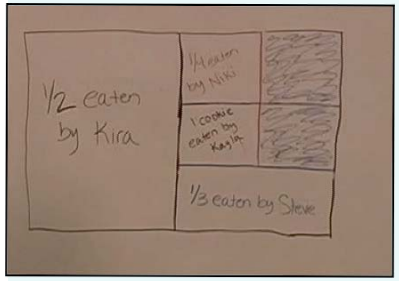
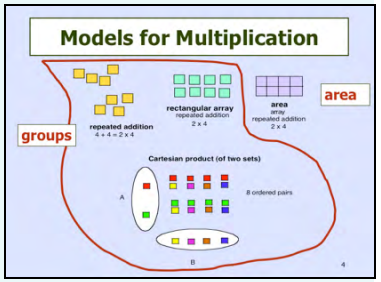
Teaching
for Mathematics Learning





Deborah Loewenberg Ball

Hyman Bass

University of Michigan

Center for Proficiency in Teaching Mathematics

The Future of Mathematics Education and Mathematics Learning

BIFEB, Strobl,Österreich

14 -19 August 2005

www-personal.umich.edu/
~dball/

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Problems of Mathematics Education in the U.S.

- Too many students not learning enough worthwhile mathematics well enough for
  - Continued mathematics study
  - Use in everyday settings
  - Workplace
- Pervasive inequality: Unequal distribution of mathematical success by race and social class

GOAL: DEVELOPING USEFUL AND USABLE RESOURCES FOR SOLVING THESE PROBLEMS

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Overview of Session

1. Looking in a mathematics classroom what do “we” “see”?
2. Seeing teaching as mathematical work
3. Knowing mathematics in and for teaching
4. Developing and studying teachers’ knowing of mathematics for teaching



© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission



A SHORT VISIT TO A CLASSROOM

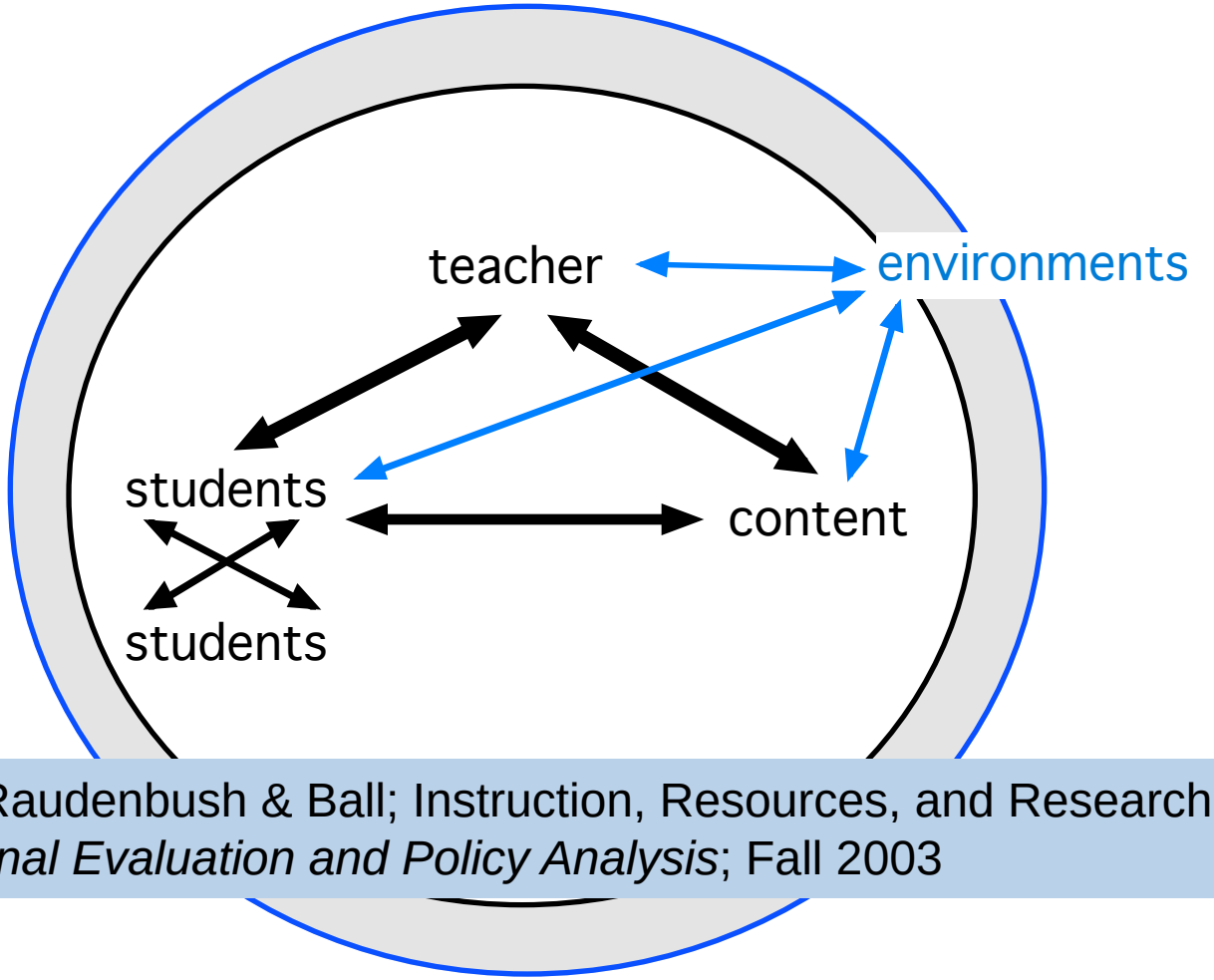
© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Viewing #1

