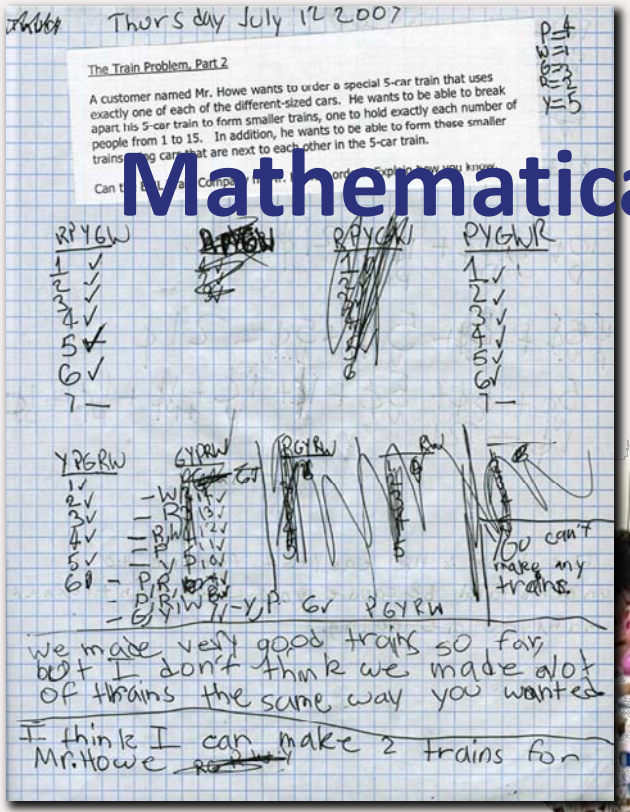



# Developing Teachers' Mathematical Knowledge for Teaching


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University of Michigan




University of North Carolina • Mathematics Focus Session
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# Overview of session

1. A larger perspective: why are we doing this, and what will it take?
2. Mathematical knowledge for teaching (MKT)
3. Framework for MKT task design
4. Examining sample MKT tasks together
5. Discussion: Issues for teacher development focused on MKT



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# 1. Mathematical knowledge for teaching (MKT)



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# The problem

How can we improve students' learning?

. . .

Teachers' mathematical knowledge is a key factor shaping what they are able do.

What mathematical knowledge do teachers need?



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# Different questions

**From teacher knowledge to knowledge for teaching:**

1. What mathematics do teachers need to know?
2. What mathematics do teachers know?
3. What mathematics do teachers use?
4. What mathematics does teaching entail?



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# Elements of our "practice-based" approach

1. Study instruction and identify the mathematical work of teaching
2. Analyze what mathematical knowledge is entailed by the work (MKT)
3. Test the working hypotheses based on these analyses by developing measures of MKT, validating teacher scores against practice and against student achievement gains
4. Develop and evaluate approaches to helping teachers learn mathematical knowledge for teaching



