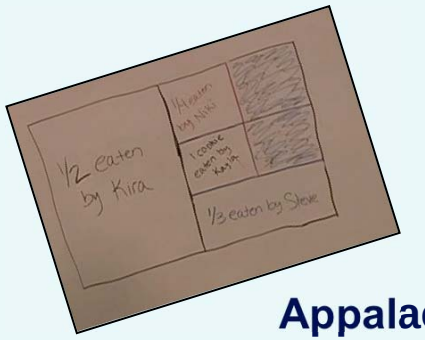
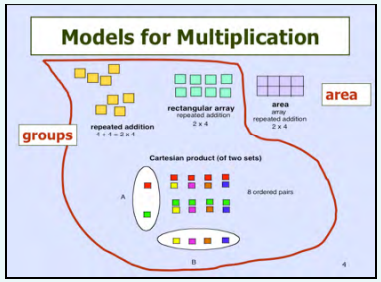
Learning the Mathematical Work of Teaching





Deborah Loewenberg Ball

Hyman Bass

University of Michigan

Center for Proficiency in Teaching Mathematics

Appalachian Association of Mathematics Teacher Educators

September 23, 2005

www-personal.umich.edu/
~dball/

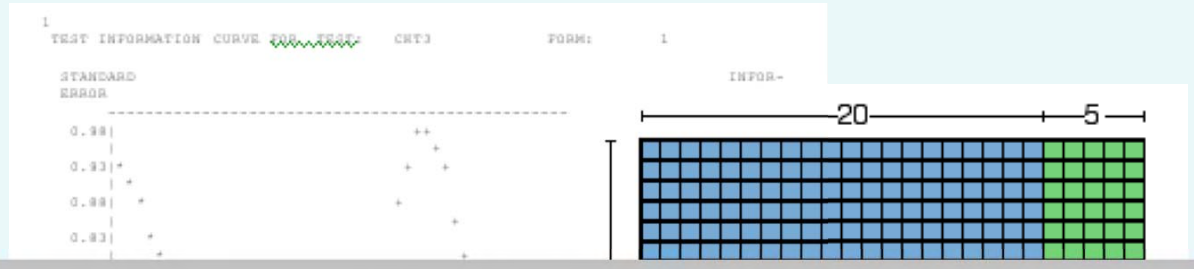
© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

