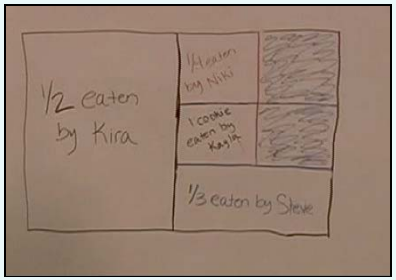
Knowing Mathematics for Teaching

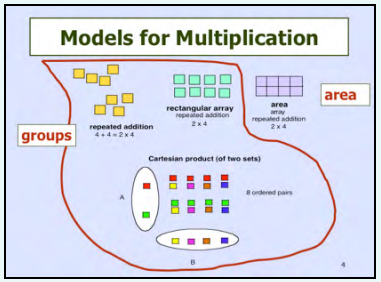
Deborah Loewenberg Ball and Hyman Bass

Imani Masters Goffney

University of Michigan

Center for Proficiency in Teaching Mathematics





IHE Network Conference

Milwaukee Mathematics Partnership

August 25 & 26, 2005

www-personal.umich.edu/~dball/

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Overview of Workshop

1. Foundations: Major problems of mathematics education, and the need to focus on instruction
2. Teaching mathematics for proficiency and equity: Focus on reasoning and language
3. Knowing mathematics for teaching
4. Assessing mathematical knowledge for teaching
5. Developing teachers' mathematical knowledge

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

1. Foundations: Problems of Mathematics Education and the Need to Focus on Instruction

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

What are Central Problems of Mathematics Education?

- Ineffective curriculum: Too many students not learning mathematics well enough for
  - Continued mathematics study
  - Use in everyday settings
  - Workplace
- Pervasive inequality: Unequal distribution of mathematical success by race, social class
- Lack of capacity for improvement
  - Public understanding, support, investment
  - Teacher shortages, teacher development and support
  - Teacher educator development and resources

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

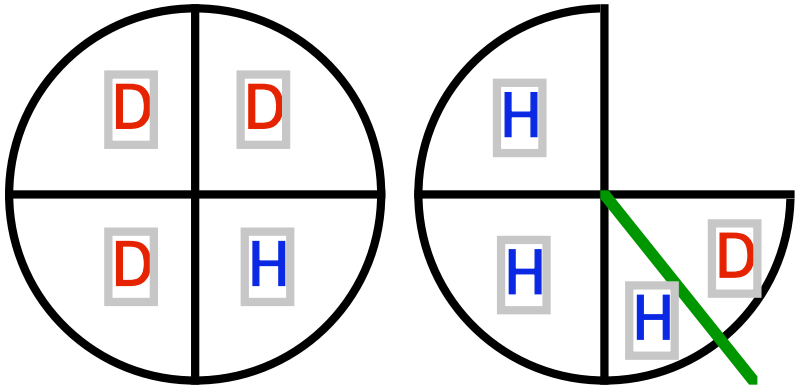
A Closer Look

$$1\frac{3}{4} \div \frac{1}{2}$$

1. Calculate the answer.
2. Write a story problem, or describe a situation, that corresponds to the calculation. Make sure the answer to your story is the same as the answer to your calculation.

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

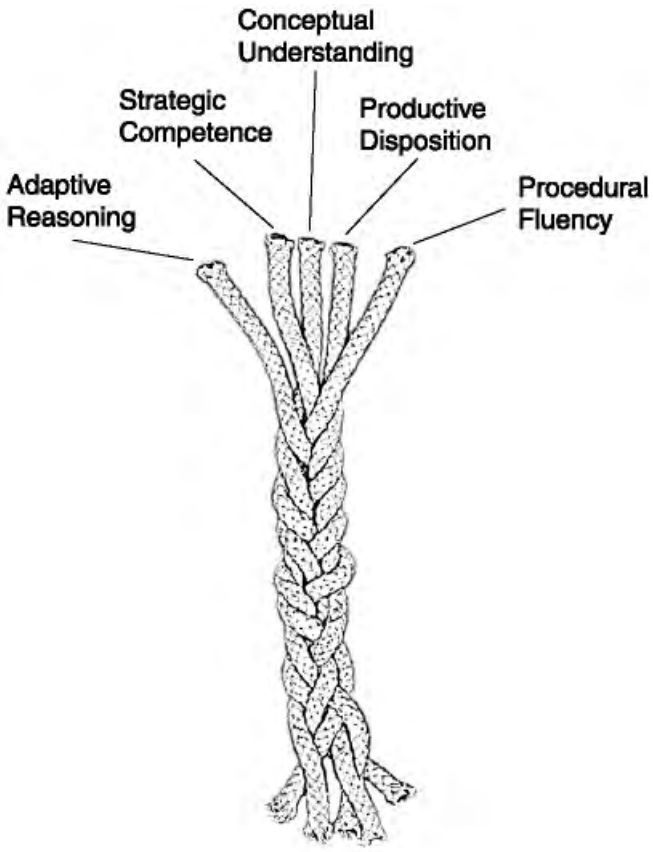
I have two pizzas. My friend eats one quarter of one of the pizzas. I have one and three quarters pizzas left. Then I split it evenly between two of my other friends. Each person gets three and a half pieces of pizza.