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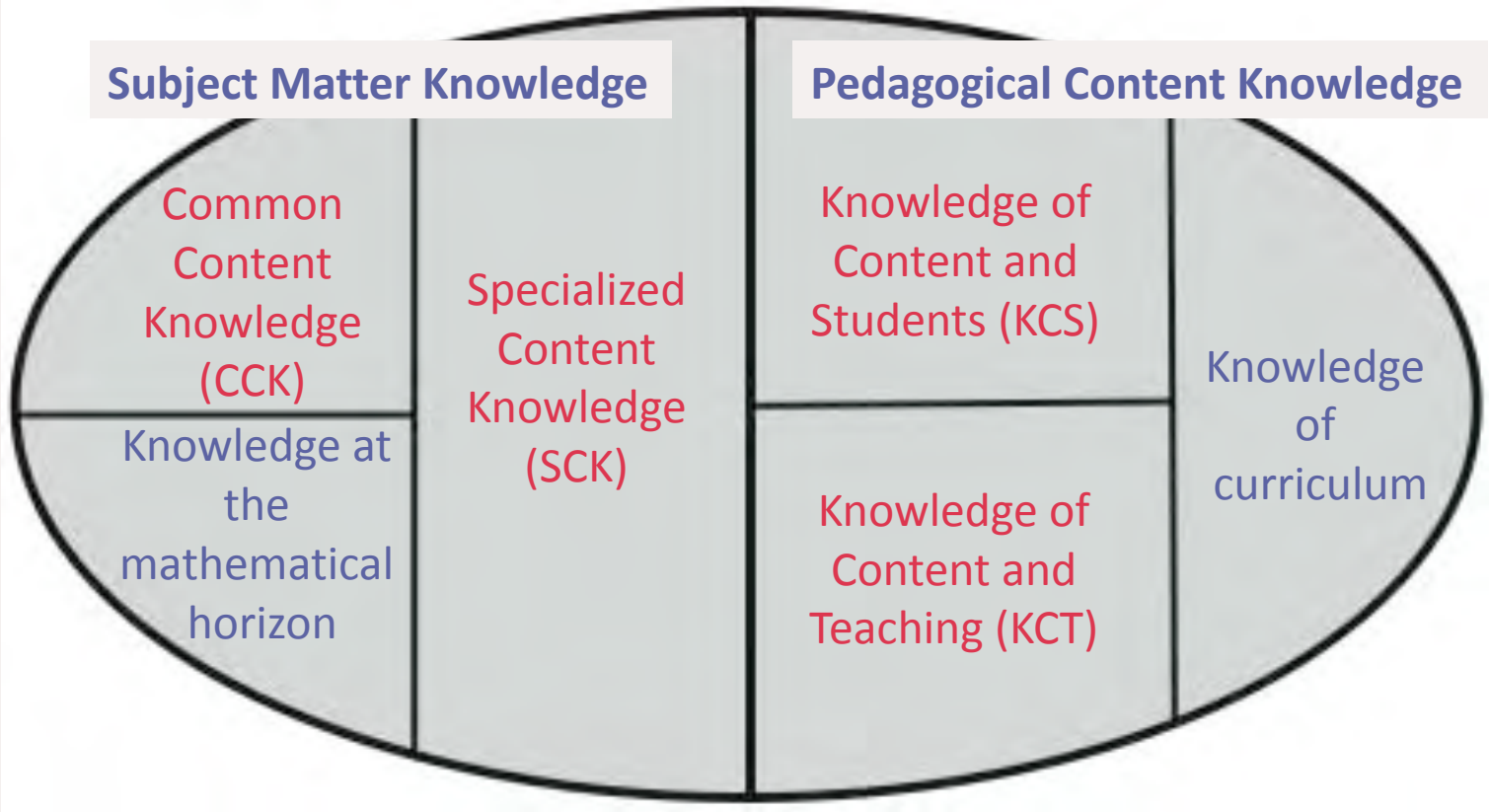
# Examples of work of teaching

1. Selecting/designing tasks
2. Identifying and working toward the mathematical goal of the lesson
3. Listening to and interpreting students' responses
4. Teaching students what counts as "mathematics" and mathematical practice
5. Making error a fruitful site for mathematical work
6. Attending to ambiguity of "big rectangle"
7. Deciding what to clarify, what to make more precise, what to leave in student's own language



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# Mathematical knowledge for teaching






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# Common content knowledge (CCK)

Calculate:

$$\frac{5}{6} \div \frac{1}{3}$$



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# Specialized content knowledge (SCK)

$$\frac{5}{6} \div \frac{1}{3} = \frac{10}{12} \div \frac{4}{12} = 10 \div 4 = 2\frac{1}{2}$$

Is this a fluke?

Does it work in general?

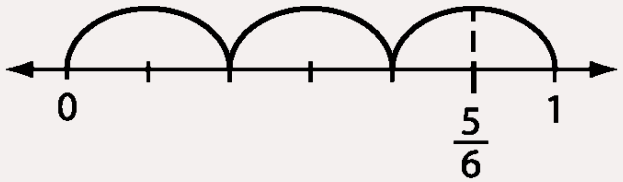
If so, why does it work?



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# Specialized content knowledge (SCK)

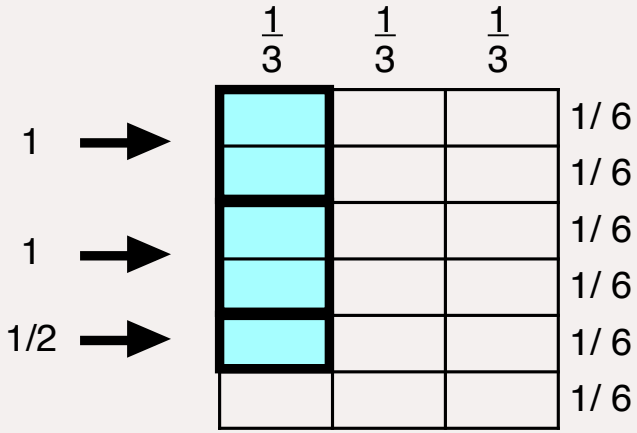
Which of these can be used to represent $\frac{5}{6} \div \frac{1}{3}$ ?