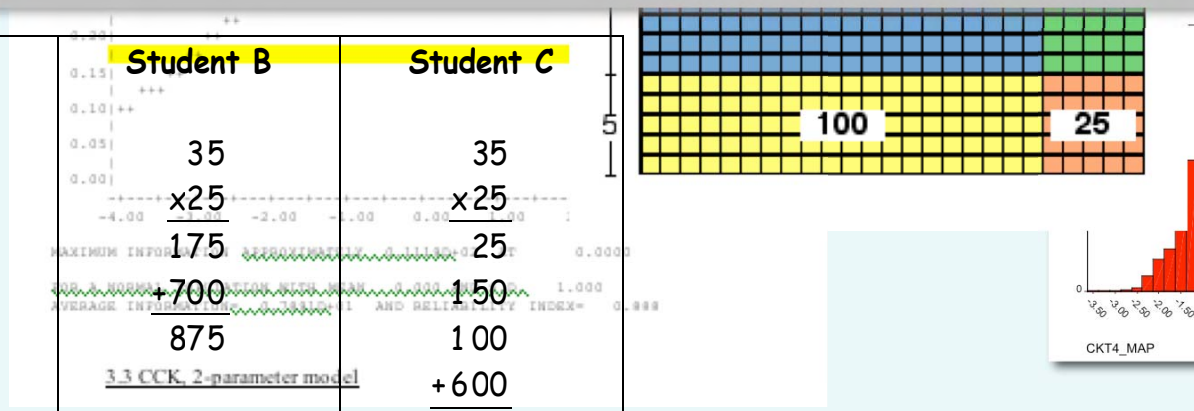
Overview of Session

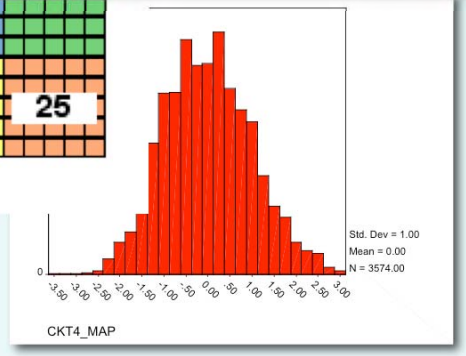
1. Mathematical knowledge for teaching: A theory in progress, and recent developments
2. Developing mathematical knowledge for teaching: Exploring the design and teaching of explicit opportunities for teachers to learn

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

1. Mathematical Knowledge for Teaching: A Theory in Progress, and Recent Developments







Student A	Student B	Student C
35 x25 125 +75 875	35 x25 175 +700 875	35 x25 25 150 100 +600 875

3.3 CCK, 2-parameter model

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Clarifying the Problem

The quality of mathematics teaching and learning

Teachers' knowledge of mathematics

and their ability to use it in teaching

What mathematical resources do teachers need

in order to teach mathematics to all students effectively?

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

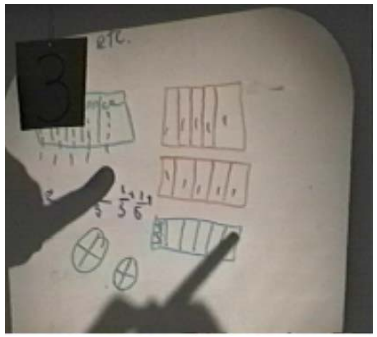
A Practice-Based Approach to Answering the Question

1. **Study instruction**, and identify the mathematical work of teaching
2. Analyze **what mathematical knowledge** is needed to do that work effectively, and **how** it must be understood in order to be useful for the work
3. Develop, test, and refine **approaches to helping teachers** develop and use mathematical knowledge for teaching, study results

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

What Do we Mean By "Mathematical Knowledge for Teaching"?





© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Mathematical Knowledge for Teaching

What do we mean when we use this term, "**mathematical knowledge for teaching**"?

- Mathematical knowledge, skill, habits of mind that are entailed by the work of teaching

What do we mean by the "**work of teaching**"?

- The tasks in which teachers engage, and the responsibilities they have, to teach mathematics, both inside and outside of the classroom

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

What *is* "Mathematical Knowledge for Teaching"? An Example from Subtraction

Subtract:

$$\begin{array}{r} 307 \\ -\ 168 \\ \hline \end{array}$$

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Analyzing Incorrect Answers for $\begin{array}{r} 307 \\ -\ 168 \\ \hline \end{array}$

(a)

$$\begin{array}{r} 307 \\ -\ 168 \\ \hline 261 \end{array}$$

(b)

$$\begin{array}{r} 307 \\ -\ 168 \\ \hline 169 \end{array}$$

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Analyzing Correct Answers for $\begin{array}{r} 307 \\ -\ 168 \\ \hline \end{array}$

(a)

$$\begin{array}{r} 307 \\ -\ 168 \\ \hline -1 \\ -60 \\ 200 \\ \hline 139 \end{array}$$

(b)

$$\begin{array}{r} 307 \\ -\ 168 \\ \hline 2 \\ 30 \\ 107 \\ \hline 139 \end{array}$$

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Other Work of Teaching

- Choosing and using representations
- Choosing numerical examples
- Using and defining terms
- Using notation
- Explaining

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Hypotheses About Knowledge of Mathematics for Teaching

PREVALENT HYPOTHESES

1. Teachers need knowledge of the curriculum, plus N levels more knowledge
2. "Pedagogical content knowledge" (— but what is it?)

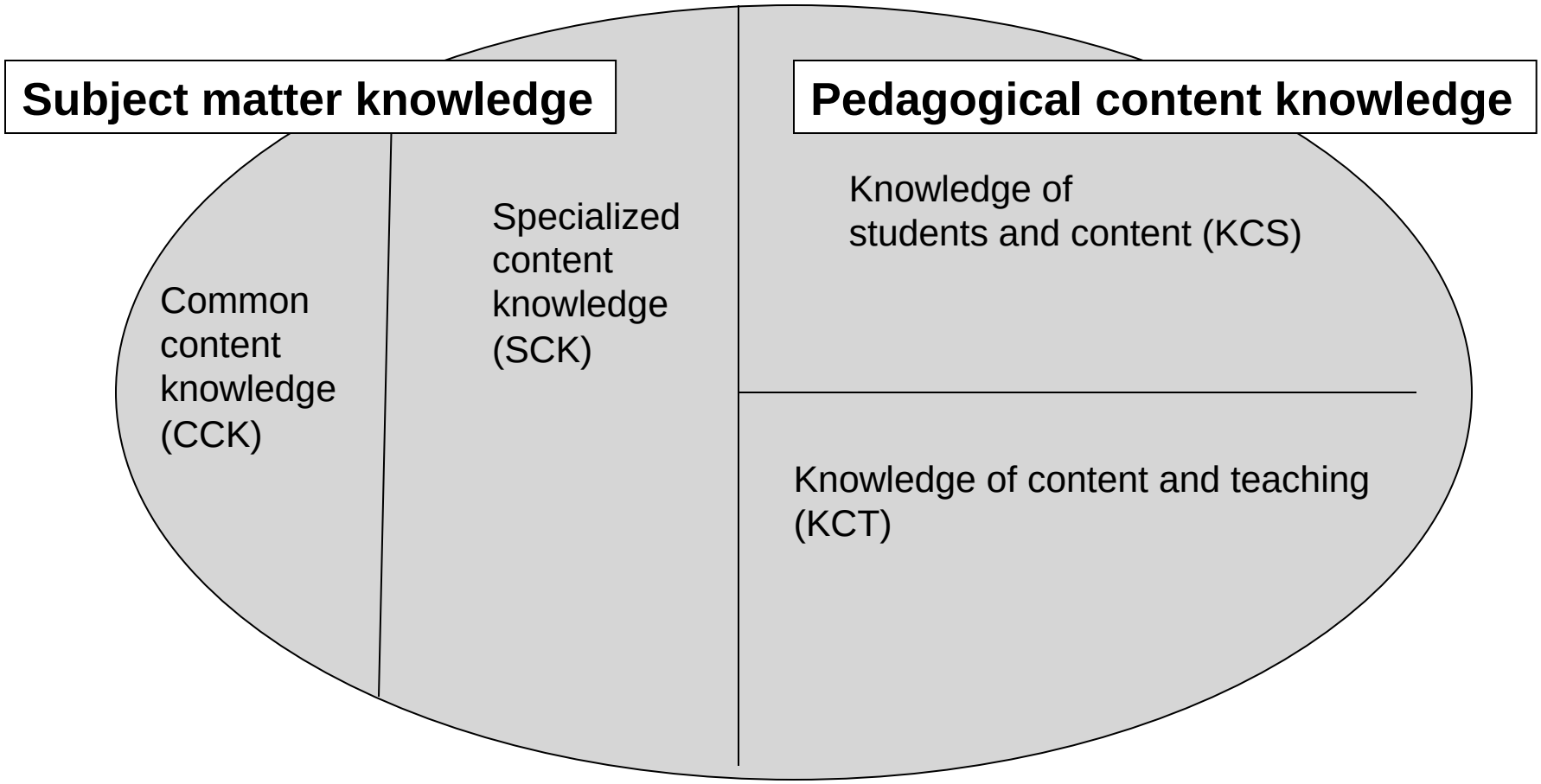
OUR CURRENT HYPOTHESES

- **Common** content knowledge
- **Specialized** content knowledge
- Knowledge of content and **students**
- Knowledge of content and **teaching and curriculum**

What does this have to do with "pedagogical content knowledge"?

