DEVELOPING TEACHERS' MATHEMATICAL KNOWLEDGE FOR TEACHING

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UNIVERSITY OF MICHIGAN

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OVERVIEW

1. What do we mean by “mathematical knowledge for teaching” (MKT)?
2. Exploring some examples of the “special mathematical work of teaching”
3. Trying four teacher education tasks designed to develop teachers’ MKT and their capacity to do the special mathematical work of teaching
4. Challenges of teaching MKT

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WHAT IS MATHEMATICAL KNOWLEDGE FOR TEACHING (MKT)?

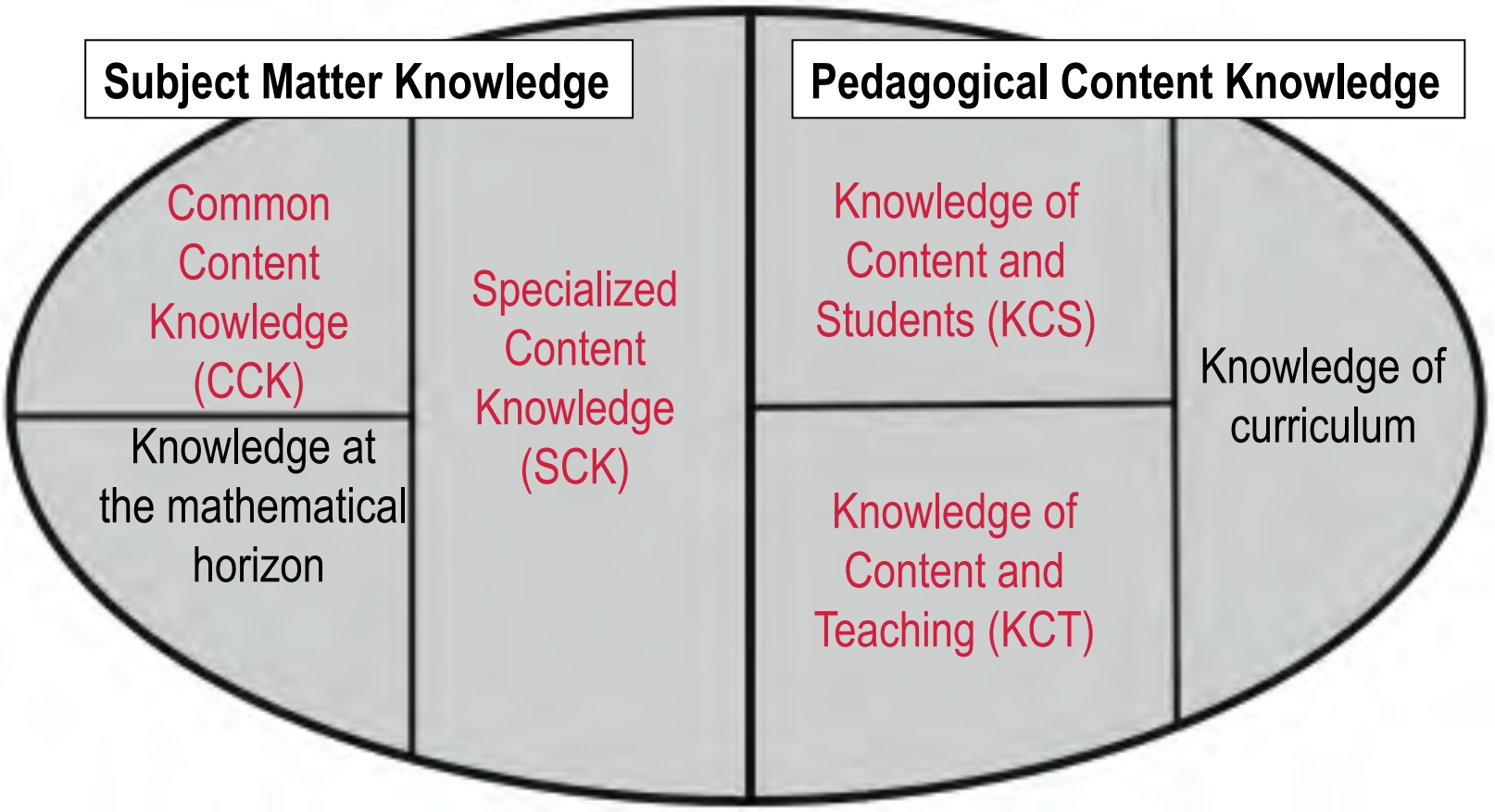
The resources needed to engage in the special mathematical work of teaching:

- By **knowledge** we mean concepts, skills, fluency, habits, practices
- By **work of teaching** we mean the tasks and practices required to help learners, both inside and outside the classroom
- By **mathematical work** we mean the tasks and practices that entail mathematical skill

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MATHEMATICAL KNOWLEDGE FOR TEACHING



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EXAMPLES OF MATHEMATICAL WORK OF TEACHING

- Responding to learners' WHY questions (e.g., Why can't we divide by 0?)
- Unpacking mathematical ideas
- Explaining and guiding explanation
- Using mathematical language and notation
- Generating examples
- Sequencing ideas
- Choosing and using representations
- Analyzing errors
- Interpreting and evaluating alternative solutions and thinking
- Analyzing mathematical treatments in textbooks
- Making mathematical practices explicit
- Attending to issues of equity (e.g., language, contexts, mathematical practices)

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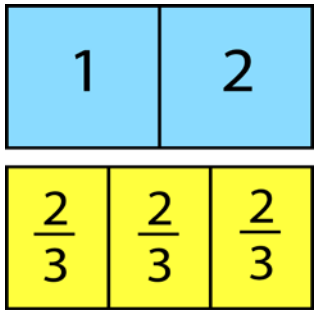
THE SPECIAL MATHEMATICAL WORK OF TEACHING: SOME EXAMPLES

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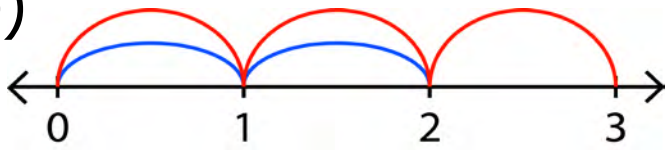
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ANALYZING REPRESENTATIONS: WHICH OF THE FOLLOWING CAN BE INTERPRETED TO REPRESENT 2 ÷ ⅔?

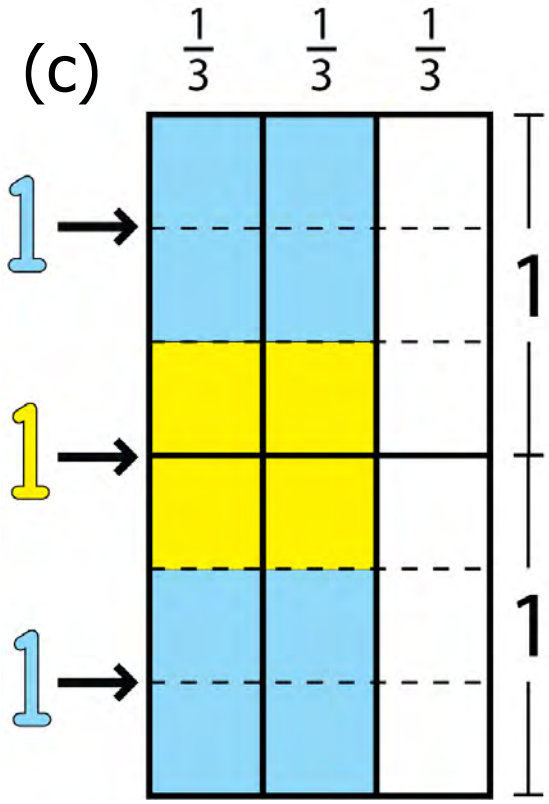
(a)