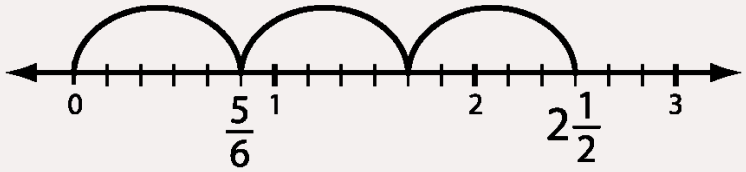











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$$\frac{5}{6} \div \frac{1}{3} = 2\frac{1}{2}$$

## Knowledge of students and content (KCS)

- What are common errors students make when dividing fractions?
- How do students' experiences with division of whole numbers support their understanding of division of fractions? How does it confuse them?
- What difficulties do students typically have interpreting the answer to a division of fractions problem?

## Knowledge of teaching and content (KCT)

- Which representation would you use to introduce the meaning of division of fractions? Or to explain the invert and multiply algorithm?
- What sequence of problems would you use to begin work on division of fractions?
- In a whole-class discussion, what solution methods would you want presented, and in what order?



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# Other tasks of teaching mathematics

- Responding to students' "why" questions
- Unpacking and decomposing mathematical ideas
- Explaining and guiding explanation
- Using mathematical language and notation
- Generating examples
- Sequencing ideas
- Choosing and using representations
- Analyzing errors
- Interpreting and evaluating alternative solutions and thinking
- Analyzing mathematical treatments in textbooks
- Making mathematical practices explicit
- Attending to issues of equity (e.g., language, contexts, mathematical practices)



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# The challenge

- How can MKT, in particular SCK, be developed? How can opportunities for learning SCK be provided?
- Easy to work on mathematics, or on students; less easy to create opportunities to develop the specialized knowledge of mathematics needed for teaching



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# Lunch

1. How can MKT, in particular SCK, be developed?
2. How can opportunities for learning SCK be provided?

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# 2. Framework for MKT task design



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# The challenge

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# Our approach to the problem

- Find/develop tasks that create opportunities for learning mathematical knowledge for teaching
- Situate teachers' opportunities to learn in the contexts of use
- Provide opportunities to practice the kinds of mathematical thinking, reasoning, and communicating used in teaching
- Enact tasks in ways that maintain the focus on developing MKT and the ability to use it in teaching



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# Designing tasks to develop SCK

What is the difference between

a good mathematics task

and one that is good for developing

mathematical knowledge for teaching?

Why does this matter?



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# Division of fractions

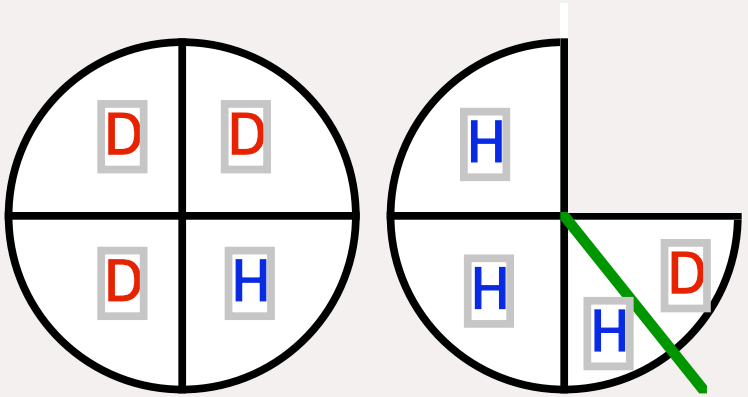
$$1\frac{3}{4} \div \frac{1}{2}$$

1. Calculate the answer.
2. Write a story problem, or describe a situation, that corresponds to 1 3/4 ÷ 1/2.



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I have two pizzas. My friend eats one quarter of one of the pizzas. I have one and three quarters pizzas left. Then I split it evenly between two of my other friends. Each person gets three and a half pieces of pizza.




1. What is wrong with this?
2. Write a story problem that correctly represents the division.



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# Features of tasks designed to develop MKT

- Unpacks, makes explicit, and develops a flexible understanding of mathematical ideas that are central to the school curriculum
- Opens opportunities to build connections among mathematical ideas
- Provokes a stumble due to a superficial "understanding" of an idea
- Lends itself to alternative/multiple representations and solution methods
- Provides opportunities to engage in mathematical practices central to teaching (explaining, representing, using mathematical language, analyzing equivalences, proving, proof analysis, posing questions, writing on the board)



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# Three SCK tasks



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# SCK task #1:
# Writing division story problems

$38 \div 4$

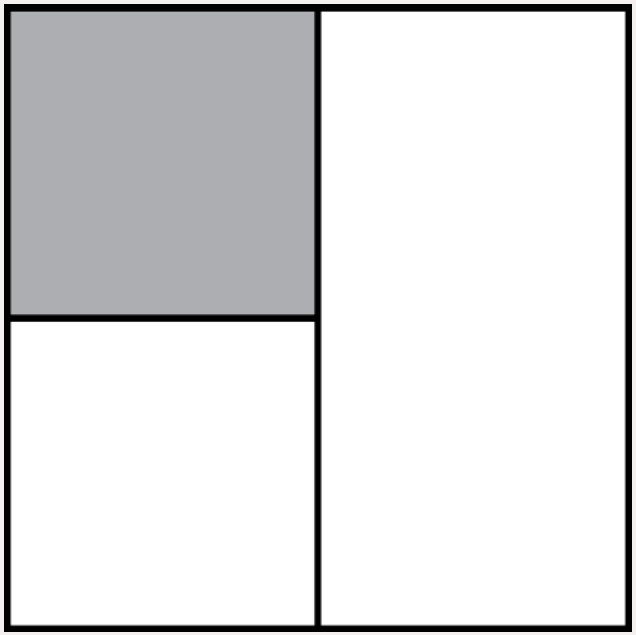
Write as many different stories as you can that correspond to this division expression and that represent different interpretations of the meaning of division or what it means in specific situations.



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# SCK task #2:
# Analyzing solutions

What fraction of the rectangle is shaded?



What reasoning could produce each of these answers?

$\frac{1}{2}$ $\frac{1}{4}$ $\frac{1}{3}$ $1\frac{1}{2}$



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# SCK task #3: Analyzing errors

(a)
$$\begin{array}{r} 42 \\ \times\ 83 \\ \hline 326 \end{array}$$

(b)
$$\begin{array}{r} 24 \\ \times\ 53 \\ \hline 72 \\ 120 \\ \hline 192 \end{array}$$

(c)
$$\begin{array}{r} 3.4 \\ \times\ 2.4 \\ \hline 136 \\ 680 \\ \hline 81.6 \end{array}$$

(d)
$$\begin{array}{r} 283 \\ \times\ \ 4 \\ \hline 2062 \end{array}$$

**For each problem:**

1. What mathematical steps likely produced each incorrect answer?
2. Do the problem correctly and explain the procedure you use and why it works.



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# Questions to consider when examining tasks

1. What opportunities to learn MKT does each task provide?
2. Looking across the tasks: What are central features of "MKT tasks"?
3. What are the challenges of enacting MKT tasks with teachers?



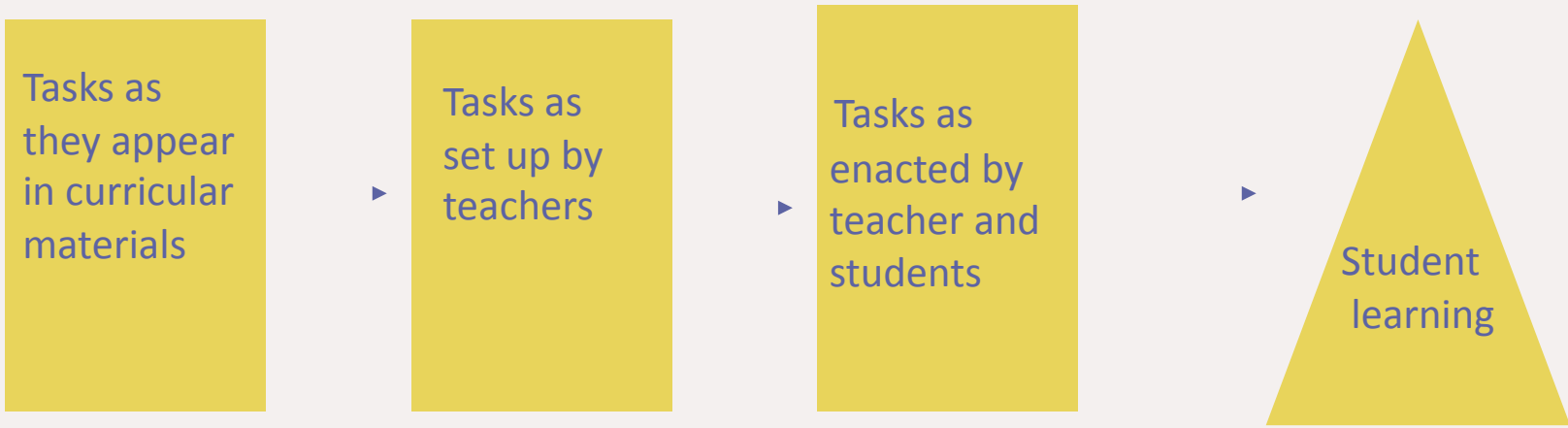
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# Conjectures about "enactment" of MKT tasks



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# The Mathematical Task Framework (MTF)

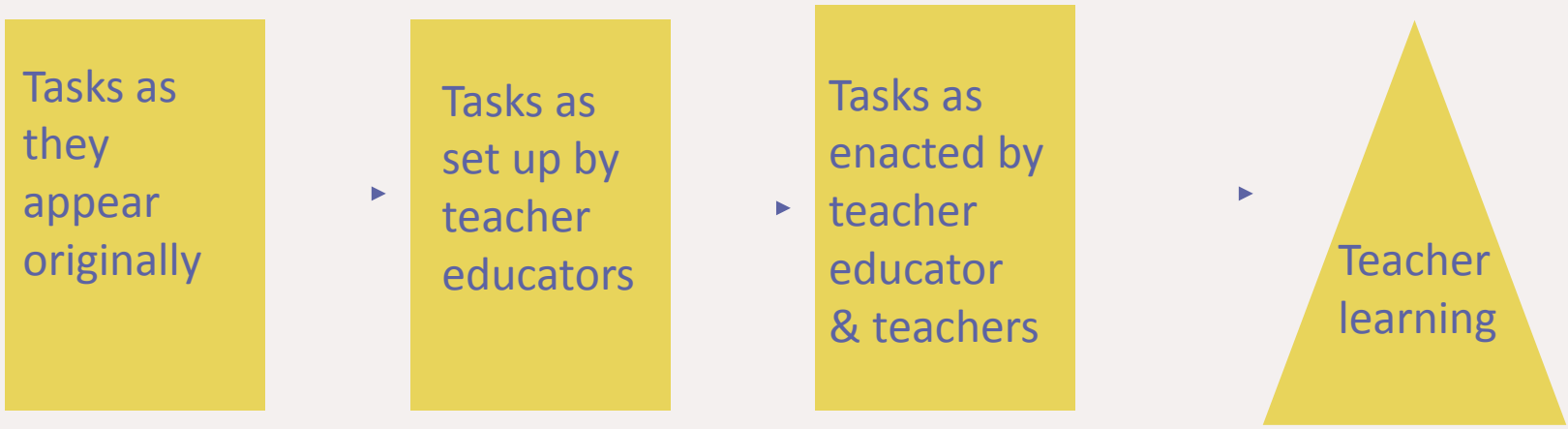



Stein, Grover & Henningsen (1996), Smith & Stein (1998); Stein, Smith, Henningsen & Silver (2000)



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# The Mathematical Task Framework adapted to teacher education






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# Challenges of teaching MKT

- Staying focused on the mathematics, and not on how to *teach* the math
- Keeping the problems focused on MKT and not just "M"
- Unpacking the mathematics sufficiently and convincingly helping them see what there is to learn and do
- Making visible the connections to the kinds of mathematical thinking, judgment, reasoning one has to do in teaching



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# Enactment:
## What are key questions and moves that can be used to keep a task focused on developing MKT?

- Asking teachers to explain their solutions to the others
- Having teachers explain what is/was confusing them
- Asking teachers to figure out what might be confusing/difficult for someone else about the problem
- Having teachers ask questions to become more clear about their colleagues' solutions
- Asking teachers to make correspondences between solutions and/or representations
- Asking teachers to explain someone else's thinking
- Providing opportunities to "talk mathematics" and write on the board
- Provoking a common error
- Narrating how something a teacher does/says relates to or is a skill used in teaching



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# Discussion

- What issues are there in developing high-quality opportunities for teachers to learn MKT?



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