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Shulman's Original Category Scheme (I985) Compared with Ours



© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Developing and Using Valid Measures of Mathematical Knowledge of Teaching

- Interdisciplinary item development
- Piloting and analysis
  - **Discriminating among kinds of mathematical knowledge for teaching**
  - **Discriminating among mathematical domains**
- Content validity checks
- Cognitive tracing interviews with teachers, non-teachers, mathematicians
  - Solving mathematical problems involving specialized content knowledge not always obvious or easy for mathematicians
- Examining videotapes of classroom teaching
- Using teachers' scores to predict student achievement

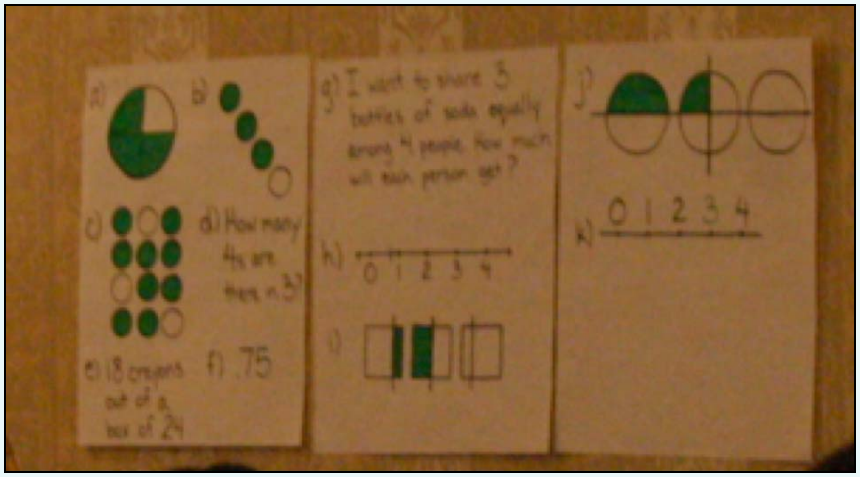
© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Linking Teacher Knowledge and Student Achievement: New Results

- Questionnaire consisting of 30 items
- Model: Student Terra Nova gains predicted by:
  - Student descriptors (family SES, absence rate)
  - Teacher characteristics (math methods/content, content knowledge)
- Teacher content knowledge significant
  - Small effect (< 1/10 standard deviation): 2 - 3 weeks of instruction
  - But student SES is also about the same size effect on achievement
    - (Hill,Rowan, and Ball, in press)

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

2. Developing Mathematical Knowledge for Teaching: Exploring the Design and Teaching of Explicit Opportunities for Teachers' Learning



www-personal.umich.edu/
~dball/

cptm lmt

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Three Kinds of Learning of Mathematics

- Learning mathematics as a student
- Learning mathematics as a teacher
- Learning to teach mathematics

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Mathematical Content and Applications for the Teaching of School Mathematics

- A mathematics content course for prospective elementary and middle school teachers
- Focused on **knowing and using mathematics for teaching**
  - ❑ Content
    - Mathematical topic: fractions
    - Mathematical practices: Using representations, using mathematical language, **making and evaluating mathematical explanations**
  - ❑ Applications
    - Using mathematics for tasks of teaching: Interpret and evaluate student work, discuss the mathematics in play in classroom lessons, analyze curriculum materials

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Making and Evaluating Mathematical Explanations

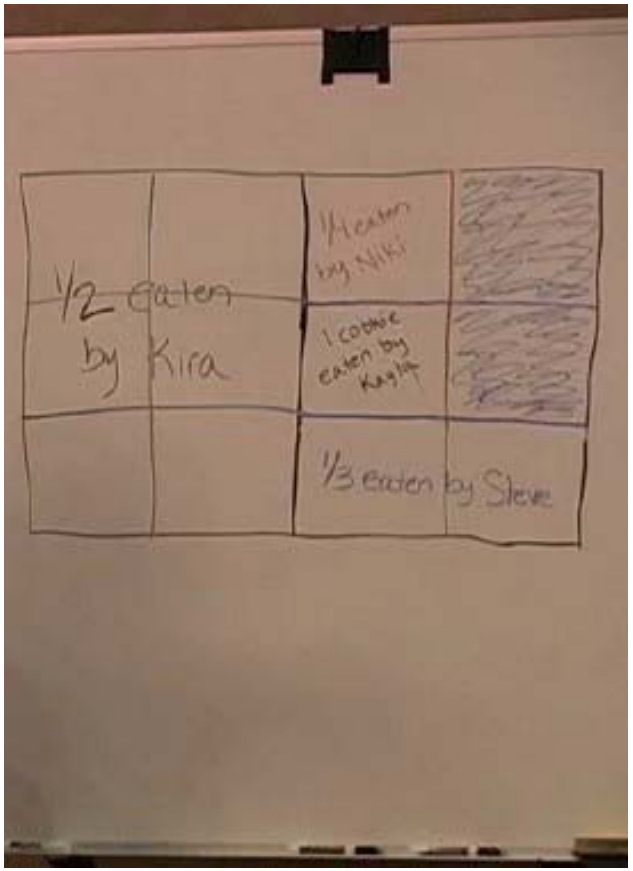
- The forms of expression that mathematically justify and help others understand why a mathematical claim is true
- Purposes and central features of mathematical explanation
- The social nature of explanation
- Providing adequate explanations
- Listening to and appraising others' explanations

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Cookie Jar Problem

There was a jar of cookies on the table. Kira was hungry because she hadn't had breakfast, so she ate half the cookies. Then Steve came along and noticed the cookies. He thought they looked good, so he ate a third of what was left in the jar. Niki came by and decided to take a fourth of the remaining cookies with her to her next class. Then Kayla came dashing up and took a cookie to munch on. When Pam looked at the cookie jar, she saw that there were two cookies left. "How many cookies were there in the jar to begin with?" she asked Kira.

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission



Anna asks: “How did you decide that the one cookie eaten by Kayla got its own square?”

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Why <u>does</u> one square equal one cookie?



© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Kira ate half the cookies



© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Steve ate a third of what was left



© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Niki took a fourth of the remaining cookies with her



© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Because Niki took a fourth of the remaining cookies, there must be three-fourths left



© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Kayla took a cookie to munch on



© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

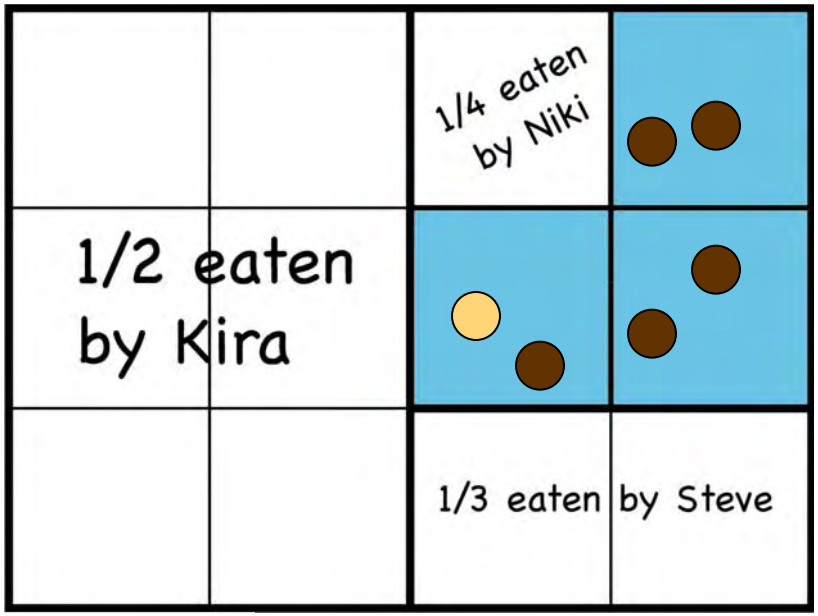
Why does one square equal one cookie?



© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Consider an alternate version of the problem:

Kayla took a cookie to munch on.
Pam saw that there were five cookies left.



© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Students' Explanations

1. What do you notice about what the students are doing, about what teacher is doing, and about the content?
2. Are these explanations? What do they have, or lack?
3. How does what students are working on relate to learning mathematical knowledge for teaching?

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Launching Explicit Work on Mathematical Explanation

STUDENTS' WORK IN CLASS

- Are you convinced that the only solution to the Cookie Jar Problem is 12? If so, what has convinced you? If not, what is leaving you skeptical?

- Are there things you have noticed that you would call "features of good mathematical explanations"?

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Features of "Good" Mathematical Explanations for the Cookie Jar Problem

- Makes clear at the outset what is being explained, and why you start there, and carefully connects the explanation to the question or idea being explained
- Starts from the beginning, and traces the logical flow of the reasoning
- Should be logical and complete, makes conclusion clear and links back to original question or claim or problem
- Might number the steps if appropriate, or label parts of a diagram
- Strives to be as simple and clear as possible
- Defines terms as needed, uses available definitions as needed
- Uses representation(s) accurately (algebraic, geometric, etc.), and combining representations
- Links the language and diagrams clearly to the steps of the argument
- Shows what something means or why is true, and is convincing to the person to whom you are explaining
- Is calibrated to the context (considers the person to whom you are explaining, and what is already established as true and does not need more explanation)

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Exam: Question #1

- DIRECTIONS: This question will be completed in pairs. You and your partner will take turns being the presenter and the evaluator. After the presenter completes the explanation, the evaluator will appraise the explanation using the criteria we developed in class, and will provide feedback. Then you will switch roles.

- You are encouraged to prepare your explanations in advance, however you may not use any notes for this question. Although you will only explain one problem, you should do the other as well, so that you are prepared to listen closely to your partner.

- Paper and pens will be available for you to use during your explanation, as will copies of the problems and the list of criteria for good mathematical explanation. You should also refer to our list of criteria for good mathematical explanations, but do not use it rigidly (i.e., you need not comment on every item).

- For this question, you will be assessed on both your ability to provide and to evaluate a mathematical explanation. When you are listening to your partner's explanation, your role is to appraise the quality of the explanation. You should not be coaching your partner, and should listen neutrally so you do not distract or otherwise intervene in your partner's reasoning.

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Problems for Explanation Performance Task on Exam

- Amy has 5 yards of beautiful imported embroidered ribbon. For a special project, she decides to cut it equally into 8 strips.
- What fraction of a yard will each strip be?
- What fraction of all of the ribbon will each strip be?

- Chris bought 7 boxes of Girl Scout chocolate mint cookies. He loves cookies. In order to exercise self-control, he decides to eat an equal amount each day, and he wants his cookie supply to last for 20 days.
- What fraction of a box of cookies does he have for lunch each day?
- What fraction of his entire cookie supply does he eat each day?

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Conclusion: Questions and Challenges

- Tackling issues of coverage
  - Choosing and justifying mathematical content (topics and practices) for **in-depth work**
  - Choosing and justifying content for **brief treatment**
  - Choosing and justifying content **to omit**
- Understanding and developing mathematical fluency for teaching
- Developing tasks and pedagogy appropriate for mathematical knowledge for teaching: Do we really think that our courses/workshops should be taught as teachers would teach their own students?
- Developing performance assessments: knowing and using mathematics (e.g., explaining, representing)
- Rethinking the intersection of content and methods
- The professional development of professional developers

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission

Slides will be available at

http://www-personal.umich.edu/~dball/

© 2005 Mathematics Teaching and Learning to Teach
School of Education • University of Michigan • Ann Arbor, MI 48109-1259 • (734) 615-7235 • mtlt@umich.edu
Please do not circulate or cite without permission