- What do you notice?
- What is important to see in this classroom segment?

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Instruction as Interaction



Cohen, Raudenbush & Ball; Instruction, Resources, and Research; *Educational Evaluation and Policy Analysis*; Fall 2003

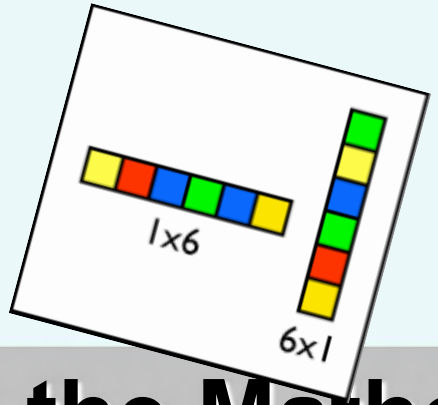
© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission



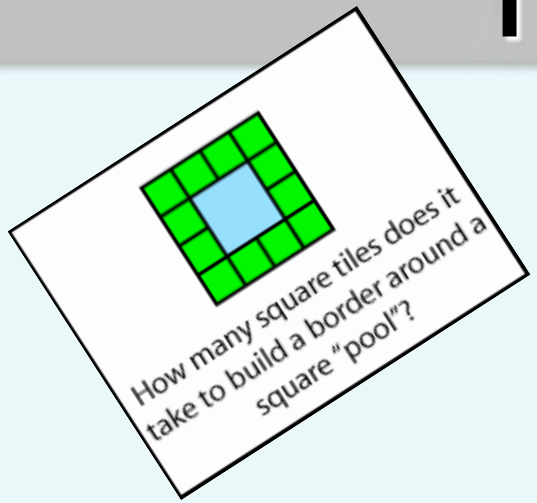
SEEING TEACHING AS MATHEMATICAL WORK

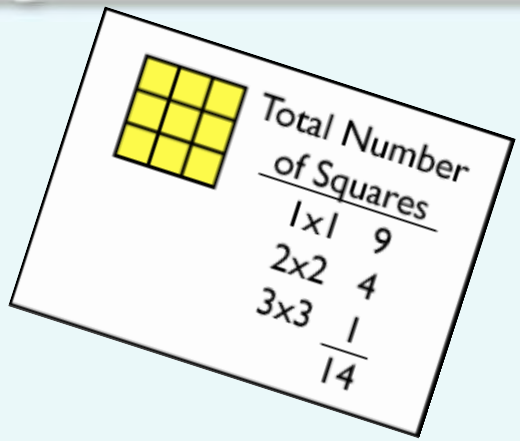
1. Investigating the Mathematical Territory
2. Investigating the Work Inside the Mathematical Territory

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission



1. Investigating the Mathematical Territory



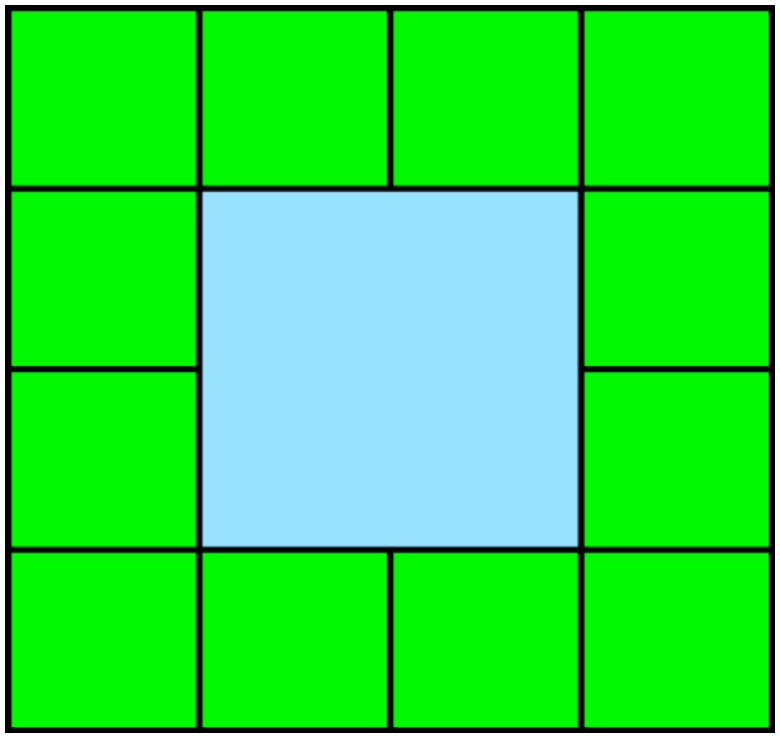


© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Pool Border Problem

How many square tiles does it take to build a border around a square "pool"?

Find a way to know the number of tiles it will take without having to count, for <u>any size</u> pool.



© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Geometric and Algebraic Solutions to the Pool Border Problem

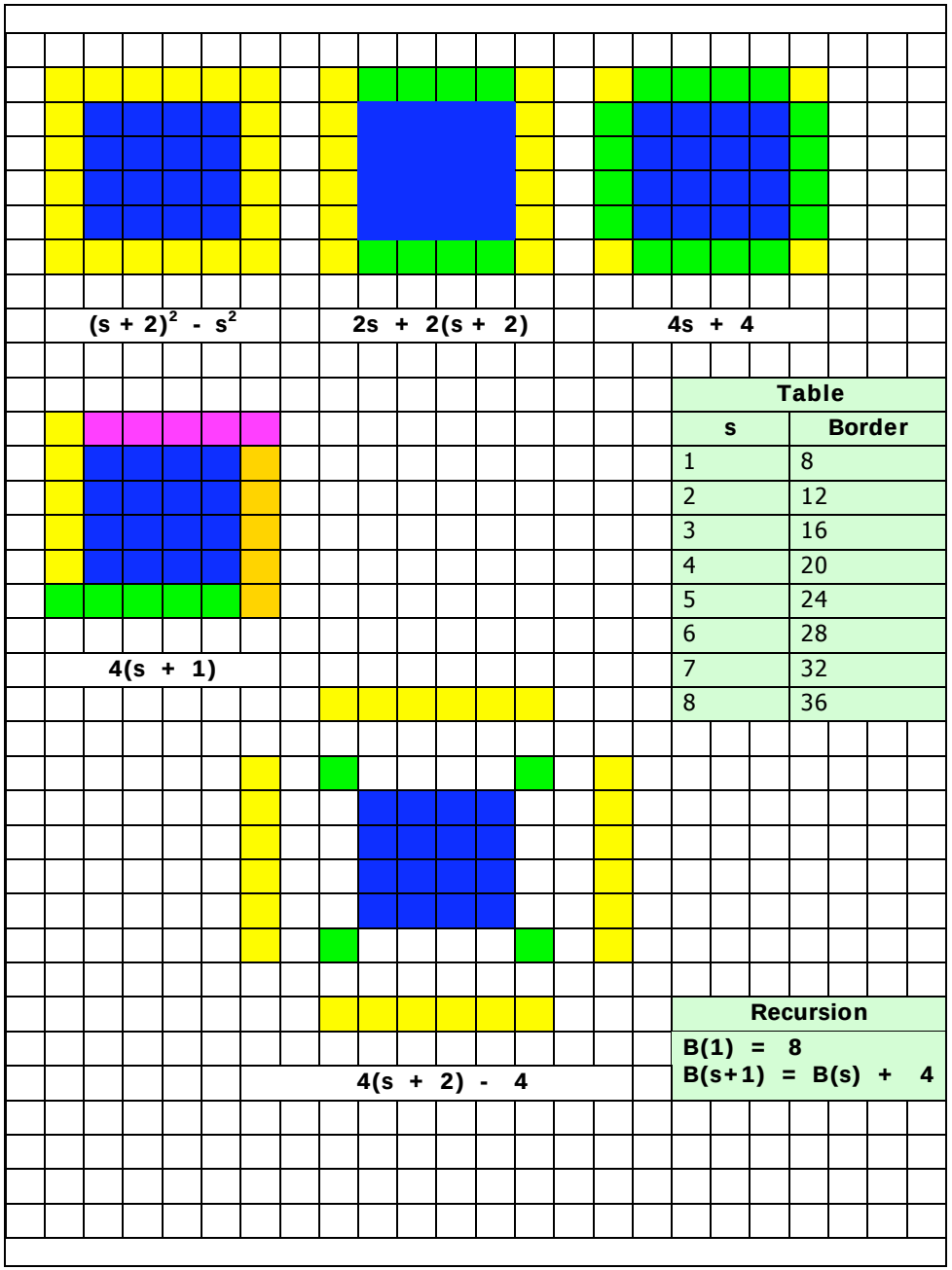


Table	
s	Border
1	8
2	12
3	16
4	20
5	24
6	28
7	32
8	36

Recursion
B(1) = 8
B(s+1) = B(s) + 4

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

What Mathematics is (Potentially) Involved Here?

TOPICS

- Combinatorial geometry — relating linear and area measure
- Linear functions/growth
- Skills with algebraic notation and geometric representations to describe situations

PRACTICES

- Checking and analyzing small cases to search for a solution in general; pattern or structure seeking
- Finding multiple forms of the solution and mapping correspondences between them
- Use of geometric and/or algebraic methods to organize and express solutions

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission



2. Investigating the Work INSIDE the Mathematical Territory

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Setting the Background

- "Laboratory" class
- Grade 5 pupils (ages 10 - 11)
- Wide range of mathematical achievement and disposition
- 20 hours of instruction:
  - intersection of geometry and arithmetic
  - skill development
  - mathematical explanation, and precision with language and representation
  - discourse of exploration, conjecture, explanation, proof

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Mathematical Content (Ideas and Practices)

- Concept of rectangle
- Defining rectangle (and other quadrilaterals) precisely
- Squares, primes, composites (and definitions)
- Multiplication
- Rotations in the plane
- Using exponents
- Using variables to express a generalization
- Formulating and seeking to prove or refute significant mathematical conjectures
- Justifying the completeness of a solution
- Factors and finding all the factors of a number (algorithm and its proof)
- Mathematical explanation
- Mathematical generalization
- Writing and representing mathematical ideas

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Basic Instructional Formats

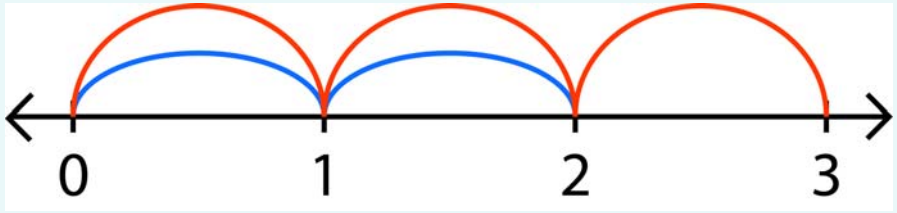
- Class segments:
  - skill review
  - independent and partner work on main problems
  - whole group discussion and development of main problems
  - individual notebook writing
- Daily homework composed of multiple genres of work
  - Practice
  - Extension
  - Next steps on something unfinished
  - Demonstrate to family

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Viewing #2: More Specific Focus

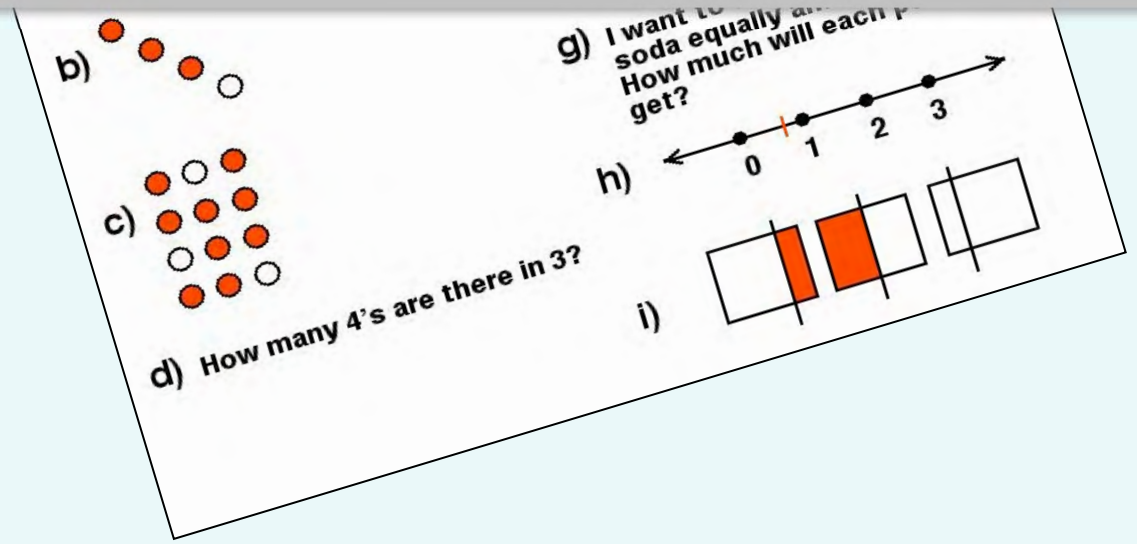
- What is the teacher doing?
- What mathematical reasoning or other knowledge is required for the teaching?

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission



$$2 \div \frac{2}{3}$$

KNOWING MATHEMATICS FOR TEACHING



© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Common Framing of the Problem
⟶ Failure to Solve the Problem

- Teachers do not know mathematics well enough (evidence and causes vary)
- Concentration of under-qualified teachers in urban and high poverty schools

- Increase number and rigor of mathematics requirements for teacher certification
- Recruit mathematically-trained people into teaching

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Clarifying the Problem

The quality of mathematics teaching and learning

Teachers' knowledge of mathematics

and their ability to use it in teaching

What mathematical resources do teachers need

in order to teach mathematics to all students effectively?

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

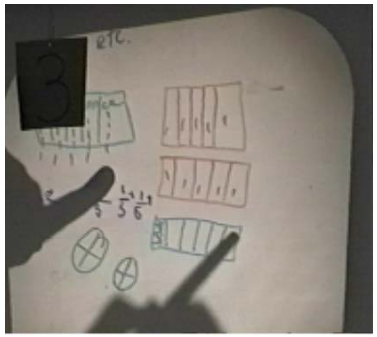
A Practice-Based Approach to Answering the Question

1. **Study instruction**, and identify the mathematical work of teaching
2. Analyze **what mathematical knowledge** is needed to do that work effectively, and **how** it must be understood in order to be useful for the work
3. Develop, test, and refine **approaches to helping teachers** develop and use mathematical knowledge for teaching, study results

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

What Do we Mean By "Mathematical Knowledge for Teaching"?





© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Mathematical Knowledge for Teaching

What do we mean when we use this term, "**mathematical knowledge for teaching**"?

- Mathematical knowledge, skill, habits of mind that are entailed by the work of teaching

What do we mean by the "**work of teaching**"?

- The tasks in which teachers engage, and the responsibilities they have, to teach mathematics, both inside and outside of the classroom

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

What *is* "Mathematical Knowledge for Teaching"? An Example from Multiplication of Decimals

Multiply:

$$\begin{array}{r} 3.5 \\ \times\ 2.5 \\ \hline \end{array}$$

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Analyzing Incorrect Answers for $\begin{array}{r} 3.5 \\ \times\ 2.5 \\ \hline \end{array}$

Why do you multiply before you add?

(a)
$$\begin{array}{r} 3.5 \\ \times\ 2.5 \\ \hline 255 \\ 80 \\ \hline 10.55 \end{array}$$

(b)
$$\begin{array}{r} 3.5 \\ \times\ 2.5 \\ \hline 62.5 \end{array}$$

Why do you have as many decimal places in the answer as the total number of decimal places in the problem?

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Analyzing Correct Answers for

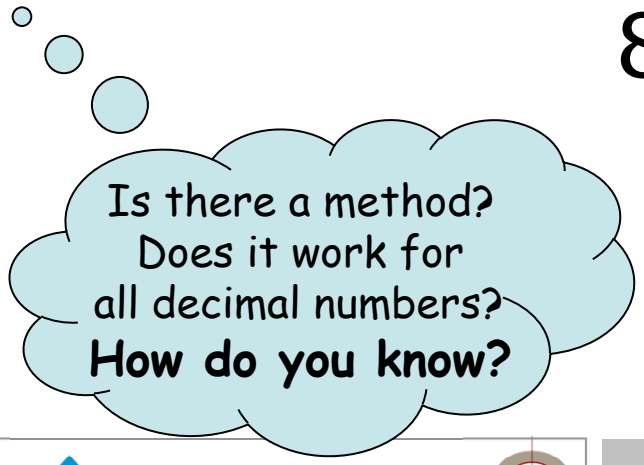
$$\begin{array}{r} 3.5 \\ \times\ 2.5 \\ \hline \end{array}$$

(a)

$$\begin{array}{r} 3.5 \\ \times\ 2.5 \\ \hline .25 \\ 1.5 \\ 1 \\ 6 \\ \hline 8.75 \end{array}$$

(b)

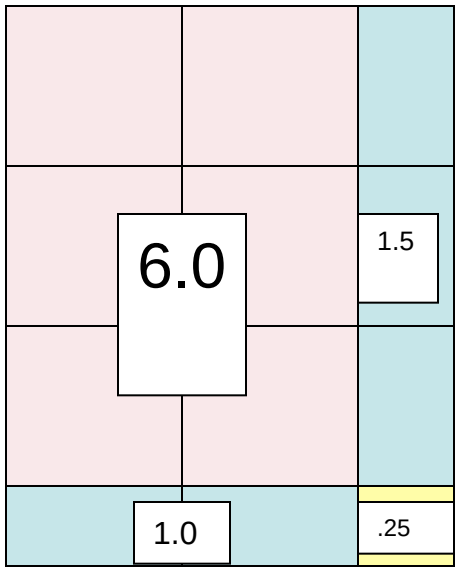
$$\begin{array}{r} 3.5 \\ \times\ 2.5 \\ \hline 1.25 \\ 7.5 \\ \hline 8.75 \end{array}$$



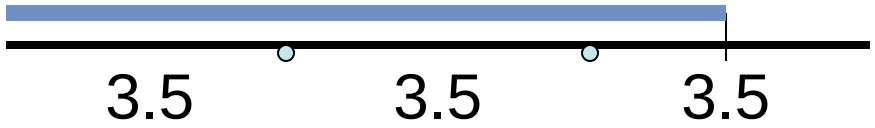
© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Using Representations for $\begin{array}{r} 3.5 \\ \times\ 2.5 \\ \hline \end{array}$

(a)



(b)



Explain:
Where is the 3.5?
Where is the 2.5?
Where is 8.75?

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Other Work of Teaching

- Choosing and using representations
- Choosing specific numerical or geometric examples
- Using and defining terms
- Using notation
- Explaining

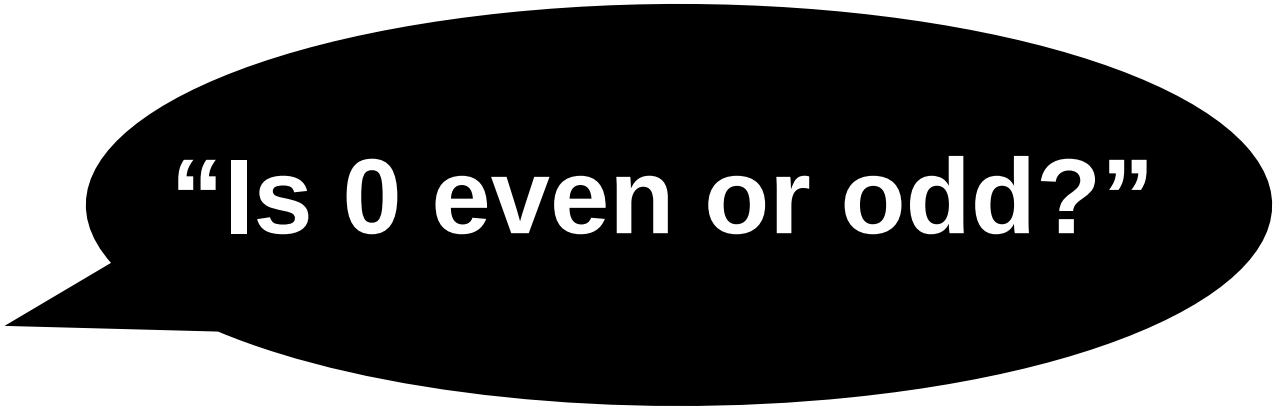
© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Choosing Numerical Examples

A.	.5	7	.01	11.4
B.	.60	2.53	3.12	.45
C.	.6	4.25	.565	2.5

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Use and Teach Definitions

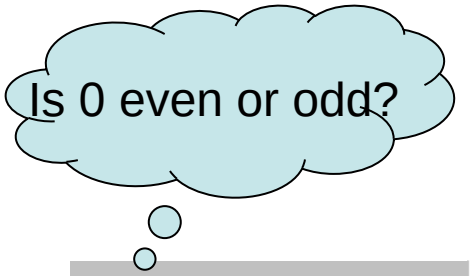


© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Examine Textbook Definitions

How well does any of these help to answer the the question?

1. An even number is a number of the form 2k, where k is an integer.
2. An even number is a natural number that is divisible by 2.
3. An even number is any multiple of 2.
4. An even number is a number that has 0, 2, 4, 6, or 8 in the ones place.



© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Define Terms

Write a definition of “even number” that is mathematically precise and also usable by third graders.

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Hypotheses About Knowledge of Mathematics for Teaching

PREVALENT HYPOTHESES

1. Teachers need knowledge of the curriculum, plus N levels more knowledge
2. "Pedagogical content knowledge" (— but what is it?)

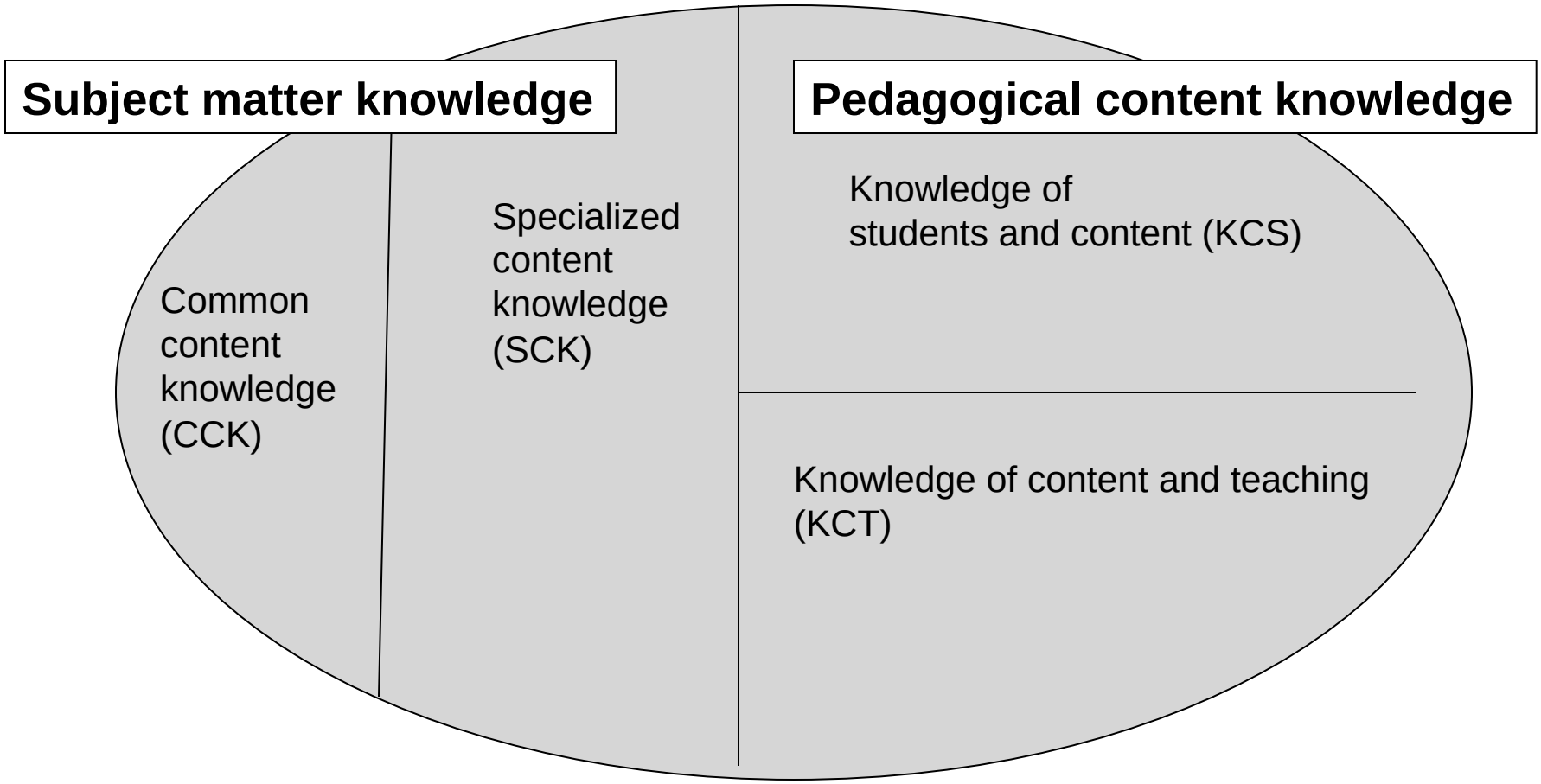
OUR CURRENT HYPOTHESES

- **Common** content knowledge
- **Specialized** content knowledge
- Knowledge of content and **students**
- Knowledge of content and **teaching and curriculum**

What does this have to do with "pedagogical content knowledge"?

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

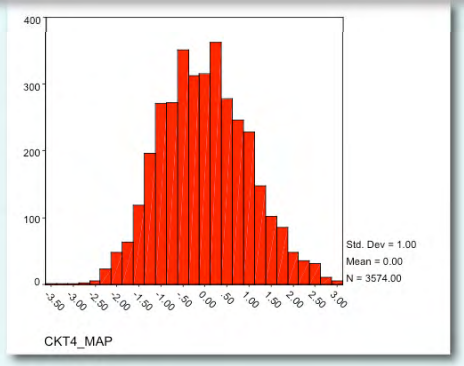
Shulman's Original Category Scheme (I985) Compared with Ours



© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

DEVELOPING AND STUDYING TEACHERS' MATHEMATICAL KNOWLEDGE FOR TEACHING

Student A	Student B	Student C
35 x25 125 +75 875	35 x25 175 +700 875	35 x25 25 150 100 +600 875



© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Can These Ideas Improve Students' Mathematics Learning?

DEVELOPING APPROACHES FOR HELPING TEACHERS LEARN MKT

STUDYING THE RELATIONSHIPS OF TEACHERS' MKT AND STUDENT ACHIEVEMENT

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Using Data to Test and Improve Theory

- Factor analyses
- Analyses of validity
- Uses of measures
  - To predict student achievement
  - To evaluate professional development

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Why Would We Want to "Measure" Teachers' Content Knowledge for Teaching?

- To understand role of teachers' content knowledge in students' performance
- To study and compare outcomes of professional development and teacher education
- To inform design of teachers' opportunities to learn content knowledge

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Measures of Teachers' Mathematical Knowledge for Teaching

OUR GOALS:

- Measure mathematical knowledge used in teaching
- Contextualize the items in tasks (or situations) that all teachers face
- Usable measures:
  - Large sample of teachers (N=5500) = multiple choice
  - Not grade-specific (grade 1 - 6 teachers)
  - Not represent any single view of teaching (e.g., "reform")
  - Items that vary in difficulty

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Factor Analysis

- Enables analyses of the number of underlying factors in a data set
- Can link items to constructs
- Ask similar question of our data:
  - How many factors? What are they named?
  - How do items group?