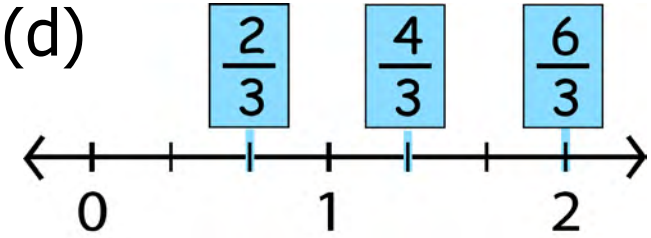
(b)



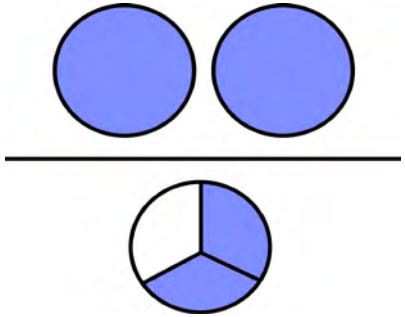
(c)



(d)



(e)



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CHOOSING EXAMPLES

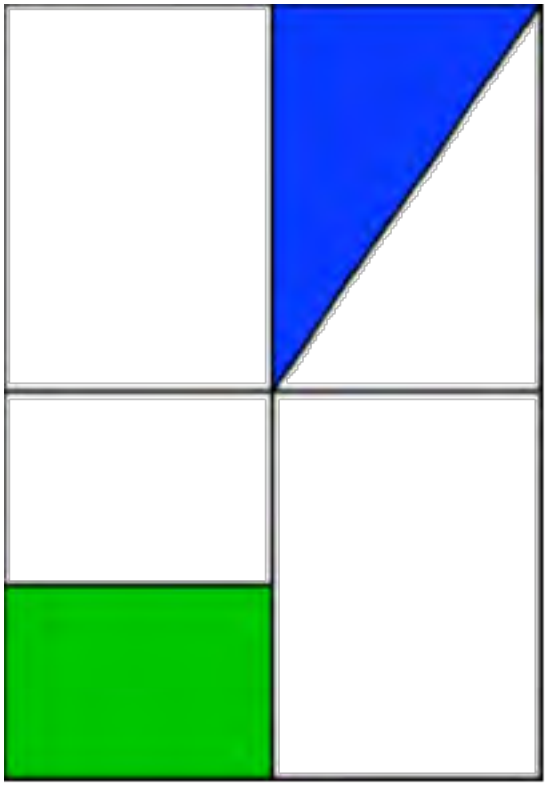
A.	.5	7	.01	11.4
B.	.60	2.53	3.12	.45
C.	.6	4.25	.565	2.5

D. These lists are all equally good for assessing whether students understand how to order decimal numbers.

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EXPLAINING MATHEMATICAL IDEAS



- What is involved in showing that the blue region and the green region are each one-eighth (1/8) of the whole rectangle?
- What is involved in showing that the green and the blue together cover one-fourth (1/4) of the whole rectangle?

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CHOOSING EXAMPLES

Which of the following is best for setting up a discussion about different solution paths for simplifying radical expressions?

(a) $\sqrt{54}$

(b) $\sqrt{156}$

(c) $\sqrt{128}$

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THE SPECIAL MATHEMATICAL WORK OF TEACHING: SOME EXAMPLES

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OUR APPROACH TO THE PROBLEM

1. Identify “high-leverage” teaching practices
2. Find/develop tasks that create opportunities for learning to do that special work
3. Provide opportunities to practice the kinds of mathematical thinking, reasoning, and communicating used in teaching

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DESIGNING TASKS TO DEVELOP MKT

What is the difference between a good mathematical task and one that is good for developing mathematical knowledge for teaching?

Why does this matter?

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FEATURES OF TASKS DESIGNED TO DEVELOP MKT

- Unpacks, makes explicit, and develops a flexible understanding of mathematical ideas that are central to the school curriculum
- Opens opportunities to build connections among mathematical ideas
- Provokes a stumble due to a superficial "understanding" of an idea
- Lends itself to alternative/multiple representations and solution methods
- Provides opportunities to engage in mathematical practices central to teaching (explaining, representing, using mathematical language, analyzing equivalences, proving, proof analysis, posing questions, writing on the board)

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FOUR MKT TASKS

1. Creating and analyzing definitions
2. Analyzing errors
3. Choosing examples
4. Seeing mathematics in teaching practice

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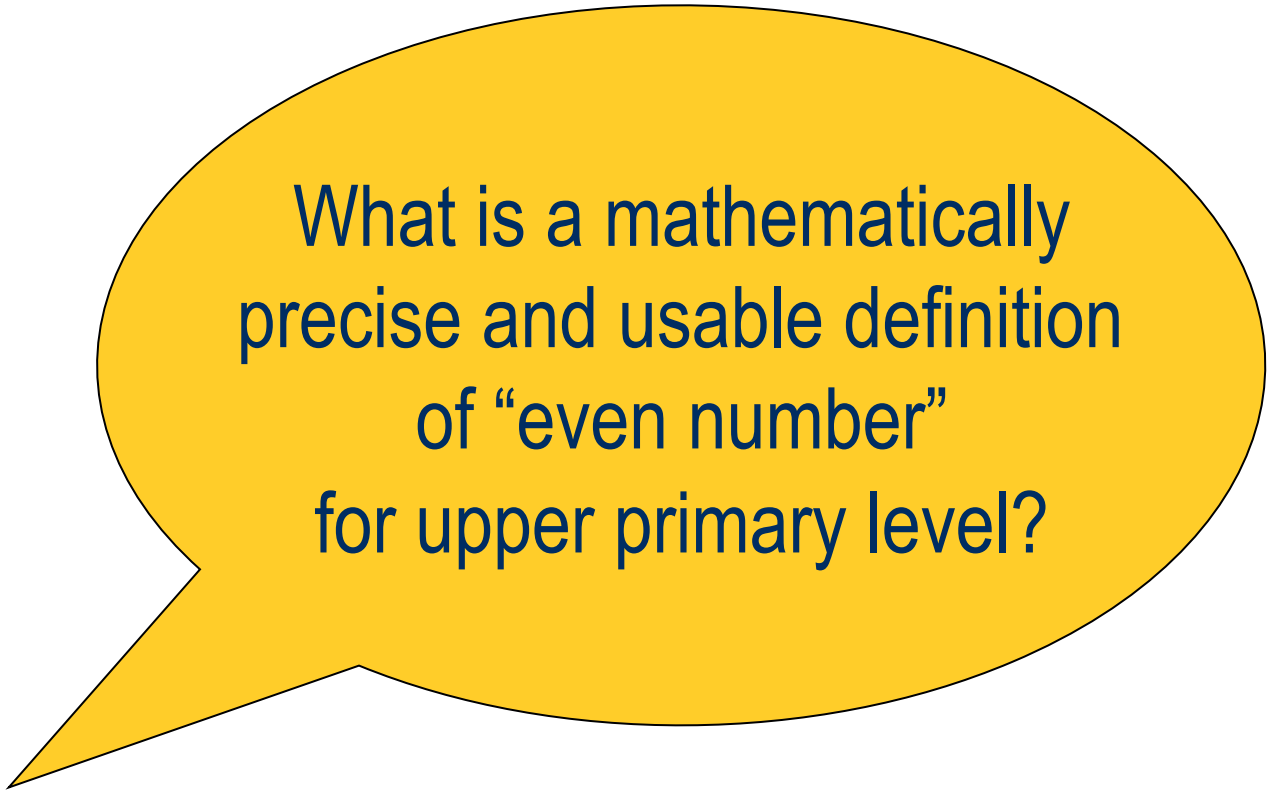
QUESTIONS TO CONSIDER WHEN EXAMINING TASKS

1. What opportunities to learn MKT does each task provide?
2. Looking across the tasks: What are central features of MKT tasks?
3. What are the challenges of enacting MKT tasks with teachers?

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TASK #1: MAKING DEFINITIONS



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PROPOSED DEFINITIONS FOR EVEN NUMBERS

1. An even number is a number that can be divided into two equal parts.
2. An even number is any multiple of 2.
3. An even number is any integer multiple of 2.
4. An even number is any number whose unit digit is 0, 2, 4, 6, or 8.
5. An even number is a whole number with zero remainder when divided by 2.
6. A whole number is even if it is the sum of a whole number with itself.
7. If you add 1 to any odd number you will obtain an even number.

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CONSEQUENCES

a) An even number is a number that can be divided into two equal parts.

b) An even number is any multiple of 2.

All numbers, for example 7, 3/5, $\sqrt{2}$, π, are even!

c) An even number is any integer multiple of 2.

This is a correct definition of even number, but not usable at the primary level.

d) An even number is any number whose unit digit is 0, 2, 4, 6, or 8.

In this case, 36.7 is an even number!

e) An even number is a whole number with zero remainder when divided by 2.

According to this, -6 is not even.

f) A whole number is even if it is the sum of a particular whole number with itself.

This is a correct definition of evenness for whole numbers, and is consistent with the general definition for integers that students will learn later.

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ANALYSIS DEFINITIONS:
A WORKING DEFINITION OF FRACTIONS

- Identify the whole
- Make *d* equal parts
- Write 1/*d* to show one of the equal parts
- If you have *d* of 1/*d*, then you have the whole
- If you have *n* of 1/*d*, then you have n/d
  - *n* and *d* are whole numbers (now, in 5th grade)
  - *d* cannot be 0

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TASK #2: ANALYZING ERRORS

What mathematical steps produced each of these answers?

What might students have been thinking?

(a)

$$\begin{array}{r} 49 \\ \times\ 25 \\ \hline 405 \\ 108 \\ \hline 1485 \end{array}$$

(b)

$$\begin{array}{r} 49 \\ \times\ 25 \\ \hline 225 \\ 100 \\ \hline 325 \end{array}$$

(c)

$$\begin{array}{r} 49 \\ \times\ 25 \\ \hline 1250 \\ 25 \\ \hline 1275 \end{array}$$

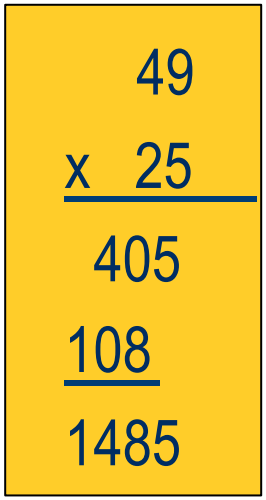
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TAKING NEXT STEPS

Consider this error:



- What are three different things that might have led to this error? What could you do to find out?
- What would you do to help learners understand the mistake, and what would you try to remedy it?

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TASK #3: CHOOSING EXAMPLES

- Comparing fractions on the number line
- Select a set of 5 numbers for pupils to place on the line and explain why you chose that set of numbers. What will each provide in terms of opportunity to learn?



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TASK #4: SEEING MATHEMATICS IN TEACHING

What mathematical issues do you see in the video?

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THE WORK OF TEACHING MATHEMATICS: ONE LOOK

"Off camera": Before this episode

1. Learn about individual pupils and what they know, care about, are worried about, can do, etc.; pay attention to who participates in what ways
2. Establish the environment to create norms and manage behavior
3. Teach intellectual habits (e.g., drawing, speaking to peers, knowing and being able to choose and make different kinds of mathematical moves)
4. Choose the specific problem: Why this diagram? Why these fractions? Why this wording, and could it have been worded better?

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DURING THESE 5 MINUTES

00:00—Open the discussion: use specific tone, body movement around the room, encourage volunteers, wait for 8 seconds, choose whom to call on, and call on that learner

00:27—Structure Mamadou's turn: request answer, listen, ask for explanation, ask him to come to the board and "show it"

00:56—While Mamadou comes to the board, structure the attention of the other pupils ("You should be thinking already about his reason"; "Who can repeat what Mamadou said?")

01:08—Dovan repeats Mamadou's answer and begins to analyze it. Stop Dovan from analysis ("Wait, let him talk")

01:25—Structure space for Mamadou's explanation; direct other pupils to watch

01:43—Support pupils to examine Mamadou's answer of ½, focus on "the whole," "equal parts"; scaffold Mamadou's reprise

02:14—Ask other pupils how Mamadou got the answer ½ (not agree or disagree); Kalvin explains; draw on new diagram to highlight different "whole"; back and forth speaking to Mamadou, to class, to Kalvin

03:00—Ask class if Mamadou is right, given the different "whole"

03:04—Point to "different question" in original problem; conclude Mamadou's turn

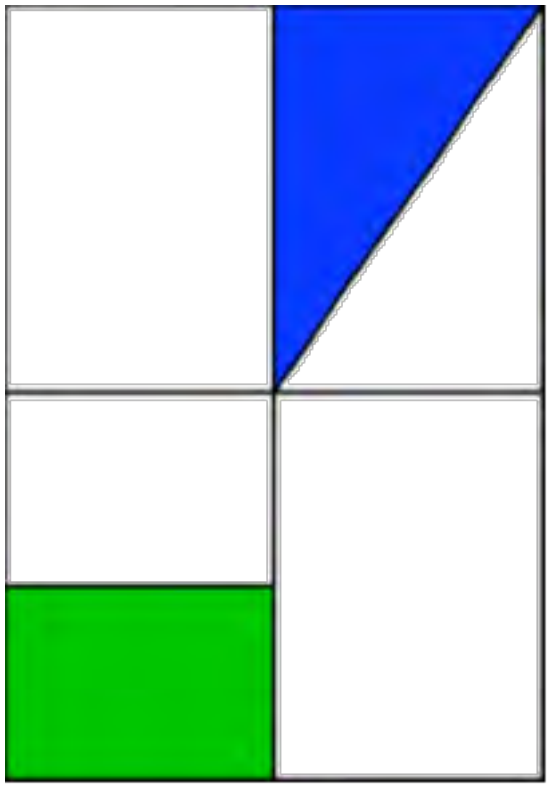
03:19—Focus class on original question, re-read, interpret "the big rectangle," ask Ariel to show; put up new copy and ask class about shaded part of "whole big rectangle"

04:27—Ask Mamadou to explain what is different about this question; correct "square" and "half" in his talk

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USING THE PROBLEM MATHEMATICALLY



- Unpacking the notion of “equal parts”: counting parts, measuring, congruence of areas
- What is involved in proving that the blue region and the green region are each one-eighth (1/8)?
- What is involved in showing that the green and the blue together cover one-fourth (1/4) of the whole rectangle?

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CHALLENGES OF TEACHING MKT

- Lack of good materials and problems
- Our own knowledge and skills as teacher educators
- Easy to slip into working “only” on the math or “only” on teaching issues or issues about learners

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CHALLENGES OF TEACHING MKT

- Staying focused on the mathematics, and not on how to *teach* the math
- Keeping the problems focused on MKT and not just "M"
- Unpacking the mathematics sufficiently and convincingly helping them see what there is to learn and do
- Making visible the connections to the kinds of mathematical thinking, judgment, reasoning one has to do in teaching

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ENACTMENT:
WHAT ARE KEY QUESTIONS AND MOVES THAT CAN BE USED TO KEEP A TASK FOCUSED ON DEVELOPING MKT?

- Asking teachers to explain their solutions to the others
- Having teachers explain what is/was confusing them
- Asking teachers to figure out what might be confusing/difficult for someone else about the problem
- Having teachers ask questions to become more clear about their colleagues' solutions
- Asking teachers to make correspondences between solutions and/or representations
- Asking teachers to explain someone else's thinking
- Providing opportunities to 'talk mathematics' and write on the board
- Provoking a common error
- Narrating how something a teacher does/says relates to or is a skill used in teaching

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