1. Explain why this story does ***not*** correspond to the problem.
2. Why do you think this error is so common?

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Mathematical Proficiency for All Students



- ***Conceptual understanding*** - comprehension of mathematical concepts, operations, and relations
- ***Procedural fluency*** - skill in carrying out procedures flexibly, accurately, efficiently, and appropriately
- ***Strategic competence*** - ability to formulate, represent, and solve mathematical problems
- ***Adaptive reasoning*** - capacity for logical thought, reflection, explanation, and justification
- ***Productive disposition*** - habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy

Kilpatrick, J., J. Swafford, and B. Findell. (2001). *Adding It Up: How Children Learn Mathematics*. Washington, DC: National Academy Press.

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Attending to Equity in Developing Mathematical Proficiency

- Attending to the relationships between everyday and mathematical language
- Care with “contexts” (everyday and mathematical)
- Broadening what it means to be successful in mathematics and the space to do so
- Assigning competence: whose mathematical work is made public
- Explaining as central, and making explicit what it means to explain

References: Ball,Hoover, Lewis, Wall, Bass (2003); E. Cohen & Lotan (1997); Boaler (2004)

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Common Remedies for Problems of Mathematics Education

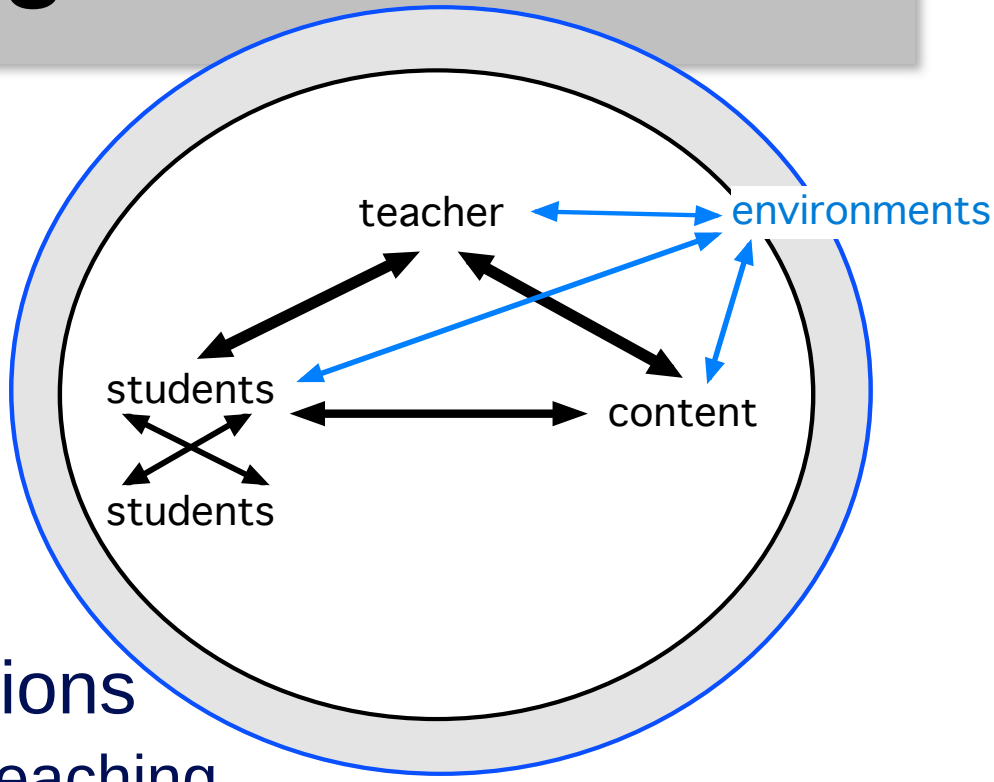
- Raise teachers' salaries
- Increase schools' performance accountability
- Build better curriculum
- Import successful practice
- Recruit smarter people into teaching

. . . but -- these may not improve equitable access to high quality learning of mathematics for every student

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Improving Instruction

- What do we mean by "instruction"?