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Overarching Findings: Factor Analyses

- Multidimensionality of mathematical knowledge for teaching
  - Knowledge of students and content different from content knowledge
  - Patterns, functions & algebra content knowledge for teaching
  - Geometry mathematical knowledge for teaching
  - Number and and operations mathematical knowledge for teaching
    - Specialized
    - Common
- Tentative conclusion: existence of professional mathematical knowledge for teaching

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Validating Our Measures

How do we interpret teachers' performance on our questions?

1. Their score reflects their mathematical thinking
   - **Cognitive interviews**
2. Higher scores mean higher-quality mathematics instruction
   - **Videotape validation study**
3. Scores reflect common and specialized knowledge of content
   - **Mathematician and non-teacher interviews**
4. Higher scores related to improved student learning
   - **Study of Instructional Improvement student gains analysis**

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Studying Links Between Teachers' Knowledge of Mathematics for Teaching and Student Achievement

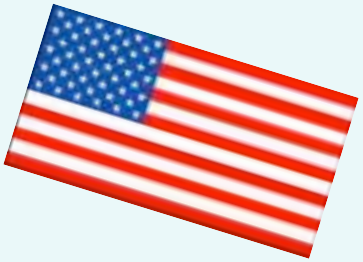
Question: Do students learn more when teachers do better on our measures?

- SII data and analysis – measured CKT on questionnaire
- SII measures highly reliable, from test construction perspective

- Model: Student Terra Nova gains predicted by:
  - Student descriptors (family SES, absence rate)
  - Teacher characteristics (math methods/content, content knowledge)
- Teacher content knowledge significantly related to achievement gains
  - Small effect (< 1/10 standard deviation)
  - But student SES is also on same order of magnitude

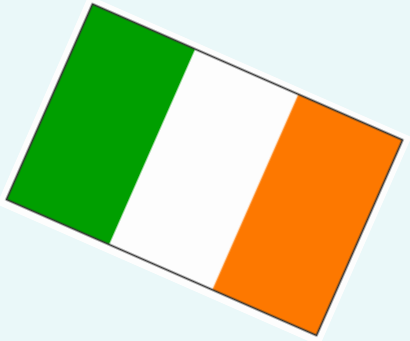
Hill, Rowan, & Ball, Effects of teachers' mathematical knowledge for teaching on student achievement; *American Educational Research Journal*, Summer 2005

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Is Mathematical Knowledge for Teaching General or Culturally-Specific?





© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Conclusions

- Teaching is an important and often invisible contributor of learning.
- A major share of teaching is mathematical work.
- Mathematical knowledge for teaching is the knowledge needed for the work of teaching.That work entails specialized content knowledge distinct from that required by other mathematically-intensive professions.
- Knowing mathematics in special ways may be important to improving achievement of students, especially for those who are most “school-dependent.”

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Slides will be available at

http://www-personal.umich.edu/~dball/

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Student Achievement Data and Analysis

- Other key measures and data sources
  - Terra Nova gain scores (fall-fall or spring-spring)
  - Teacher coursework, certification, years experience
  - Time in mathematics instruction
  - CKT-English Language Arts (ELA)

- Mixed model (SAS)
  - 3-level model (students, teachers, schools)
  - Two models: 1st and 3rd grade

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Results of Student Achievement Analysis

	Grade 1	Grade 3
Intercept	57.6 (1.32)	39.3 (.96)
Student variables		
SES	3.95*** (.94)	2.24** (.78)
….		
Teacher/classroom variables		
Math methods & content	.55 (.99)	1.69 (.92)
Certified	.28 (.87)	-.24 (.76)
Years experience	.72 (1.13)	1.04 (.65)
CKT-Mathematics	2.23* (.91)	2.23** (.77)
Math lesson length	-.13 (1.03)	1.80* (.85)
…..		

* Significant at $p < .05$
** Significant at $p < .01$
*** Significant at $p < .001$

- Teachers' CKT - M performance significantly predicts student gain scores
  - -1st and 3rd grade
- Small effect
  - -2-3 weeks of additional instruction
  - -comparable to SES
- Instructional time
  - -Positive in 3rd grade

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Correlates of Teacher Knowledge

	Math methods & content	Certified	Years experience	CKT-Math	CKT-ELA
Math methods & content	1.0	.03	.19**	-.08	-.05
Certified		1.0	.15*	.11*	.02
Years experience			1.0	-.09	.05
CKT-Mathematics				1.0	.37**
CKT-ELA					1.0

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Writing Items: Our Strategy

- Limit topic domains: number concepts; operations; patterns, functions, and algebra
- Use varied sources to identify task domains of teaching
- Involve experts in item development and review
- Pilot in California as part of development

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission