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# Engineering Notes

## On Feasible Body, Aerodynamic, and Navigation Euler Angles for Aircraft Flight

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### I. Introduction

ATTITUDE modeling is a central topic for vehicles that move in three dimensions. The orientation of a rigid body can be represented by a real $3 \times 3$ matrix $R$ that is orthogonal, that is, $RR^T = I$, and proper, that is, $\det R = 1$. Matrices of this type can be viewed as direction cosine matrices. The set of rotation matrices forms the Lie group SO(3) [1].

Historically, various representations of SO(3) have been used to model the attitude of a rigid body. These include Euler angles ([2], [3] pp. 763–765), Euler parameters (also called quaternions) [4], and Gibbs parameters (also called Rodrigues parameters) [5,6]. Despite more than two centuries devoted to parameterizations of SO(3), interest in this subject continues unabated, as evidenced by recent studies devoted to generalized Euler angles [7,8].

The present paper focuses on the classical Euler angles and explores the problem of determining the feasible values of Euler angles for *closed rotation sequences*, that is, sequences of Euler rotation matrices whose product is equal to the identity matrix [9]. For the case of up to four orthogonal rotation axes, the present paper provides an explicit characterization of the feasible rotation angles.

The present paper is motivated by aircraft kinematics and the need to determine explicit instantaneous relationships among constant or time-dependent Euler angles under steady or nonsteady flight conditions. Assuming a flat Earth and beginning from an Earth-fixed frame $\mathrm{F_E}$, the 3-2-1 Euler-angle sequence with the yaw, pitch, and roll angles $\Psi$, $\Theta$, and $\Phi$ yields the aircraft body-fixed frame $\mathrm{F_{AC}}$. These are the *body angles*. An additional 2-axis rotation through minus the angle of attack $-\alpha$ yields the stability frame $\mathrm{F_S}$, and a 3-axis rotation through the sideslip angle $\beta$ yields the wind frame $\mathrm{F_W}$, whose 1-axis is aligned with the aircraft velocity vector. These are the *aerodynamic angles*. The combined 3-2-1-2-3 sequence can be represented by [10]

$$\mathrm{F_E} \xrightarrow[3]{\Psi} \mathrm{F_{E'}} \xrightarrow[2]{\Theta} \mathrm{F_{E''}} \xrightarrow[1]{\Phi} \mathrm{F_{AC}} \xrightarrow[2]{-\alpha} \mathrm{F_S} \xrightarrow[3]{\beta} \mathrm{F_W} \tag{1}$$

where $\mathrm{F_{E'}}$ and $\mathrm{F_{E''}}$ are intermediate Earth frames.

Alternatively, the wind frame $\mathrm{F_W}$ can be reached by a different sequence. In particular, beginning again from $\mathrm{F_E}$, the 3-2-1 Euler-angle sequence with the heading, flight-path, and bank angles $\eta$, $\gamma$, and $\mu$ yields $\mathrm{F_W}$. This 3-2-1 sequence of *navigation angles* can be represented by

$$\mathrm{F_E} \xrightarrow[3]{\eta} \mathrm{F_F} \xrightarrow[2]{\gamma} \mathrm{F_G} \xrightarrow[1]{\mu} \mathrm{F_W} \tag{2}$$

where $\mathrm{F_F}$ and $\mathrm{F_G}$ are intermediate Earth frames. The bank angle $\mu$ is a rotation around the aircraft velocity vector and is not necessarily equal to the roll angle $\Phi$, which is a rotation around the body 1-axis. The body, aerodynamic, and navigation angles thus involve a total of eight frames and eight rotation angles. Merging Eqs. (1) and (2) yields the closed rotation sequence

$$\mathrm{F_E} \xrightarrow[3]{\Psi} \mathrm{F_{E'}} \xrightarrow[2]{\Theta} \mathrm{F_{E''}} \xrightarrow[1]{\Phi} \mathrm{F_{AC}} \xrightarrow[2]{-\alpha} \mathrm{F_S} \xrightarrow[3]{\beta} \mathrm{F_W} \xrightarrow[1]{-\mu} \mathrm{F_G} \xrightarrow[2]{-\gamma} \mathrm{F_F} \xrightarrow[3]{-\eta} \mathrm{F_E} \tag{3}$$

Now, let $\mathcal{O}_{\mathrm{W/E}}$ denote the direction cosine matrix that transforms physical vectors resolved in $\mathrm{F_E}$ to physical vectors resolved in $\mathrm{F_W}$. It thus follows from Eqs. (1) and (2) that

$$\mathcal{O}_{\mathrm{W/E}} = \mathcal{O}_3(\beta)\mathcal{O}_2(-\alpha)\mathcal{O}_1(\Phi)\mathcal{O}_2(\Theta)\mathcal{O}_3(\Psi) = \mathcal{O}_1(\mu)\mathcal{O}_2(\gamma)\mathcal{O}_3(\eta) \tag{4}$$

which can be rewritten as

$$\mathcal{O}_2(-\gamma)\mathcal{O}_1(-\mu)\mathcal{O}_3(\beta)\mathcal{O}_2(-\alpha)\mathcal{O}_1(\Phi)\mathcal{O}_2(\Theta)\mathcal{O}_3(\Psi-\eta) = I \tag{5}$$

Note that Eq. (5) is a 3-2-1-2-3-1-2 Euler-angle sequence (reading right to left) involving seven rather than eight angles because the single angle $\Psi - \eta$ replaces the separate angles $\Psi$ and $\eta$. Therefore, for all real numbers $a$, the angles $\Psi$ and $\eta$ can be replaced by $\Psi + a$ and $\eta + a$, respectively, without modifying the remaining angles. Physically, this means that the yaw and heading angles can be rotated by the same amount relative to the Earth without changing the aerodynamic angles and remaining body and navigation angles.

Several special cases are worth noting. If $\Psi \equiv \eta$, then Eq. (5) becomes

$$\mathcal{O}_1(-\mu)\mathcal{O}_3(\beta)\mathcal{O}_2(-\alpha)\mathcal{O}_1(\Phi)\mathcal{O}_2(\Theta-\gamma) = I \tag{6}$$

and thus the effective number of angles is five rather than six. If $\Psi \equiv \eta$ and $\Theta \equiv \gamma$, then Eq. (6) becomes

$$\mathcal{O}_3(\beta)\mathcal{O}_2(-\alpha)\mathcal{O}_1(\Phi-\mu) = I \tag{7}$$

and thus the effective number of angles is three rather than four. If $\Psi \equiv \eta$ and $\Phi \equiv 0$, then Eq. (6) becomes

$$\mathcal{O}_1(-\mu)\mathcal{O}_3(\beta)\mathcal{O}_2(\Theta-\gamma-\alpha) = I \tag{8}$$

and again the effective number of angles is three rather than four. If $\Phi \equiv 0$, then Eq. (5) becomes

$$\mathcal{O}_2(-\gamma)\mathcal{O}_1(-\mu)\mathcal{O}_3(\beta)\mathcal{O}_2(\Theta-\alpha)\mathcal{O}_3(\Psi-\eta) = I \tag{9}$$

and the effective number of angles is again five rather than six. If $\Phi \equiv 0$ and $\Theta \equiv \alpha$, then Eq. (9) becomes

$$\mathcal{O}_2(-\gamma)\mathcal{O}_1(-\mu)\mathcal{O}_3(\Psi-\eta+\beta) = I \tag{10}$$

and thus the effective number of angles is three rather than four. If $\beta \equiv \mu \equiv 0$, then Eq. (9) becomes

$$\mathcal{O}_2(-\gamma-\alpha)\mathcal{O}_1(\Phi)\mathcal{O}_2(\Theta)\mathcal{O}_3(\Psi-\eta) = I \tag{11}$$

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and thus the effective number of angles is four rather than five. If $\alpha \equiv \beta \equiv 0$, then Eq. (9) becomes

$$\mathcal{O}_2(-\gamma)\mathcal{O}_1(\Phi-\mu)\mathcal{O}_2(\Theta)\mathcal{O}_3(\Psi-\eta) = I \tag{12}$$

and thus the effective number of angles is four rather than five.

The fact that a product of seven Euler rotation matrices equals the identity matrix implies that the angles $\Psi-\eta$, $\Theta$, $\Phi$, $\alpha$, $\beta$, $\mu$, and $\gamma$ cannot be independent. It is thus of interest to determine the feasible values of these seven angles. Note that the constraint (5) must be satisfied at each instant of time whether or not the angles are constant; however, the time argument is omitted in Eq. (5) for simplicity. Note that all of the navigation angles are constant if and only if the aircraft is flying in a straight line relative to the Earth frame. This case is discussed in Sec. VII. Relationships among these angles for maneuvering flight are studied in [11] using spherical trigonometry.

All of the results in this paper are stated for the full possible range of angles. In particular, each body, aerodynamic, and navigation angle can be equal to $\pi$. For example, if either $\alpha = \beta = 0$ or $\alpha = \beta = \pi$, then the aircraft velocity vector is pointed in the direction of the body 1-axis. Furthermore, if either $\alpha = \pi$ and $\beta = 0$ or $\alpha = 0$ and $\beta = \pi$, then the aircraft velocity vector is pointed in the direction that is opposite to the body 1-axis. In this case, the aircraft is flying backward. Although this is nonphysical for fixed-wing aircraft, it is meaningful for quadrotors.

The contents of this paper are as follows. Section II provides preliminary material. Section III considers products of two and three Euler rotation matrices that are equal to the identity matrix. Section IV considers products of four Euler rotation matrices. Section V discusses extensions to five or more Euler rotation matrices. Section VI applies the results of Secs. III and IV to the body, aerodynamic, and navigation angles.

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## II. Preliminaries

A rotation matrix $R$ is a real $3 \times 3$ matrix that is orthogonal, that is, $R^T R = I$, and proper, that is, $\det R = 1$. For a real number $a$, define the *Euler rotation matrices*

$$\mathcal{O}_1(a) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos a & \sin a \\ 0 & -\sin a & \cos a \end{bmatrix},$$

$$\mathcal{O}_2(a) = \begin{bmatrix} \cos a & 0 & -\sin a \\ 0 & 1 & 0 \\ \sin a & 0 & \cos a \end{bmatrix},$$

$$\mathcal{O}_3(a) = \begin{bmatrix} \cos a & \sin a & 0 \\ -\sin a & \cos a & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{13}$$

each of which is a rotation matrix.

For trigonometric functions, it suffices to confine angles to $(-\pi, \pi]$. However, sums and differences of angles can violate this constraint, and thus it is convenient to represent angles by arbitrary real numbers. Hence, for $a, b \in \mathbb{R}$, the notation $a \equiv b$ means that $a - b$ is an integer multiple of $2\pi$, and thus $\sin a = \sin b$ and $\cos a = \cos b$. Note that $\pi \equiv -\pi$.

Some basic properties of Euler rotation matrices are given by the following result.

*Lemma 1:* The following statements hold:

i) Let $a \in \mathbb{R}$ and $i \in \{1, 2, 3\}$. Then, $a \equiv 0$ if and only if $\mathcal{O}_i(a) = I$.

ii) Let $a \in \mathbb{R}$. Then, the following statements are equivalent:

a) $a \equiv \pi$.

b) $\mathcal{O}_1(a) = \text{diag}(1, -1, -1)$.

c) $\mathcal{O}_2(a) = \text{diag}(-1, 1, -1)$.

d) $\mathcal{O}_3(a) = \text{diag}(-1, -1, 1)$.

iii) Let $i, j, k \in \{1, 2, 3\}$ be distinct. Then, $\mathcal{O}_i(\pi)\mathcal{O}_j(\pi)\mathcal{O}_k(\pi) = I_3$ and $\mathcal{O}_i(\pi) = \mathcal{O}_j(\pi)\mathcal{O}_k(\pi)$.

iv) Let $a \in \mathbb{R}$ and $i \in \{1, 2, 3\}$. Then, the following statements are equivalent:

a) Either $a \equiv 0$ or $a \equiv \pi$.

b) $\mathcal{O}_i(a)$ is symmetric.

c) $\mathcal{O}_i(a)$ is diagonal.

v) Let $a \in \mathbb{R}$ and $i \in \{1, 2, 3\}$. Then, $\mathcal{O}_i(a)e_i = e_i$, where $e_i$ is the $i$th column of $I_3$.

vi) Let $a \in \mathbb{R}$ and $i \in \{1, 2, 3\}$. Then, $\mathcal{O}_i(-a) = \mathcal{O}_i(a)^{-1} = \mathcal{O}_i(a)^T$.

vii) Let $a \in \mathbb{R}$, and let $i, j \in \{1, 2, 3\}$ be distinct. Then, $\mathcal{O}_i(\pi)\mathcal{O}_j(a)\mathcal{O}_i(\pi) = \mathcal{O}_j(-a)$.

viii) Let $a \in \mathbb{R}$, and let $i, j, k \in \{1, 2, 3\}$ be distinct. Then,

$$\mathcal{O}_i(\pm\pi/2)\mathcal{O}_j(a)\mathcal{O}_i(\mp\pi/2) = \begin{cases} \mathcal{O}_k(\pm a), & (i,j) \in \{(1,3),(2,1),(3,2)\}, \\ \mathcal{O}_k(\mp a), & (i,j) \in \{(1,2),(2,3),(3,1)\} \end{cases} \tag{14}$$

ix) Let $a, b \in \mathbb{R}$ and $i \in \{1, 2, 3\}$. Then, $\mathcal{O}_i(a)\mathcal{O}_i(b) = \mathcal{O}_i(a + b)$.

x) Let $a, b \in \mathbb{R}$, and let $i, j \in \{1, 2, 3\}$ be distinct. Then, the following statements are equivalent:

a) $a \equiv b \equiv 0$.

b) $\mathcal{O}_i(a) = \mathcal{O}_j(b)$.

c) $\mathcal{O}_i(a)\mathcal{O}_j(b) = I$.

Lemma 1 can be used to show that the angles 0 and $\pi$ can be interchanged by suitably modifying additional angles. For example, suppose that $\Theta = \pi$, so that Eq. (5) has the form

$$\mathcal{O}_2(-\gamma)\mathcal{O}_1(-\mu)\mathcal{O}_3(\beta)\mathcal{O}_2(-\alpha)\mathcal{O}_1(\Phi)\mathcal{O}_2(\pi)\mathcal{O}_3(\Psi-\eta) = I \tag{15}$$

It follows from iii of Lemma 1 that $\mathcal{O}_2(\pi) = \mathcal{O}_1(\pi)\mathcal{O}_2(0)\mathcal{O}_3(\pi)$. Therefore, Eq. (15) can be written as

$$\mathcal{O}_2(-\gamma)\mathcal{O}_1(-\mu)\mathcal{O}_3(\beta)\mathcal{O}_2(-\alpha)\mathcal{O}_1(\Phi+\pi)\mathcal{O}_2(0)\mathcal{O}_3(\Psi-\eta+\pi) = I \tag{16}$$

Hence $\Theta = \pi$ is replaced by $\Theta = 0$, and the angles $\Phi$ and $\Psi - \eta$ are replaced by $\Phi + \pi$ and $\Psi - \eta + \pi$, respectively. Conversely, suppose that $\Theta = 0$, so that Eq. (5) has the form

$$\mathcal{O}_2(-\gamma)\mathcal{O}_1(-\mu)\mathcal{O}_3(\beta)\mathcal{O}_2(-\alpha)\mathcal{O}_1(\Phi)\mathcal{O}_2(0)\mathcal{O}_3(\Psi-\eta) = I \tag{17}$$

It follows from iii and ix of Lemma 1 that $\mathcal{O}_2(0) = I = \mathcal{O}_1(\pi)\mathcal{O}_2(\pi)\mathcal{O}_3(\pi)$. Therefore, Eq. (17) can be written as

$$\mathcal{O}_2(-\gamma)\mathcal{O}_1(-\mu)\mathcal{O}_3(\beta)\mathcal{O}_2(-\alpha)\mathcal{O}_1(\Phi+\pi)\mathcal{O}_2(\pi)\mathcal{O}_3(\Psi-\eta+\pi) = I \tag{18}$$

Hence $\Theta = 0$ is replaced by $\Theta = \pi$, and the angles $\Phi$ and $\Psi - \eta$ are replaced by $\Phi + \pi$ and $\Psi - \eta + \pi$, respectively. This technique can be applied to $\mu$, $\Theta$, $\alpha$, $\gamma$, $\beta$, but not to $\Phi$ and $\Psi - \eta$ because these two angles occur between pairs of 2-axis rotations.

## III. Euler-Angle Permutations

Considering all permutations of $i, j, k \in \{1, 2, 3\}$, there exist 12 distinct sequences of 3 Euler rotation matrices [3] p. 764). However, by relabeling axes, these 12 sequences can be represented by 2 sequences, for example, 3-2-1 and 1-2-1. Consequently, a rotation represented in terms of the axes $i, j, k$ can be equivalently represented in terms of an arbitrary permutation of $i, j, k$. For example, a 1-2-1 sequence can be applied as a 3-1-3 sequence involving precession, nutation, and spin used in spacecraft kinematics. Note, however, that, although a 2-1-2 sequence can be applied as a 3-1-3 sequence by relabeling the 2-axis as the 3-axis, the resulting 3-axis must be reflected in order to retain the right-handedness of the coordinate

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frame. In characterizing all closed rotation sequences, it thus suffices to disregard the effect of axis relabeling and reflection.

In addition to axis relabeling, by choosing an alternative starting point, sequence cycling can be disregarded in the sense that the sequences 2-1-2-1-3 and 1-2-1-3-2 are identical. Finally, by multiplying each angle by $-1$, sequence reversal can be disregarded in the sense that 2-1-2-1-3 and 3-1-2-1-2 are identical. Consequently, in characterizing all closed rotation sequences of a given length, a pair of sequences can be viewed as identical if one sequence can be obtained from the other by axis relabeling (with axis reflection to retain right-handedness), sequence cycling, and sequence reversal.

To make this idea more precise, let $\sigma: \{1, 2, 3\} \to \{1, 2, 3\}$ be a permutation, and define the matrix $S_\sigma \in \mathrm{SO}(3)$ by $(S_\sigma)_{i,j} = 1$ if $\sigma(i) = j$ and 0 otherwise. Hence, if $\sigma$ maps the rotation sequence (1, 2, 3) to the rotation sequence (2, 1, 3), then

$$S_\sigma = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \tag{19}$$

Note that, for all $\sigma$, $\det S_\sigma = \pm 1$.

*Lemma 2:* Let $i \in \{1, 2, 3\}$, and let $a \in \mathbb{R}$. Then,

$$S_\sigma^T \mathcal{O}_i(a) S_\sigma = \mathcal{O}_{\sigma(i)}((\det S_\sigma)a) \tag{20}$$

The following result, which follows directly from Lemma 2, shows that an arbitrary product of Euler rotation matrices can be equivalently represented as a product of Euler rotation matrices with axis relabeling.

*Proposition 1:* Let $n$ be a positive integer, let $i_1, \ldots, i_n$ be elements of $\{1, 2, 3\}$, let $a_1, \ldots, a_n$ be real numbers, let $R \in \mathrm{SO}(3)$, and assume that

$$\mathcal{O}_{i_1}(a_1) \cdots \mathcal{O}_{i_n}(a_n) = R \tag{21}$$

Then,

$$\mathcal{O}_{\sigma(i_1)}((\det S_\sigma)a_1) \cdots \mathcal{O}_{\sigma(i_n)}((\det S_\sigma)a_n) = S_\sigma^T R S_\sigma \tag{22}$$

*Definition 1:* Let $n$ be a positive integer, and let $(i_1, \ldots, i_n)$ and $(j_1, \ldots, j_n)$ be sequences of elements of $\{1, 2, 3\}$ with distinct adjacent components and distinct first and last components. Then, $(i_1, \ldots, i_n)$ and $(j_1, \ldots, j_n)$ are *permutationally distinct* if there does not exist a permutation $\sigma$ such that $(\sigma(i_1), \ldots, \sigma(i_n))$ can be transformed to $(j_1, \ldots, j_n)$ by sequence cycling and sequence reversal.

## IV. Products of Two and Three Euler Rotation Matrices Equal to the Identity Matrix

There is one permutationally distinct case where a product of two Euler rotation matrices is equal to the identity matrix, namely, a 1-2 product.

*Proposition 2:* Let $a, b \in \mathbb{R}$. Then,

$$\mathcal{O}_1(a)\mathcal{O}_2(b) = I \tag{23}$$

if and only if $a \equiv b \equiv 0$.

*Proof:* The result follows from $x$ of Lemma 1. □

There is one permutationally distinct case where a product of three Euler rotation matrices is equal to the identity matrix, namely, a 1-2-3 product.

The following result is a special case of the Rodrigues-Hamilton theorem [9]; an animation of this result appears in the online version of [9].

*Proposition 3:* Let $a, b, c \in \mathbb{R}$. Then,

$$\mathcal{O}_1(a)\mathcal{O}_2(b)\mathcal{O}_3(c) = I \tag{24}$$

if and only if either $a \equiv b \equiv c \equiv 0$ or $a \equiv b \equiv c \equiv \pi$.

*Proof:* Sufficiency is immediate. To prove necessity, note that, by rewriting Eq. (24) as (writing S$a$ for $\sin a$)

$$\begin{bmatrix} \mathrm{C}b\mathrm{C}c & \mathrm{C}b\mathrm{S}c & -\mathrm{S}b \\ \mathrm{C}c\mathrm{S}a\mathrm{S}b - \mathrm{C}a\mathrm{S}c & \mathrm{C}a\mathrm{C}c + \mathrm{S}a\mathrm{S}b\mathrm{S}c & \mathrm{C}b\mathrm{S}a \\ \mathrm{S}a\mathrm{S}c + \mathrm{C}a\mathrm{C}c\mathrm{S}b & \mathrm{C}a\mathrm{S}b\mathrm{S}c - \mathrm{C}c\mathrm{S}a & \mathrm{C}a\mathrm{C}b \end{bmatrix} = I$$

it follows from the (1,1) entry that $\mathrm{C}b \neq 0$, and thus from the (1,2), (1,3), and (2,3) entries that $\mathrm{S}a = \mathrm{S}b = \mathrm{S}c = 0$. Hence, it follows from the (2,2) and (3,3) entries that $\mathrm{C}a\mathrm{C}c = \mathrm{C}a\mathrm{C}b = 1$, and thus either $\mathrm{C}a = \mathrm{C}b = \mathrm{C}c = 1$ or $\mathrm{C}a = \mathrm{C}b = \mathrm{C}c = -1$. Hence, either $a \equiv b \equiv c \equiv 0$ or $a \equiv b \equiv c \equiv \pi$.

As an alternative proof of necessity, it follows from Eq. (24) using v and ix of Lemma 1 that

$$0 = e_1^T I_3 e_3 = e_1^T \mathcal{O}_1(a)\mathcal{O}_2(b)\mathcal{O}_3(c)e_3 = e_1^T \mathcal{O}_2(b)e_3 = -\sin b$$

Hence, either $b \equiv 0$ or $b \equiv \pi$. In the case where $b \equiv 0$, Eq. (24) implies that $\mathcal{O}_1(a)\mathcal{O}_3(c) = I_3$, and thus Proposition 2 implies that $a \equiv c \equiv 0$. In the case where $b \equiv \pi$, it follows from Eq. (24) that

$$\begin{aligned} I_3 &= \mathcal{O}_1(a)\mathcal{O}_2(\pi)\mathcal{O}_3(c) = \mathcal{O}_1(a)\mathcal{O}_1(\pi)\mathcal{O}_3(\pi)\mathcal{O}_3(c) \\ &= \mathcal{O}_1(a+\pi)\mathcal{O}_3(c+\pi) \end{aligned}$$

and thus Proposition 2 implies that $a \equiv c \equiv \pi$. □

## V. Products of Four Euler Rotation Matrices Equal to the Identity Matrix

There are two permutationally distinct cases where a product of four Euler rotation matrices is equal to the identity matrix, namely, 1-2-3-2 and 1-2-1-2 products. The following result considers the case of a 1-2-3-2 product.

*Proposition 4:* Let $a, b, c, d \in \mathbb{R}$. Then,

$$\mathcal{O}_1(a)\mathcal{O}_2(b)\mathcal{O}_3(c)\mathcal{O}_2(d) = I \tag{25}$$

if and only if either i) $b \equiv -d \equiv \pi/2$ and $a \equiv c$, ii) $b \equiv -d \equiv -\pi/2$ and $a \equiv -c$, iii) $a \equiv c \equiv 0$ and $b \equiv -d$, or iv) $a \equiv c \equiv \pi$ and $b \equiv d + \pi$.

*Proof:* Sufficiency is immediate. To prove necessity, note that Eq. (25) implies

$$\begin{bmatrix} \mathrm{C}b\mathrm{C}c\mathrm{C}d - \mathrm{S}b\mathrm{S}d & \mathrm{C}b\mathrm{S}c & -\mathrm{C}d\mathrm{S}b - \mathrm{C}b\mathrm{C}c\mathrm{S}d \\ \mathrm{C}b\mathrm{S}a\mathrm{S}d - \mathrm{C}d(\mathrm{C}a\mathrm{S}c - \mathrm{C}c\mathrm{S}a\mathrm{S}b) & \mathrm{C}a\mathrm{C}c + \mathrm{S}a\mathrm{S}b\mathrm{S}c & \mathrm{S}d(\mathrm{C}a\mathrm{S}c - \mathrm{C}c\mathrm{S}a\mathrm{S}b) + \mathrm{C}b\mathrm{C}d\mathrm{S}a \\ \mathrm{C}d(\mathrm{S}a\mathrm{S}c + \mathrm{C}a\mathrm{C}c\mathrm{S}b) + \mathrm{C}a\mathrm{C}b\mathrm{S}d & \mathrm{C}a\mathrm{S}b\mathrm{S}c - \mathrm{C}c\mathrm{S}a & \mathrm{C}a\mathrm{C}b\mathrm{C}d - \mathrm{S}d(\mathrm{S}a\mathrm{S}c + \mathrm{C}a\mathrm{C}c\mathrm{S}b) \end{bmatrix} = I$$

Since $\mathrm{C}b\mathrm{S}c = 0$, it follows that either $\mathrm{C}b = 0$ or $\mathrm{S}c = 0$. Therefore, either i) $b \equiv \pi/2$, ii) $b \equiv -\pi/2$, iii) $c \equiv 0$, or iv) $c \equiv \pi$.

*Case i:* $b \equiv \pi/2$. In this case,

$$\begin{bmatrix} -\mathrm{S}d & 0 & -\mathrm{C}d \\ -\mathrm{C}d(\mathrm{C}a\mathrm{S}c - \mathrm{C}c\mathrm{S}a) & \mathrm{C}a\mathrm{C}c + \mathrm{S}a\mathrm{S}c & \mathrm{S}d(\mathrm{C}a\mathrm{S}c - \mathrm{C}c\mathrm{S}a) + \mathrm{C}b\mathrm{C}d\mathrm{S}a \\ \mathrm{C}d(\mathrm{S}a\mathrm{S}c + \mathrm{C}a\mathrm{C}c) & \mathrm{C}a\mathrm{S}c - \mathrm{C}c\mathrm{S}a & -\mathrm{S}d(\mathrm{S}a\mathrm{S}c + \mathrm{C}a\mathrm{C}c) \end{bmatrix} = I$$

Since $\mathrm{S}d = -1$ and $\mathrm{C}d = 0$, it follows that $d \equiv -\pi/2$. Hence,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \mathrm{C}a\mathrm{C}c + \mathrm{S}a\mathrm{S}c & -(\mathrm{C}a\mathrm{S}c - \mathrm{C}c\mathrm{S}a) \\ 0 & \mathrm{C}a\mathrm{S}c - \mathrm{C}c\mathrm{S}a & \mathrm{S}a\mathrm{S}c + \mathrm{C}a\mathrm{C}c \end{bmatrix} = I$$

which can be written as

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(a-c) & \sin(a-c) \\ 0 & -\sin(a-c) & \cos(a-c) \end{bmatrix} = I$$

Hence, $a - c \equiv 0$.

*Case ii:* $b \equiv -\pi/2$. In this case,

$$\begin{bmatrix} \mathrm{S}d & 0 & \mathrm{C}d \\ -\mathrm{C}d(\mathrm{C}a\mathrm{S}c + \mathrm{C}c\mathrm{S}a) & \mathrm{C}a\mathrm{C}c - \mathrm{S}a\mathrm{S}c & \mathrm{S}d(\mathrm{C}a\mathrm{S}c + \mathrm{C}c\mathrm{S}a) \\ \mathrm{C}d(\mathrm{S}a\mathrm{S}c - \mathrm{C}a\mathrm{C}c) & -\mathrm{C}a\mathrm{S}c - \mathrm{C}c\mathrm{S}a & -\mathrm{S}d(\mathrm{S}a\mathrm{S}c - \mathrm{C}a\mathrm{C}c) \end{bmatrix} = I$$

Since $\mathrm{S}d = 1$ and $\mathrm{C}d = 0$, it follows that $d \equiv \pi/2$. Hence,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \mathrm{C}a\mathrm{C}c - \mathrm{S}a\mathrm{S}c & \mathrm{C}a\mathrm{S}c + \mathrm{C}c\mathrm{S}a \\ 0 & -\mathrm{C}a\mathrm{S}c - \mathrm{C}c\mathrm{S}a & -(\mathrm{S}a\mathrm{S}c - \mathrm{C}a\mathrm{C}c) \end{bmatrix} = I$$

which can be written as

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(a+c) & \sin(a+c) \\ 0 & -\sin(a+c) & \cos(a+c) \end{bmatrix} = I$$

Hence, $a + c \equiv 0$.

*Case iii:* $c \equiv 0$. In this case,

$$\begin{bmatrix} \mathrm{C}b\mathrm{C}d - \mathrm{S}b\mathrm{S}d & 0 & -\mathrm{C}d\mathrm{S}b - \mathrm{C}b\mathrm{S}d \\ \mathrm{C}b\mathrm{S}a\mathrm{S}d + \mathrm{C}d\mathrm{S}a\mathrm{S}b) & \mathrm{C}a & -\mathrm{S}d\mathrm{S}a\mathrm{S}b + \mathrm{C}b\mathrm{C}d\mathrm{S}a \\ \mathrm{C}d\mathrm{C}a\mathrm{S}b + \mathrm{C}a\mathrm{C}b\mathrm{S}d & -\mathrm{S}a & \mathrm{C}a\mathrm{C}b\mathrm{C}d - \mathrm{S}d\mathrm{C}a\mathrm{S}b \end{bmatrix} = I$$

Since $\mathrm{C}a = 1$ and $\mathrm{S}a = 0$, it follows that $a \equiv 0$. Hence,

$$\begin{bmatrix} \mathrm{C}b\mathrm{C}d - \mathrm{S}b\mathrm{S}d & 0 & -\mathrm{C}d\mathrm{S}b - \mathrm{C}b\mathrm{S}d \\ 0 & 1 & 0 \\ \mathrm{C}d\mathrm{S}b + \mathrm{C}b\mathrm{S}d & 0 & \mathrm{C}b\mathrm{C}d - \mathrm{S}d\mathrm{S}b \end{bmatrix} = I$$

which can be written as

$$\begin{bmatrix} \cos(b+d) & 0 & -\sin(b+d) \\ 0 & 1 & 0 \\ \sin(b+d) & 0 & \cos(b+d) \end{bmatrix} = I$$

Hence, $b + d \equiv 0$.

*Case iv:* $c \equiv \pi$. In this case,

$$\begin{bmatrix} -\mathrm{C}b\mathrm{C}d - \mathrm{S}b\mathrm{S}d & 0 & -\mathrm{C}d\mathrm{S}b + \mathrm{C}b\mathrm{S}d \\ \mathrm{C}b\mathrm{S}a\mathrm{S}d - \mathrm{C}d\mathrm{S}a\mathrm{S}b & -\mathrm{C}a & \mathrm{S}d\mathrm{S}a\mathrm{S}b + \mathrm{C}b\mathrm{C}d\mathrm{S}a \\ -\mathrm{C}d\mathrm{C}a\mathrm{S}b + \mathrm{C}a\mathrm{C}b\mathrm{S}d & \mathrm{S}a & \mathrm{C}a\mathrm{C}b\mathrm{C}d + \mathrm{S}d\mathrm{C}a\mathrm{S}b \end{bmatrix} = I$$

Since $\mathrm{C}a = -1$ and $\mathrm{S}a = 0$, it follows that $a \equiv \pi$. Hence,

$$\begin{bmatrix} -\mathrm{C}b\mathrm{C}d - \mathrm{S}b\mathrm{S}d & 0 & -\mathrm{C}d\mathrm{S}b + \mathrm{C}b\mathrm{S}d \\ 0 & 1 & 0 \\ \mathrm{C}d\mathrm{S}b - \mathrm{C}b\mathrm{S}d & 0 & -\mathrm{C}b\mathrm{C}d - \mathrm{S}d\mathrm{S}b \end{bmatrix} = I$$

which can be written as

$$\begin{bmatrix} -\cos(b-d) & 0 & -\sin(b-d) \\ 0 & 1 & 0 \\ \sin(b-d) & 0 & -\cos(b-d) \end{bmatrix} = I$$

Hence, $b - d \equiv \pi$.

As an alternative proof of necessity, note that it follows from Eq. (25) using v of Lemma 1 that

$$\begin{aligned} 0 = e_1^T I e_2 &= e_1^T \mathcal{O}_1(a)\mathcal{O}_1(b)\mathcal{O}_1(c)\mathcal{O}_1(d)e_2 = e_1^T \mathcal{O}_1(b)\mathcal{O}_1(c)e_2 \\ &= (\cos b)\sin c \end{aligned}$$

Hence, either $b \equiv \pm\pi/2$, $c \equiv 0$, or $c \equiv \pi$.

*Cases i and ii:* $b \equiv \pm\pi/2$. In this case, it follows from Eq. (25) using v, viii, and ix of Lemma 1 that

$$\begin{aligned} I &= \mathcal{O}_1(a)\mathcal{O}_2(\pm\pi/2)\mathcal{O}_3(c)\mathcal{O}_2(d) \\ &= \mathcal{O}_1(a)\mathcal{O}_2(\pm\pi/2)\mathcal{O}_3(c)\mathcal{O}_2(\mp\pi/2)\mathcal{O}_2(\pm\pi/2)\mathcal{O}_2(d) \\ &= \mathcal{O}_1(a)\mathcal{O}_1(\mp c)\mathcal{O}_2(\pm\pi/2)\mathcal{O}_2(d) \\ &= \mathcal{O}_1(a \mp c)\mathcal{O}_2(d \pm \pi/2) \end{aligned}$$

and thus Proposition 2 implies that $a \equiv \pm c$ and $d \equiv \mp\pi/2$.

*Case iii:* $c \equiv 0$. In this case, it follows from Eq. (25) using ix of Lemma 1 that

$$I = \mathcal{O}_1(a)\mathcal{O}_2(b)\mathcal{O}_2(d) = \mathcal{O}_1(a)\mathcal{O}_2(b+d)$$

and thus Proposition 2 implies that $a \equiv 0$ and $b \equiv -d$.

*Case iv:* $c \equiv \pi$. In this case, it follows from Eq. (25) using iii, vi, and ix of Lemma 1 that

$$\begin{aligned} I &= \mathcal{O}_1(a)\mathcal{O}_2(b)\mathcal{O}_3(\pi)\mathcal{O}_2(d) \\ &= \mathcal{O}_1(a)\mathcal{O}_1(\pi)\mathcal{O}_1(-\pi)\mathcal{O}_2(b)\mathcal{O}_1(\pi)\mathcal{O}_2(\pi)\mathcal{O}_2(d) \\ &= \mathcal{O}_1(a+\pi)\mathcal{O}_2(-b)\mathcal{O}_2(d+\pi)\mathcal{O}_1(a+\pi) \\ &= \mathcal{O}_1(a+\pi)\mathcal{O}_2(d+\pi-b) \end{aligned}$$

and thus Proposition 2 implies that $a \equiv \pi$ and $b \equiv d + \pi$. □

The following result considers the case where a 1-2-3 product of Euler rotation matrices is equal to a 2-axis Euler rotation matrix.

*Corollary 1:* $a, b, c, d \in \mathbb{R}$ satisfy

$$\mathcal{O}_2(d) = \mathcal{O}_1(a)\mathcal{O}_2(b)\mathcal{O}_3(c) \tag{26}$$

if and only if either i) $a \equiv c$ and $b \equiv d \equiv \pi/2$, ii) $a \equiv -c$ and $b \equiv d \equiv -\pi/2$, iii) $a \equiv c \equiv 0$ and $b \equiv d$, or iv) $a \equiv c \equiv \pi$ and $b \equiv \pi - d$.

The following result considers the case of a 1-2-1-2 product. The proof is similar to the proof of Proposition 4 and thus is omitted.

*Proposition 5:* Let $a, b, c, d \in \mathbb{R}$. Then,

$$\mathcal{O}_1(a)\mathcal{O}_2(b)\mathcal{O}_1(c)\mathcal{O}_2(d) = I \tag{27}$$

if and only if either i) $b \equiv d \equiv 0$ and $a \equiv -c$, ii) $a \equiv c \equiv 0$ and $b \equiv -d$, iii) $b \equiv d \equiv \pi$ and $a \equiv c$, or iv) $a \equiv c \equiv \pi$ and $b \equiv d$.

The following result considers the case where a 1-2-1 product of Euler rotation matrices is equal to a 2-axis rotation.

*Corollary 2:* Let $a, b, c, d \in \mathbb{R}$. Then,

$$\mathcal{O}_2(d) = \mathcal{O}_1(a)\mathcal{O}_2(b)\mathcal{O}_1(c) \tag{28}$$

if and only if either i) $b \equiv -d \equiv 0$ and $a \equiv -c$, ii) $a \equiv c \equiv 0$ and $b \equiv d$, iii) $b \equiv -d \equiv \pi$ and $a \equiv c$, or iv) $a \equiv c \equiv \pi$ and $b \equiv -d$.

Proposition 5 yields the following result on commuting Euler rotation matrices.

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*Corollary 3:* Let $a, b \in \mathbb{R}$. Then,

$$\mathcal{O}_1(a)\mathcal{O}_2(b) = \mathcal{O}_2(b)\mathcal{O}_1(a) \tag{29}$$

if and only if either $a \equiv 0$, $b \equiv 0$, or $a \equiv b \equiv \pi$.

## VI. Extensions to Products of Five or More Euler Rotation Matrices

For the purpose of characterizing closed rotation sequences of a given length, it suffices to consider permutationally distinct products, which accounts for axis relabeling, cycling, and reversal. It has been shown that there is one permutationally distinct product of two Euler rotation matrices whose product is the identity; one permutationally distinct product of three Euler rotation matrices whose product is the identity; and two permutationally distinct products of four Euler rotation matrices whose product is the identity. Products of five or more Euler rotation matrices can be considered. For example, there is one permutationally distinct product of five Euler rotation matrices whose product is the identity, namely, 1-2-1-2-3; there are four permutationally distinct products of six Euler rotation matrices whose product is the identity, namely, 1-2-1-2-1-2, 1-2-1-2-1-3, 1-2-1-3-2-3, and 1-2-3-1-2-3; and there are three permutationally distinct products of seven Euler rotation matrices whose product is the identity, namely, 1-2-1-2-1-2-3, 1-2-1-2-3-1-3, and 1-2-1-3-2-1-3. Hence, Eq. (5) is one of three permutationally distinct products of seven Euler rotation matrices, in particular, 1-2-1-3-2-1-3. Furthermore, there are eight permutationally distinct products of eight Euler rotation matrices whose product is the identity.

For a rotation sequence of arbitrary length $n$, the number of permutationally distinct products of $n$ Euler rotation matrices is given by sequence A114438 of the On-line Encyclopedia of Integer Sequences (OEIS). It can be shown that the number of permutationally distinct products of $n$ Euler rotation matrices with or without allowing reversals is the same for all $n \leq 8$ [12].

## VII. Feasible Euler Angles for Aircraft Frames

This section considers sequences of body, aerodynamic, and navigation angles that satisfy Eq. (5). Simplification arises from the repeated axes in Eq. (5). In particular, by setting certain angles to zero, rotations about the same axis become adjacent in Eq. (5) and thus can be combined into a single rotation, thereby reducing the number of factors. For example, by setting $\Phi \equiv 0$ (wings-level flight), Eq. (5) can be written as

$$\mathcal{O}_2(-\gamma)\mathcal{O}_1(-\mu)\mathcal{O}_3(\beta)\mathcal{O}_2(\Theta - \alpha)\mathcal{O}_3(\Psi - \eta) = I \tag{30}$$

and thus the effective number of angles is five rather than six. Because Eq. (30) involves five angles, setting one of these to zero yields a product of four Euler rotation matrices, which is amenable to either Proposition 4 or Proposition 5. This can be done in five different ways. For example, setting either $\Theta \equiv \alpha$ or $\Psi \equiv \eta$ yields a product of three Euler rotation matrices. These cases are considered below by conditions i and ii of Corollary 5, respectively.

More generally, by setting three of the seven angles in Eq. (5) to zero, which can be done in $\binom{7}{3} = 35$ different ways, the remaining four angles can be determined by Proposition 4 and Proposition 5. For example, assuming that all angles are constant with $\Phi = \gamma = \beta = 0$ yields wings-level, horizontal, zero-sideslip, straight-line flight. This case is typically considered when linearizing the aircraft equations of motion [10]. With these values of $\gamma$, $\Phi$, and $\beta$, Eq. (5) specializes to

$$\mathcal{O}_1(-\mu)\mathcal{O}_2(\Theta - \alpha)\mathcal{O}_3(\Psi - \eta) = I \tag{31}$$

and thus the effective number of angles is three. The case where all angles are constant, $\Phi = \beta = 0$, and $\gamma \neq 0$ yields wings-level, zero-sideslip, straight-line flight with a nonzero flight-path angle.

The following result considers various special cases of Eq. (5) that entail two undetermined angles. These cases are thus consequences of Proposition 2.

*Corollary 4:* Let $\Psi$, $\Theta$, $\Phi$, $\alpha$, $\beta$, $\eta$, $\gamma$, $\mu$ satisfy Eq. (5). Then, the following statements hold:

i) Assume that $\alpha \equiv \beta \equiv 0$. Then, the following statements are equivalent:
   a) $\Phi \equiv \mu$.
   b) $\Psi \equiv \eta$.
If these conditions hold, then $\Theta \equiv \gamma$.

ii) Assume that $\Psi \equiv \eta$. Then, the following statements are equivalent:
   a) $\Phi \equiv 0$ and $\Theta \equiv \alpha + \gamma$.
   b) $\mu \equiv \beta \equiv 0$.

iii) Assume that $\Psi \equiv \eta$ and $\beta = 0$. Then, the following statements are equivalent:
   a) $\Phi \equiv 0$.
   b) $\mu \equiv 0$.
If these conditions hold, then $\Theta \equiv \alpha + \gamma$.

iv) Assume that $\Psi + \beta \equiv \eta$. Then, the following statements are equivalent:
   a) $\Phi \equiv 0$ and $\Theta \equiv \alpha$.
   b) $\mu \equiv \gamma \equiv 0$.

v) Assume that $\Phi \equiv \mu \equiv 0$. Then, the following statements are equivalent:
   a) $\Psi \equiv \eta$.
   b) $\beta \equiv 0$.
If these conditions hold, then $\Theta \equiv \alpha + \gamma$.

vi) Assume that $\Phi \equiv \mu \equiv 0$. Then, the following statements are equivalent:
   a) $\Theta \equiv \alpha$.
   b) $\gamma \equiv 0$.
If these conditions hold, then $\Psi + \beta \equiv \eta$.

vii) Assume that $\Theta \equiv \alpha \equiv \gamma \equiv 0$. Then, the following statements are equivalent:
   a) $\Phi \equiv 0$.
   b) $\mu \equiv 0$.
If these conditions hold, then $\Psi + \beta \equiv \eta$.

viii) Assume that $\Theta \equiv \alpha \equiv \gamma \equiv 0$. Then, the following statements are equivalent:
   a) $\Psi \equiv \eta$.
   b) $\beta \equiv 0$.
If these conditions hold, then $\Phi \equiv \mu$.

The following result considers various special cases of Eq. (5) that entail three undetermined angles. These cases are thus consequences of Proposition 3.

*Corollary 5:* The following statements hold:

i) Assume that $\Phi \equiv 0$ and $\Theta \equiv \alpha$. Then, $\Psi, \beta, \eta, \gamma, \mu$ satisfy Eq. (5) if and only if either $\gamma \equiv \mu \equiv \beta + \Psi - \eta \equiv 0$ or $\gamma \equiv \mu \equiv \beta + \Psi - \eta \equiv \pi$.

ii) Assume that $\Phi \equiv 0$ and $\Psi \equiv \eta$. Then, $\Theta, \alpha, \beta, \gamma, \mu$ satisfy Eq. (5) if and only if either $\mu \equiv \beta \equiv \Theta - \alpha - \gamma \equiv 0$ or $\mu \equiv \beta \equiv \Theta - \alpha - \gamma \equiv \pi$.

iii) Assume that $\Phi \equiv 0$, $\Theta \equiv \alpha$, and $\Psi \equiv \eta$. Then, $\beta$, $\gamma$, $\mu$ satisfy Eq. (5) if and only if either $\beta \equiv \gamma \equiv \mu \equiv 0$ or $\beta \equiv \gamma \equiv \mu \equiv \pi$.

iv) Assume that $\Theta \equiv \gamma$ and $\Psi \equiv \eta$. Then, $\Phi, \alpha, \beta, \mu$ satisfy Eq. (5) if and only if either $\Phi - \mu \equiv \alpha \equiv \beta \equiv 0$ or $\Phi - \mu \equiv \alpha \equiv \beta \equiv \pi$.

v) Assume that $\Phi \equiv \Theta + \gamma \equiv \Psi - \eta \equiv 0$. Then, $\alpha$, $\beta$, $\mu$ satisfy Eq. (5) if and only if either $\mu \equiv \alpha \equiv \beta \equiv 0$ or $\mu \equiv \alpha \equiv \beta \equiv \pi$.

vi) Assume that $\Phi \equiv \Theta + \gamma \equiv \beta \equiv 0$. Then, $\Psi$, $\alpha$, $\eta$, $\mu$ satisfy Eq. (5) if and only if either $\mu \equiv \gamma \equiv \Psi \equiv 0$ or $\mu \equiv \gamma \equiv \Psi \equiv \pi$.

vii) Assume that $\Phi \equiv \alpha \equiv \gamma \equiv \Psi \equiv 0$. Then, $\Theta$, $\beta$, $\eta$, $\mu$ satisfy Eq. (5) if and only if either $\mu \equiv \Theta \equiv \beta \equiv 0$ or $\mu \equiv \Theta \equiv \beta \equiv \pi$.

viii) Assume that $\Phi \equiv \Theta \equiv \gamma \equiv \beta \equiv 0$. Then, $\Psi$, $\alpha$, $\eta$, $\mu$ satisfy Eq. (5) if and only if either $\mu \equiv \alpha \equiv \Psi - \eta \equiv 0$ or $\mu \equiv \alpha \equiv \Psi - \eta \equiv \pi$.

ix) Assume that $\Phi \equiv \alpha \equiv \gamma \equiv \beta \equiv 0$. Then, $\Psi$, $\Theta$, $\eta$, $\mu$ satisfy Eq. (5) if and only if either $\mu \equiv \Theta \equiv \Psi - \eta \equiv 0$ or $\mu \equiv \Theta \equiv \Psi - \eta \equiv \pi$.

x) Assume that $\mu \equiv \alpha + \gamma \equiv \Psi \equiv 0$. Then, $\Theta$, $\Phi$, $\beta$, $\eta$ satisfy Eq. (5) if and only if either $\Phi \equiv \Theta \equiv \beta \equiv 0$ or $\Phi \equiv \Theta \equiv \beta \equiv \pi$.

xi) Assume that $\mu \equiv \alpha + \gamma \equiv \beta \equiv 0$. Then, $\Psi, \Theta, \Phi, \eta$ satisfy Eq. (5) if and only if either $\Phi \equiv \Theta \equiv \Psi - \eta \equiv 0$ or $\Phi \equiv \Theta \equiv \Psi - \eta \equiv \pi$.

xii) Assume that $\mu \equiv \Theta \equiv \alpha \equiv \beta \equiv 0$. Then, $\Psi$, $\Phi$, $\eta$, $\gamma$ satisfy Eq. (5) if and only if either $\Phi \equiv \gamma \equiv \Psi - \eta \equiv 0$ or $\Phi \equiv \gamma \equiv \Psi - \eta \equiv \pi$.

xiii) Assume that $\mu \equiv \Theta \equiv \alpha \equiv \Psi - \eta \equiv 0$. Then, $\Phi$, $\beta$, $\gamma$ satisfy Eq. (5) if and only if either $\Phi \equiv \gamma \equiv \beta \equiv 0$ or $\Phi \equiv \gamma \equiv \beta \equiv \pi$.

xiv) Assume that $\mu \equiv \Theta \equiv \gamma \equiv \beta \equiv 0$. Then, $\Psi$, $\Phi$, $\alpha$, $\eta$ satisfy Eq. (5) if and only if either $\Phi \equiv \alpha \equiv \Psi - \eta \equiv 0$ or $\Phi \equiv \alpha \equiv \Psi - \eta \equiv \pi$.

Propositions 3 and 4 can be applied to various special cases of Eq. (5) that entail four undetermined angles. For example, setting either $\Phi = \gamma = 0$, $\Phi = \beta = 0$, $\mu = \gamma = 0$, $\mu = \beta = 0$, $\mu = \Psi - \eta = 0$, $\alpha = \beta = 0$, or $\alpha = \Psi - \eta = 0$ yields a product of four Euler rotations that is amenable to Proposition 4. Likewise, setting either $\Phi = \mu = 0$ or $\beta = \Psi - \eta = 0$ yields a product of four Euler rotations that is amenable to Proposition 5. Two of these nine cases are given by the following two results. In particular, the next result considers wings-level, zero-sideslip flight with possibly nonzero flight path angle.

*Corollary 6:* Assume that $\Phi \equiv \beta \equiv 0$. Then, $\Psi$, $\Theta$, $\alpha$, $\eta$, $\gamma$, $\mu$ satisfy Eq. (5) if and only if i) $\mu \equiv 0$, $\Psi \equiv \eta$, and $\Theta \equiv \alpha + \gamma$, ii) $\mu \equiv \Psi - \eta \equiv \pi$ and $-\gamma \equiv \Theta - \alpha + \pi$, iii) $\gamma \equiv \Theta - \alpha \equiv \pi/2$ and $\mu \equiv \Psi - \eta$, or iv) $\gamma \equiv \Theta - \alpha \equiv -\pi/2$ and $-\mu \equiv \Psi - \eta$. If, in addition, $\gamma \equiv 0$, then $\Psi$, $\Theta$, $\alpha$, $\eta$, $\mu$ satisfy Eq. (5) if and only if i) $\mu \equiv \Psi - \eta \equiv 0$ and $\Theta \equiv \alpha$, ii) $\mu \equiv \Psi - \eta \equiv \pi$ and $\Theta - \alpha \equiv \pi$, iii) $\Theta - \alpha \equiv \pi/2 \equiv 0$ and $\mu \equiv \Psi - \eta$, or iv) $\Theta - \alpha \equiv \pi/2$ and $-\mu \equiv \Psi - \eta$.

*Proof:* The result follows by applying Proposition 4 to

$$\mathcal{O}_2(-\gamma)\mathcal{O}_1(-\mu)\mathcal{O}_2(\Theta - \alpha)\mathcal{O}_3(\Psi - \eta) = I \qquad \square$$

Next, we consider the case of wings-tilted, zero-sideslip flight with possibly nonzero flight path angle.

*Corollary 7:* Assume that $\Psi \equiv \eta$ and $\beta \equiv 0$. Then, $\Theta$, $\alpha$, $\gamma$, $\mu$ satisfy Eq. (5) if and only if either i) $\alpha \equiv 0$, $\Theta \equiv \gamma$ and $\mu \equiv \Phi$, ii) $\mu \equiv \Phi \equiv 0$ and $\alpha \equiv \Theta - \gamma$, iii) $\alpha \equiv \Theta - \gamma \equiv \pi$ and $-\mu \equiv \Phi$, or iv) $\mu \equiv \Phi \equiv \pi$ and $-\alpha \equiv \Theta - \gamma$.

*Proof:* The result follows by applying Proposition 5 to

$$\mathcal{O}_1(-\mu)\mathcal{O}_2(-\alpha)\mathcal{O}_1(\Phi)\mathcal{O}_2(\Theta - \gamma) = I \qquad \square$$

Finally, sufficiency of condition i of the following result coincides with necessity in condition iii of Corollary 5.

*Corollary 8:* Assume that either $\mu \equiv \gamma \equiv \beta \equiv 0$ or $\mu \equiv \gamma \equiv \beta \equiv \pi$. Then, $\Psi$, $\Theta$, $\Phi$, $\alpha$, $\eta$ satisfy Eq. (5) if and only if i) $\Phi \equiv \Psi - \eta \equiv 0$ and $\Theta \equiv \alpha$, ii) $\Phi \equiv \Psi - \eta \equiv \pi$ and $-\alpha \equiv \Theta + \pi$, iii) $\Theta \equiv \alpha \equiv \pi/2$ and $\Phi \equiv \eta - \Psi$, or iv) $\Theta \equiv \alpha \equiv -\pi/2$ and $\Phi \equiv \Psi - \eta$.

*Proof:* The result follows by applying Proposition 4 to

$$\mathcal{O}_2(-\alpha)\mathcal{O}_1(\Phi)\mathcal{O}_2(\Theta)\mathcal{O}_3(\Psi - \eta) = I \qquad \square$$

## VIII. Conclusions

A complete characterization was given of products of two, three, and four Euler rotation matrices equal to the identity matrix. These results were used to characterize feasible values of the body, aerodynamic, and navigation angles relating aircraft frames. For constant Euler angles, these results were used to determine Euler angles for various special cases of straight-line flight. Products of 5, 6, and 7 Euler rotation matrices can be considered to provide a more complete solution of all possible feasible Euler angles. The relationships among these angles are remarkably intricate, and this paper is only a first step toward fully elucidating these relationships. Beyond aircraft dynamics, these results are applicable to the Euler angles that relate spacecraft attitude and orbital frames [3].

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