- Goal of our investigations
  - Studying the work of teaching
  - With an eye on mathematical reasoning and equity
  - What is ***mathematical*** about the work of teaching?

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Definitions, Language, and Representations

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

How Do You Answer These Questions?

What is 7 divided by 0?

Is 0 even or odd?

An even number is an integer multiple of 2.

For any a and b (real numbers), a/b = c if and only if cb = a.

A polygon is a simple closed plane curve formed by straight line segments.

An exterior angle is the angle formed between the side of a polygon and the extension of an adjacent side.

What is the sum of the exterior angles around a polygon?

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Defining Definition

A definition is the formal attempt to answer the question,

“What is it?”

L. Davidson, (1885) The Logic of Definition

A definition is a precise statement of the nature of a thing (e.g., object, process, property).

Oxford English Dictionary

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Five Investigations

1. What might undergraduate students think the derivative “is”?
2. Why are definitions not just nice, but necessary, in mathematics?
3. What do definitions enable?
4. What makes a definition “good”?
5. How do definitions emerge and develop?

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Mathematics Teaching Seminar
Department of Mathematics

- Meets biweekly in the Mathematics Department since January 2002
- On the regular seminar schedule for the department
- Participants: about 30 people — faculty, post-docs, graduate students

Question: What are students actually learning in our courses?

Step #1: What do we want students to be like mathematically — to care about, to know, and to be able to do — after taking our courses?

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

What Do You Want Students Who Have Studied Mathematics to Be Like, to Know, to Be Able to Do?

a) be able to be surprised and to appreciate the surprises, the counterintuitive things
b) not be afraid to try a problem, draw a picture, put some numbers in
c) be motivated about what they are doing
d) have a concept of what an interesting problem is
e) have a notion of what an aesthetic solution is
f) read carefully and analyze the content of the statement
g) be able to evaluate their own work to some extent
h) use math skills to make sense of the world like when they get to their taxes
i) learn basic skills appropriate to the level we are working at
j) look forward to taking more mathematics
k) learn math on their own so that they can come back when they forget
l) never to believe a contradiction
m) ask why and so what
n) expect mathematical connections, strive to see a bigger picture
o) understand major concepts and ideas – e.g., concept of derivative
p) see different sides of a concept or an idea and be able to think about it in many different ways, and be able to make use of it in different ways
q) have a repertoire and be able to use it -- different interpretations, representations, models
r) be able to choose an appropriate method for a problem, think before doing, problem solving skills
s) be able to size up a problem and decide on an approach
t) be able to make a mathematical argument

Step #2: Investigate what students know and how they reason about the concept of the derivative.

www-personal.umich.edu/
~dball/



© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Investigation #1
What Might Undergraduates Think the Derivative "Is"?

1. Suppose someone who had not studied calculus asked you what a derivative is, what would you do or say?

2. On a more formal basis, how would you define the derivative of a function $y = f(x)$ at a point $x = a$?

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Problems with Learning and Teaching Definitions

- Taught as fixed or given; to be memorized
- Or — not taught at all
- Treated parenthetically
- Develop impression of one canonical definition; lack flexibility with equivalent definitions
- Not used in ways that make them part of the doing of mathematics

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Investigation #2

Why are Definitions Not Just Nice, but Necessary?[1]

[1]A. Ortony (1975)

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Learning Even and Odd Numbers in Third Grade: Developing and Reconciling Definitions

Thursday, January 18

Meeting with fourth grade class:
Even and odd numbers

Friday, January 19

Debriefing the meeting
(Return to conjectures about even and odd numbers
Odd + odd = even
Even + even = even
Odd + even = odd
Shea brings up idea: 6 could be even or odd

Class attempts to refute, using three different definitions of even numbers: alternating (even, odd, even, odd . . .), fair share (2k or k + k), pairing by twos (k2 or 2 + 2 + . . . +2 +2)

Lin generalizes Shea's claim in an effort to refute it:
"What about 10? Why don't you call 10 an even and an odd number?"

Ogechi develops clearer expression of definition of odd numbers ("numbers with a 1 in the middle 2 + 2 + 2 +. . . +1+ . . .+2 + 2 + 2)

Rania observes pattern: 14, 18, 22, 26 . . .

Teacher labels the generalized class of even numbers "Shea Numbers"; class defines as " even numbers that have an odd number of groups of twos"

Definitions of even and odd number are reconciled and made more precise: k2 (even) and k2 +1 (odd)

• • •

Friday, January 26

Jillian and Shekira argue that it is not possible to prove Bernadette's conjecture: An odd number plus an odd number equals an even number.

Tuesday, January 30

Bernadette offers a proof of the conjecture.

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

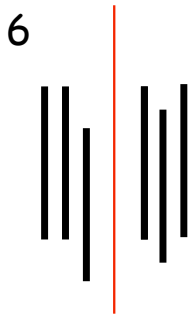
	Shea	Shekira
Claim	0 is not even (but not that it is odd).	0 is even.
Definition	An even number is a number that can be made up of two (equal) things.	Even and odd numbers alternate on the number line.
Prior knowledge		1 and –1 are odd numbers (fourth graders said so).
Argument	What two things can make it (0)?	0 is situated between –1 and 1.
	Implied: No two things can make it.	
Conclusion	Therefore 0 is not even.	Therefore 0 is even.

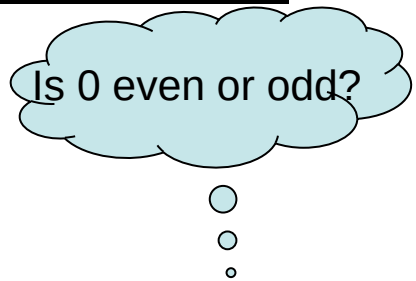
© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Considering Students' Definitions of Even Number

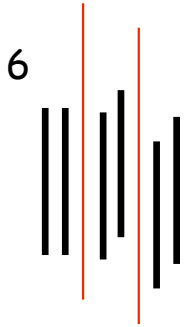
How well does any of these help to answer the question?

1. An even number is a number you can split in two equal parts without having to break anything in half.

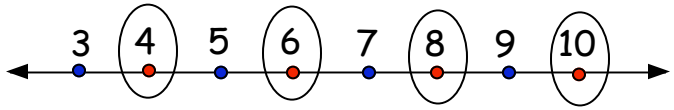




2. An even number is a number that when you group it in twos, there are none left.



3. The even numbers are every other number on the number line, like 2, 4, 6, and so on.



© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Investigation #3

What Do Definitions Enable?

Thursday, January 18

Meeting with fourth grade class:
Even and odd numbers

Friday, January 19

Debriefing the meeting

(Return to conjectures about even and odd numbers
Odd + odd = even
Even + even = even
Odd + even = odd

Shea brings up idea: 6 could be even or odd

Class attempts to refute, using three different definitions of even numbers: alternating (even, odd, even, odd. . .), fair share (2k or k + k), pairing by twos (k2 or 2 + 2 + . . +2 + 2)

Lin generalizes Shea's claim in an effort to refute it:
"What about 10? Why don't you call 10 an even and an odd number?"

Ogechi develops clearer expression of definition of odd numbers ("numbers with a 1 in the middle 2 + 2 + 2 +. . . +1+ . . .+2 + 2 + 2)

Rania observes pattern: 14, 18, 22, 26 . . .

Teacher labels the generalized class of even numbers "Shea Numbers"; class defines as " even numbers that have an odd number of groups of twos"

Definitions of even and odd number are reconciled and made more precise: k2 (even) and k2 +1 (odd)

• • •

Friday, January 26

Jillian and Shekira argue that it is not possible to prove Bernadette's conjecture: An odd number plus an odd number equals an even number.

Tuesday, January 30

Bernadette offers a proof of the conjecture.

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Investigation #4
What Makes A Definition "Good"?



- Mathematically precise —correctly identifies the kind of object, process, property
- Usable by user community — based on already-defined and understood terms
- Able to expand to the mathematical contexts into which students will move (i.e., doesn't become false).

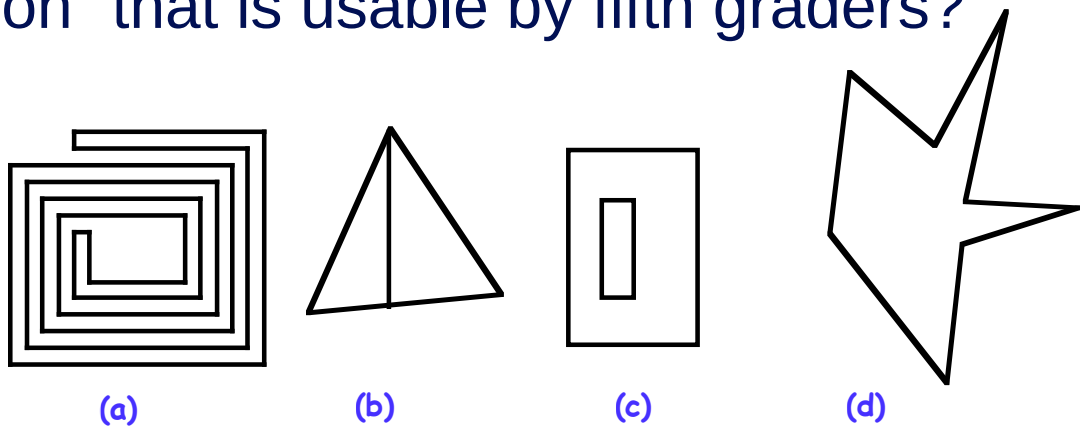
© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

What is a mathematically appropriate and usable definition of "even number" for third graders?

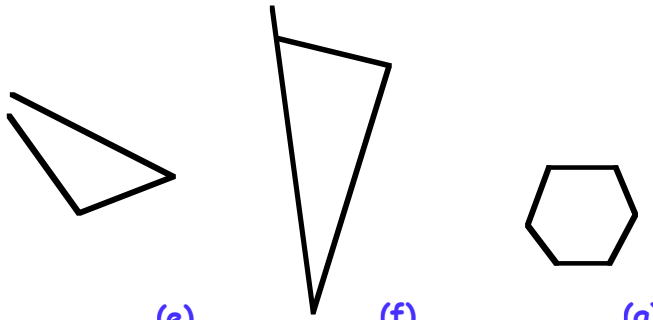
© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Definition of polygon:
A simple closed plane curve formed by straight line segments.

But — what's a mathematically acceptable definition of "polygon" that is usable by fifth graders?



(a) (b) (c) (d)



(e) (f) (g)

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Examining Textbook Definitions

How well does any of these help to answer the question?

1. An even number is a number of the form 2k, where k is an integer.
2. An even number is a natural number that is divisible by 2.
3. An even number is any multiple of 2.
4. An even number is a number that has 0, 2, 4, 6, or 8 in the ones place.

Is 0 even or odd?

www-personal.umich.edu/~dball/

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

What is a Definition of Even Number that Can Be Used with Third Graders?

- A number is even if it is the sum of two equal numbers
- A number is even if when the objects are counted out, they can be put into two equal piles, with none of the objects broken up.
- A whole number is even if it can be split into two equal sized groups.
- An even number is divisible by two without remainder.
- A number of objects is even if it can be split up into pairs with none left over.

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Looking Back: Definitions

- Definitions emerge from a need to become clearer about things.
- Definitions are crucial for mathematical communication and reasoning.
- Different definitions must be reconciled for communication within a community.
- Definitions depend on other already-defined or commonly shared terms, and must be both mathematically precise and usable.
- Definitions must have integrity in anticipating learners' expanding mathematical worlds.

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Looking Back: Equity

- Attending to the relationships between everyday and mathematical language
- Care with “contexts” (everyday and mathematical)
- Broadening what it means to be successful in mathematics and the space to do so
- Assigning competence: whose mathematical work is made public
- Explaining as central, and making explicit what it means to explain

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Language and Representations

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Viewing Focus

- What do you notice about mathematical language or representation?
- In what ways is mathematical representation being developed?
- What is the teacher doing to structure students' work?

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

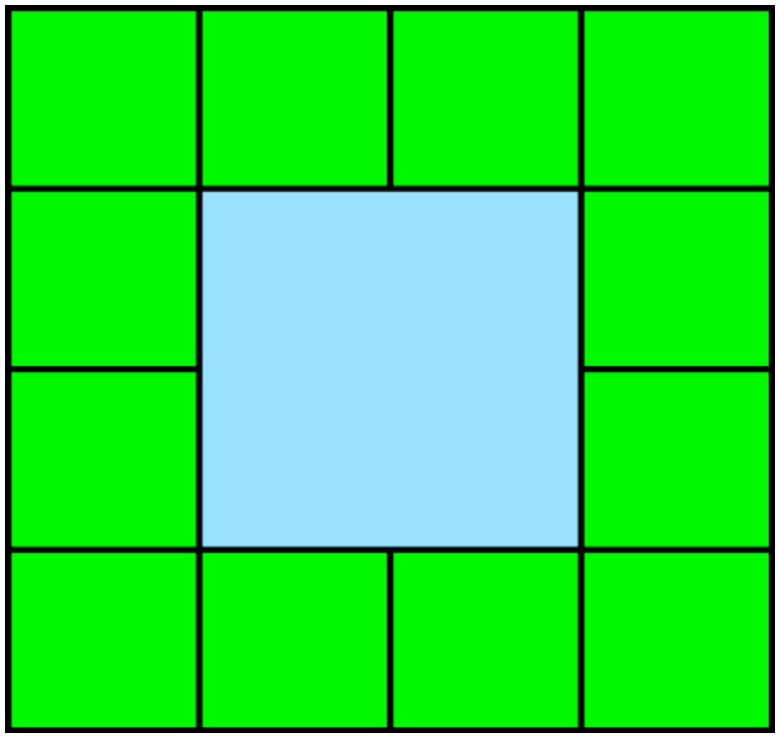
Problem #1:
Pool Border Problem

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Pool Border Problem

How many square tiles does it take to build a border around a square "pool"?

Find a way to know the number of tiles it will take without having to count, for <u>any size</u> pool.



© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Geometric and Algebraic Solutions to the Pool Border Problem

