how it must be understood to be useful for the work
3. Develop and evaluate approaches to **helping teachers learn mathematical knowledge for teaching**
4. Develop, test, and refine **measures of mathematical knowledge for teaching** using multiple methods as a means to evaluate professional education, investigate effects on students' learning, and improve theory

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Mathematical Knowledge for Teaching

What do we mean when we use this term, "**mathematical knowledge for teaching**"?

- Mathematical knowledge, skill, habits of mind that are entailed by the work of teaching

What do we mean by the "**work of teaching**"?

- The tasks in which teachers engage, and the responsibilities they have, to teach mathematics, both inside and outside of the classroom

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Knowing Multiplication

Multiply:

$$\begin{array}{r} 49 \\ \times\ 25 \\ \hline \end{array}$$

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Knowing Multiplication for Teaching:

Analyzing Student Errors

How was each answer produced?

What might lead a student to make these errors?

(a)
$$\begin{array}{r} 49 \\ \times\ 25 \\ \hline 405 \\ 108 \\ \hline 1485 \end{array}$$

(b)
$$\begin{array}{r} 49 \\ \times\ 25 \\ \hline 225 \\ 100 \\ \hline 325 \end{array}$$

(c)
$$\begin{array}{r} 49 \\ \times\ 25 \\ \hline 1250 \\ 25 \\ \hline 1275 \end{array}$$

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Knowing Radical Expressions

Simplify: $\sqrt{150}$

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Knowing Radical Expressions for Teaching:
Choosing Numerical Examples

Which of the following is best for setting up a discussion about different solution paths for simplifying radical expressions?

(a) $\sqrt{54}$

(b) $\sqrt{156}$

(c) $\sqrt{128}$

(d) These examples all work equally well.

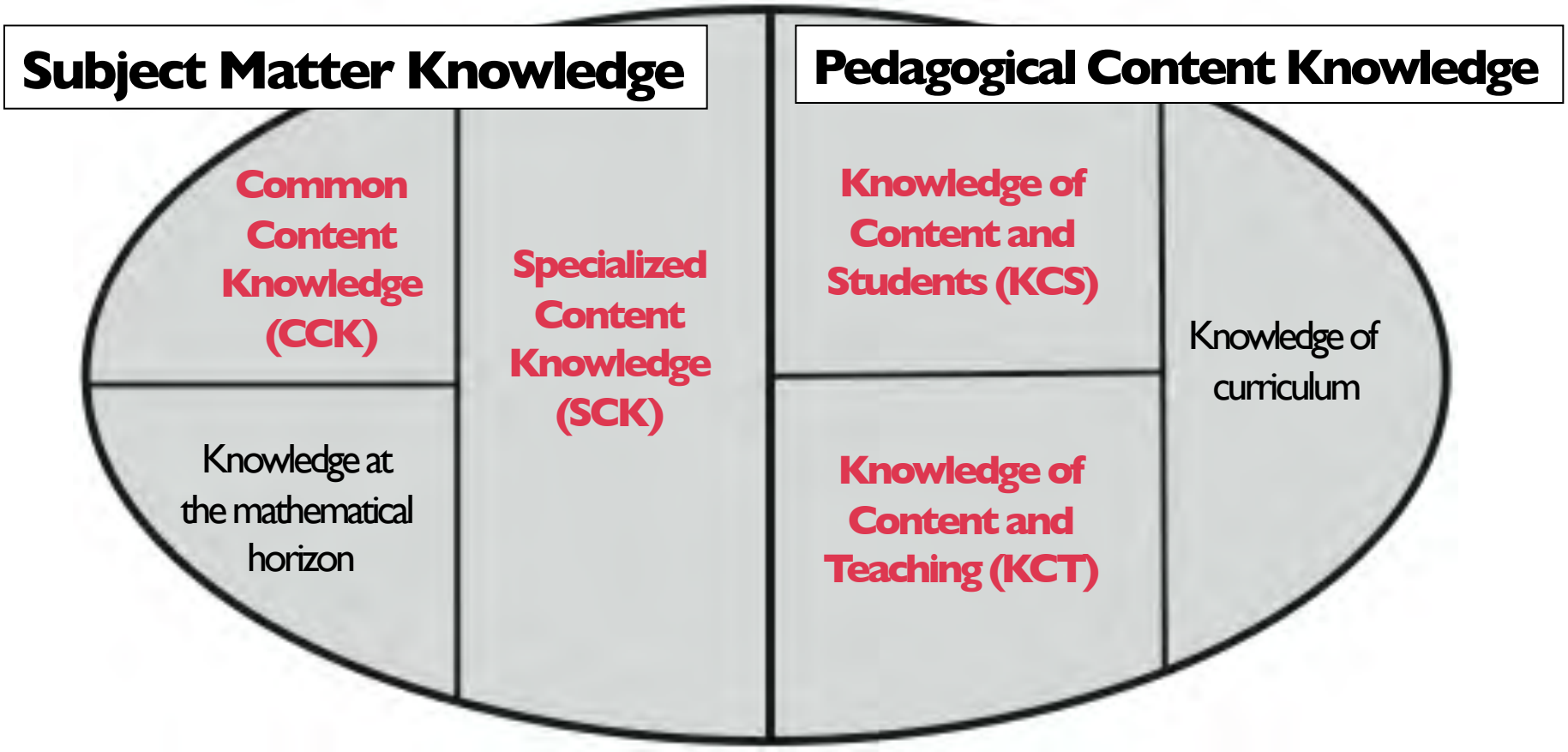
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Other Work of Teaching

- Examining the mathematical quality of a textbook treatment of a topic
- Using and defining terms
- Supporting the linguistic demands for English language learners (ELLs)
- Producing and evaluating mathematical explanations
- Using notation
- Interpreting and evaluating alternative solutions and thinking
- Choosing contexts with care for mathematical integrity, diversity, and transparency for learning
- Explaining goals and mathematical purposes to others
- Designing homework and quizzes; Selecting and modifying mathematics problems
- Choosing and using representations
- Building correspondences between a model and a concept or procedure
- Evaluating students' work and statements (often quickly)
- Posing questions
- Choosing examples

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Shulman's Original Category Scheme (1985) Compared with Ours



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How well do you know mathematics for teaching?

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Overview: Trying Your Hand at Using Mathematics in Teaching

1. Noticing and identifying the mathematics in students' productions
2. Posing strategic mathematical questions to students
3. Composing usable and precise mathematical definitions
4. Writing a quiz and choosing examples

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Case 1:
Noticing and identifying the mathematics in students' productions

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Classroom Context

- Third graders (ages 8 - 10)
- Mathematics: even and odd numbers, definitions, showing why something is true
- Multi-lingual and multi-cultural class

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Mathematical Context

- At the beginning, students could identify a small number as even or odd, but did not know definition
- They had earlier debated whether 0 was even, or odd, or neither.
- One of the students, Shea, comes up with a radical idea about the number six; 6 can be even, and it can be odd! The rest of the class challenges him.
- The teacher navigates this complicated discussion.

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What do you hear and notice mathematically?

- What is going on mathematically?
- What definitions of even and odd do you hear (implicit) in children's talk?

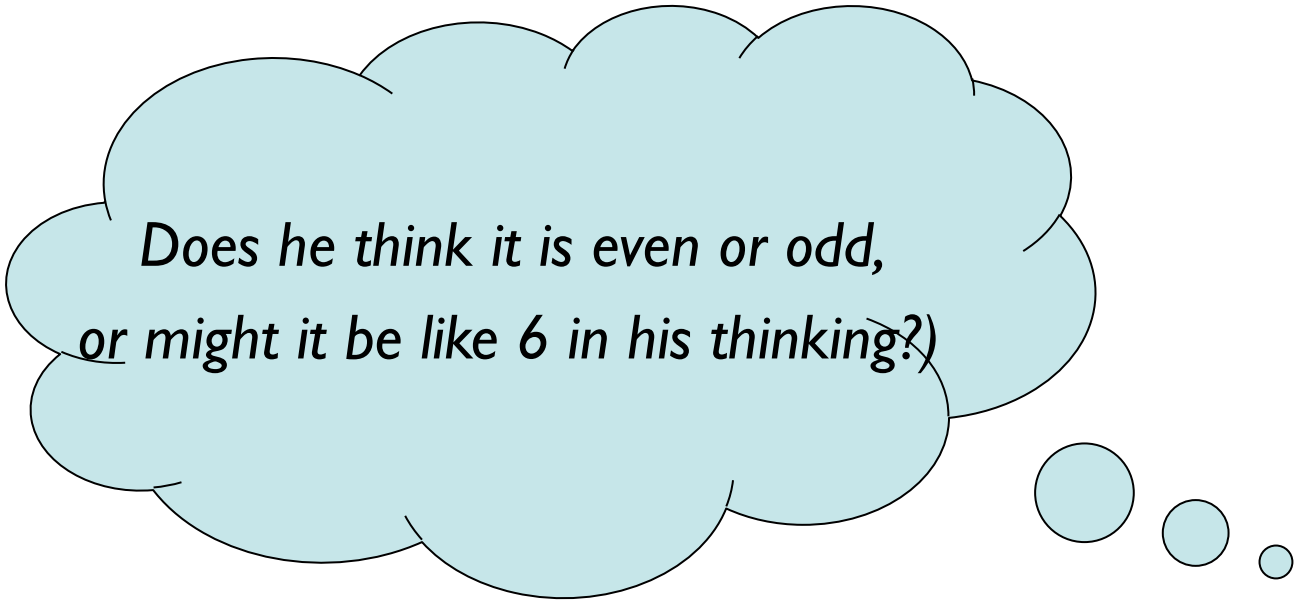
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Case 2:
Posing mathematical questions to students

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What would be good to ask Shea?

"What about the number ____?"



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Case 3:
Composing usable and precise mathematical definitions

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What makes a "good" definition?

- Mathematically precise — correctly identifies the kind of object, process, property
- Usable by user community — based on already-defined and understood term

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What is a mathematically precise and usable definition of "even number" for third graders?

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Proposed Definitions for Even Numbers

a) An even number is a number that can be divided into two equal parts.

b) An even number is any multiple of 2.

c) An even number is any integer multiple of 2.

d) An even number is any number whose unit digit is 0, 2, 4, 6, or 8.

e) An even number is a whole number with zero remainder when divided by 2.

f) A whole number is even if it is the sum of a whole number with itself.

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Consequences

a) An even number is a number that can be divided into two equal parts.

b) An even number is any multiple of 2.

All numbers, for example 7, 3/5, √2, p, are even!

c) An even number is any integer multiple of 2.

This is a correct definition of even number, but not usable by third graders.

d) An even number is any number whose unit digit is 0, 2, 4, 6, or 8.

In this case, 36.7 is an even number!

e) An even number is a whole number with zero remainder when divided by 2.

According to this, -6 is not even.

f) A whole number is even if it is the sum of a particular whole number with itself.

This is a correct definition of evenness for whole numbers, and is consistent with the general definition for integers that students will learn later.

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Case 4: Writing a Quiz and Choosing Examples

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Designing a Quiz on Even and Odd Numbers for Shea and his Classmates

- Choose a list of numbers that you would like Shea and his classmates to classify correctly so that you can learn whether they understand what makes a number even or odd
  - Does it matter what numbers you pose? Why or why not?
- How could you assess their knowledge of the definition of "even number" (and "odd number")?

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Opportunity: Study of Instructional Improvement

- Study of three Comprehensive School Reforms; teacher knowledge a key variable
- Instrument development goals:
  - Usable with 5000 teachers: survey, multiple choice
  - Measure content knowledge teachers *use* in teaching – not just what they *know*
  - Differentiate among teachers
  - Non-partisan

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Overarching Findings: Factor Analyses

- Multidimensionality of mathematical knowledge for teaching
  - Knowledge of students and content different from "pure" content knowledge
  - Knowledge differentiable by topic domain (e.g., algebra, geometry)
  - Number and and operations mathematical knowledge for teaching
    - Specialized
    - Common
- Tentative conclusion: existence of a kind of "professional" knowledge for teaching

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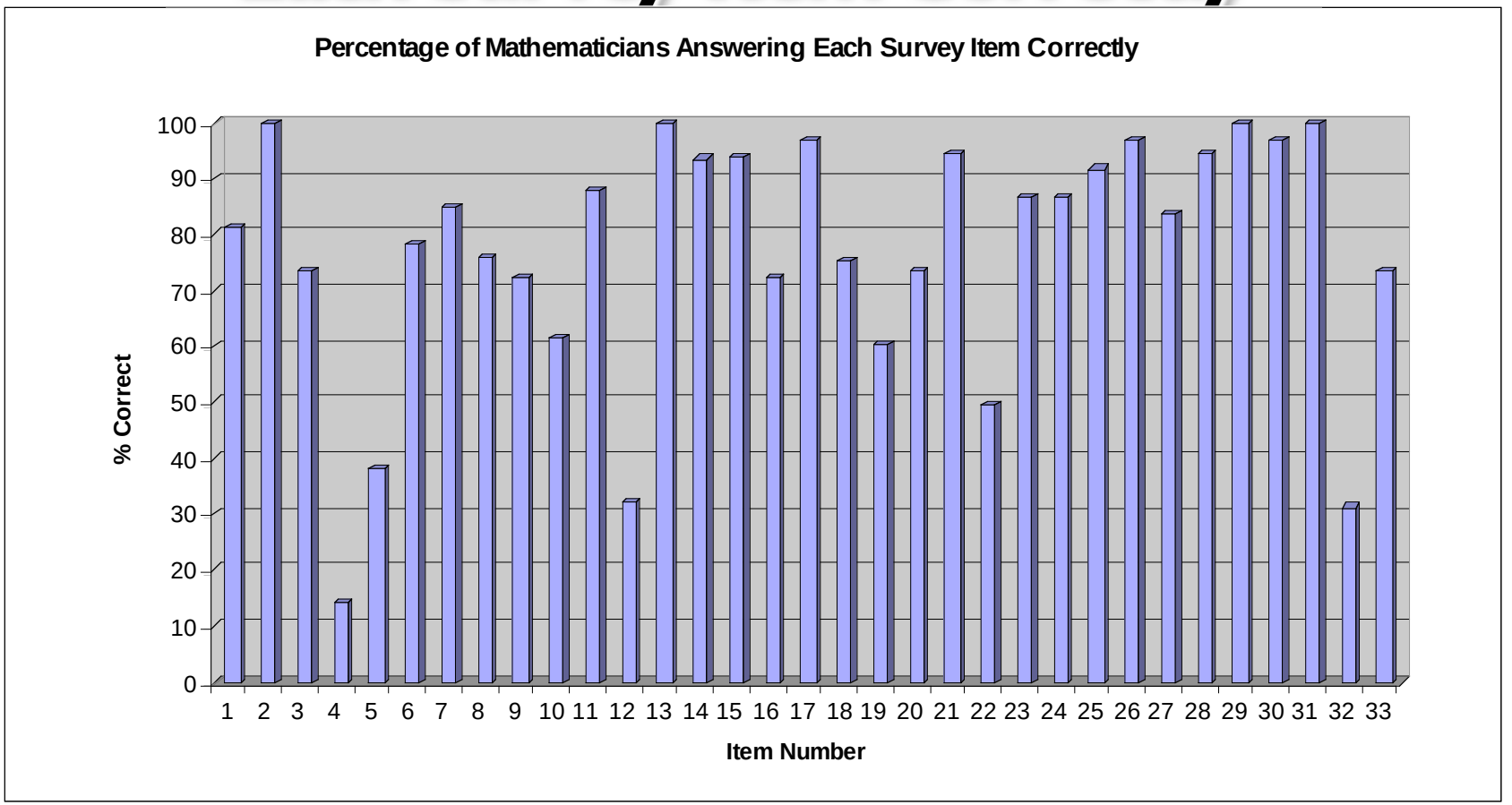
Validating Our Measures

How do we interpret teachers' performance on our questions?

1. Their score reflects their mathematical thinking
   - **Cognitive interviews**
2. Higher scores mean higher-quality mathematics instruction
   - **Videotape validation study**
3. Scores reflect common and specialized knowledge of content
   - **Mathematician and non-teacher interviews**
4. Higher scores related to improved student learning
   - **Study of Instructional Improvement student gains analysis**

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Percentage of Mathematicians Answering Each Survey Item Correctly



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Why Did Mathematicians Get Items Wrong?

- Items had mathematical flaws.
- Items required knowledge of learners.
- Items demanded mathematical knowledge unique to the work of teaching:
  - Making sense of non-standard solutions or ideas
  - Choosing numerical examples
  - Choosing representations

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Linking Teacher Knowledge and Student Achievement

- Questionnaire consisting of 30 items (scale reliability .88)
- Model: Student Terra Nova gains predicted by:
  - Student descriptors (family SES, absence rate)
  - Teacher characteristics (math methods/content, content knowledge)
- Teacher content knowledge significant
  - Small effect (< 1/10 standard deviation): 2 - 3 weeks of instruction
  - But student SES is also about the same size effect on achievement

    (Hill,Rowan, and Ball, AERJ, 2005)

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How Can Professional Development Help Teachers Learn Mathematical Knowledge for Teaching?

- Focus professional development on mathematical content, and build explicit connections to the work of teaching
  - Study content and do application problems that apply that knowledge to teaching
  - Examine student work and analyze it mathematically
  - Keep a list of mathematical terms that have other uses, and learning to distinguish which ones require careful attention, and how
  - Analyze and write mathematical definitions
  - Analyze a textbook lesson for its mathematical quality and language, and modify it with mathematical care
  - Write worksheets and quizzes
  - Analyze assessment items for their mathematical language demands
- Provide opportunities to teachers to develop fluency with mathematical knowledge for teaching
- Connect direct work on achievement disparities with mathematical resources
  - Mathematical language
  - Care with use of contexts
  - Opportunities to broaden what it means to be "good at" mathematics

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Conclusions

1. Mathematical knowledge is crucial to teaching effectiveness.
2. The mathematical knowledge needed is specialized and is more than common content knowledge known by any well-educated adult.
3. Teachers need opportunities to learn mathematics for teaching and to practice using it to teach effectively and to attend to issues of achievement disparities.
4. Professional development providers need professional development, too.
5. Professional development should be evaluated by its effects on instruction and student achievement.

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