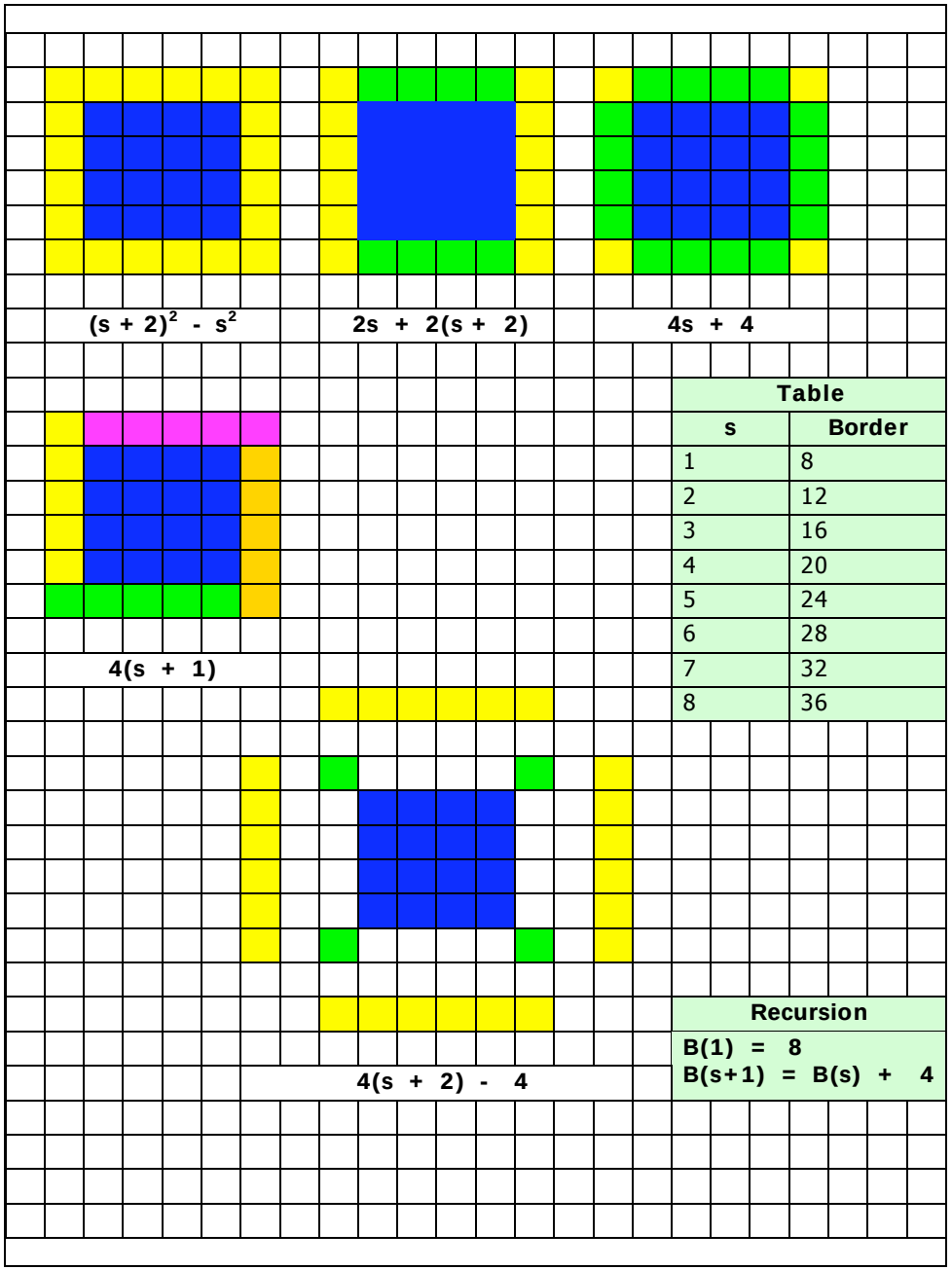


Table	
s	Border
1	8
2	12
3	16
4	20
5	24
6	28
7	32
8	36

Recursion
B(1) = 8
B(s+1) = B(s) + 4

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

What Mathematics is (Potentially) Involved Here?

TOPICS

- Combinatorial geometry — relating linear and area measure
- Linear functions/growth
- Skills with algebraic notation and geometric representations to describe situations

PRACTICES

- Checking and analyzing small cases to search for a solution in general; pattern or structure seeking
- Finding multiple forms of the solution and mapping correspondences between them
- Use of geometric and/or algebraic methods to organize and express solutions

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Setting the Background

- "Laboratory" class
- Grade 5 pupils (ages 10 - 11)
- Wide range of mathematical achievement and disposition
- 20 hours of instruction:
    - intersection of geometry and arithmetic
    - skill development
    - mathematical explanation, and precision with language and representation
    - discourse of exploration, conjecture, explanation, proof

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Mathematical Content (Ideas and Practices)

- Concept of rectangle
- Defining rectangle (and other quadrilaterals) precisely
- Squares, primes, composites (and definitions)
- Multiplication
- Rotations in the plane
- Using exponents
- Using variables to express a generalization
- Formulating and seeking to prove or refute significant mathematical conjectures
- Justifying the completeness of a solution
- Factors and finding all the factors of a number (algorithm and its proof)
- Mathematical explanation
- Mathematical generalization
- Writing and representing mathematical ideas

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Basic Instructional Formats

- Class segments:
  - skill review
  - independent and partner work on main problems
  - whole group discussion and development of main problems
  - individual notebook writing
- Daily homework composed of multiple genres of work
  - Practice
  - Extension
  - Next steps on something unfinished
  - Demonstrate to family

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Viewing Focus

- What do you notice about mathematical language or representation?
- In what ways is mathematical representation being developed?
- What is the teacher doing to structure students' work?

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Explaining Mathematical Ideas

- Convincing yourself, convincing others
- Use of language (definitions)
- Use of drawings (and other representations) and making correspondences to ideas
- Making clear what things mean and why they make sense; showing why something is correct or “true”

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Wrap Up/ Preview for Day 2

- Seeing teaching as mathematical work; seeing attending to equity also as deeply mathematical
- Knowing mathematics for teaching
- Assessing/measuring teachers' knowledge of mathematics -- why, how, what learned?
- Developing teachers' knowledge of mathematics for teaching
- Homework: Explain the divisibility rule for 4 (specify audience)
- Comment cards: What is going well? What is not?

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission