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Primes in arithmetic progressions — III

Our best unconditional bound for $\psi(x,\chi)$ (cf Theorem 11.16) is not very good, owing to our rather limited knowledge of the zero-free region of $L(s,\chi)$. If we assume GRH, then we have a much better estimate (cf Theorem 13.7). In some situations, a good bound for an average of $|\psi(x,\chi)|$ is all that is required, and such bounds can be obtained by combining Vaughan's identity with the large sieve.

1. Averages of $|\psi(x,\chi)|$

Let $\sum^*_{\chi\,(\mathrm{mod}\,q)}$ denote a sum over the primitive characters modulo q. When the conductor q may be inferred from the context, we write simply $\sum^*_\chi$. In this notation, we have

Theorem 20.1 *For arbitrary $Q \ge 1$ and $x \ge 2$,*

$$\sum_{q\le Q} \frac{q}{\varphi(q)} \sum_\chi^* \max_{y\le x} |\psi(y,\chi)| \ \ll\ \big(x + x^{5/6}Q + x^{1/2}Q^2\big)(\log x)^3\,. \tag{20.1}$$

The term $q = 1$ contributes an amount $\sim x$, but otherwise we expect that $|\psi(y,\chi)|$ is usually of the size $x^{1/2}(\log q)^{1/2}$. Thus we expect that the left hand side above is somewhere between $x + Q^2x^{1/2}(\log Q)^{1/2}$ and $x + Q^2x^{1/2}(\log Qx)^2$ in size.

Proof By Cauchy's inequality and the large sieve (Theorem 19.14), we see that

$$\begin{aligned}
&\sum_{q\le Q} \frac{q}{\varphi(q)} \sum_\chi^* \Big| \sum_{m=1}^M \sum_{n=1}^N a_m b_n \chi(mn)\Big| \\
&\qquad \le \Big(\sum_{q\le Q} \frac{q}{\varphi(q)} \sum_\chi^* \Big| \sum_{m=1}^M a_m\chi(m)\Big|^2\Big)^{1/2} \Big(\sum_{q\le Q} \frac{q}{\varphi(q)} \sum_\chi^* \Big| \sum_{n=1}^N b_n\chi(n)\Big|^2\Big)^{1/2} \\
&\qquad \ll \big(M+Q^2\big)^{1/2}\big(N+Q^2\big)^{1/2}\Big(\sum_{m=1}^M |a_m|^2\Big)^{1/2}\Big(\sum_{n=1}^N |b_n|^2\Big)^{1/2}.
\end{aligned} \tag{20.2}$$

In order to truncate this to $mn \le y$, we use the device discussed in Appendix E.2. Specifically, by (E.17) we find that

$$\max_y \Big| \sum_{\substack{1\le m\le M\\ 1\le n\le N\\ mn\le y}} a_m b_n \chi(mn)\Big| \ll \int_{-T}^{T} \Big| \sum_{\substack{1\le m\le M\\ 1\le n\le N}} a_m b_n \chi(mn)(mn)^{-it}\Big| \min(\log MN, 1/|t|)\, dt + \frac{MN}{T} \sum_{\substack{1\le m\le M\\ 1\le n\le N}} |a_m b_n|\,.$$

By Caucy's inequality, the last term is

$$\ll \frac{M^{3/2}N^{3/2}}{T}\Big(\sum_{m=1}^{M} |a_m|^2\Big)^{1/2}\Big(\sum_{n=1}^{N} |b_n|^2\Big)^{1/2}.$$

In order that this term should not be troublesome, we take $T = (MN)^{3/2}$. Since

$$\int_{-T}^{T} \min(\log MN, 1/|t|)\, dt \ll \log(T\log 2MN),$$

it follows by (20.2) that

$$\begin{aligned}\sum_{q\le Q} \frac{q}{\varphi(q)} \sideset{}{^*}\sum_{\chi} \max_y \Big| \sum_{\substack{1\le m\le M\\ 1\le n\le N\\ mn\le y}} a_m b_n \chi(mn)\Big| \\ \ll \big(M+Q^2\big)^{1/2}\big(N+Q^2\big)^{1/2}\Big(\sum_{m=1}^{M} |a_m|^2\Big)^{1/2}\Big(\sum_{n=1}^{N} |b_n|^2\Big)^{1/2} \log 2MN\,.\end{aligned} \tag{20.3}$$

If $Q^2 > x$, then we obtain (20.1) from (20.3) by taking $M = 1$, $a_1 = 1$, $N = [x]$, and $b_n = \Lambda(n)$. Suppose that $Q \le x^{1/2}$. By Vaughan's identity (18.3) we find that $\psi(y,\chi) = S_1 + S_2 + S_3 + S_4$ where

$$S_1(y,\chi) = \sum_{n\le U} \Lambda(n)\chi(n), \tag{20.4}$$

$$S_2(y,\chi) = \sum_{t\le UV} b(t) \sum_{r\le y/t} \chi(rt) \tag{20.5}$$

where $|b(t)| \le \log UV$,

$$S_3(y,\chi) \ll (\log y) \sum_{d\le V} \max_w \Big| \sum_{w\le h\le y/d} \chi(h)\Big|, \tag{20.6}$$

and

$$S_4(y,\chi) = \sum_{U<m\le y/V} \mu(m) \sum_{V<k\le y/m} c(k)\chi(mk) \tag{20.7}$$

where $c(k) \ll \log k$. Thus by (20.3),

$$\sum_{q\le Q} \frac{q}{\varphi(q)} \sideset{}{^*}\sum_{\chi} \max_{y\le x} \Big| \sum_{\substack{M<m\le 2M\\ U<m\le y/V}} \mu(m) \sum_{V<k\le y/m} c(k)\chi(mk)\Big|$$
$$\ll \big(x + QxM^{-1/2} + Qx^{1/2}M^{1/2} + Q^2x^{1/2}\big)(\log x)^2\,.$$

On summing this over $M = 2^\ell$ for $U/2 \le M = 2^\ell \le x/V$, we deduce that

$$\sum_{q\le Q} \frac{q}{\varphi(q)} \sideset{}{^*}\sum_{\chi} \max_{y\le x} |S_4(y,\chi)| \ll \big(x + QxU^{-1/2} + QxV^{-1/2} + Q^2x^{1/2}\big)(\log x)^3\,. \tag{20.8}$$

We write

$$S_2(y,\chi) = \sum_{t\le UV} = \sum_{t\le U} + \sum_{U<t\le UV} = S_2'(y,\chi) + S_2''(y,\chi), \tag{20.9}$$

and treat S_2'' in the same way that we treated S_4. Thus

$$\sum_{q\le Q} \frac{q}{\varphi(q)} \sideset{}{^*}\sum_{\chi} \max_{y\le x} |S_2''(y,\chi)| \ll \big(x + QxU^{-1/2} + Qx^{1/2}U^{1/2}V^{1/2} + Q^2x^{1/2}\big)(\log x)^3\,. \tag{20.10}$$

For $q = 1$, $S_2'(y,\chi) \ll y(\log U)^2$. For $q > 1$ we apply the Pólya–Vinogradov inequality (Theorem 9.18) to see that $S_2'(y,\chi) \ll q^{1/2}U(\log qU)^2$. Hence

$$\sum_{q\le Q} \frac{q}{\varphi(q)} \sideset{}{^*}\sum_{\chi} \max_{y\le x} |S_2'(y,\chi)| \ll \big(x + Q^{5/2}U\big)(\log Ux)^2\,. \tag{20.11}$$

We treat S_3 in the same way that we treated S_2', and hence find that

$$\sum_{q\le Q} \frac{q}{\varphi(q)} \sideset{}{^*}\sum_{\chi} \max_{y\le x} |S_3(y,\chi)| \ll \big(x + Q^{5/2}V\big)(\log Vx)^2\,. \tag{20.12}$$

Finally, it is trivial that

$$\sum_{q\le Q} \frac{q}{\varphi(q)} \sideset{}{^*}\sum_{\chi} \max_{y\le x} |S_1(y,\chi)| \ll Q^2U\,. \tag{20.13}$$

On combining (20.8)–(20.13), we conclude that

$$\sum_{q\le Q} \frac{q}{\varphi(q)} \sideset{}{^*}\sum_{\chi} \max_{y\le x} |\psi(y,\chi)| \ll \big(x + QxU^{-1/2} + QxV^{-1/2} + Q^2x^{1/2}$$
$$+ U^{1/2}V^{1/2}Qx^{1/2} + Q^{5/2}U + Q^{5/2}V\big)(\log xUV)^3\,.$$

By allowing U and V to vary with UV held constant, we see that $U = V$ is optimal. For $x^{1/3} \le Q \le x^{1/2}$, we obtain the stated bound by taking $U = V = x^{2/3}/Q$. For $1 \le Q \le x^{1/3}$, we obtain the stated bound by taking $U = V = x^{1/3}$. □

20.1.1 Exercises

1. Let $\pi(x,\chi)$, $\pi(x;q,a)$, $\vartheta(x,\chi)$, and $\vartheta(x;q,a)$ be defined as in (11.20) and (11.21).
   (a) Show that $|\psi(x,\chi)-\vartheta(x,\chi)| \le \psi(x)-\vartheta(x) \ll x^{1/2}$.
   (b) Show that

$$\sum_{q\le Q}\frac{q}{\varphi(q)}\sideset{}{^*}\sum_{\chi}\max_{y\le x}|\vartheta(y,\chi)| \ll \big(x+x^{5/6}Q+x^{1/2}Q^2\big)(\log x)^4\,.$$

   (c) Show that

$$\pi(x,\chi)=\frac{\psi(x,\chi)}{\log x}+\int_2^x\frac{\psi(u,\chi)}{u(\log u)^2}\,du\,.$$

   (d) Show that

$$\pi(x,\chi)\ll\frac{1}{\log x}\max_{x^{1/2}\le y\le x}|\psi(y,\chi)|+x^{1/2}\,.$$

   (e) Show that

$$\max_{y\le x}|\pi(y,\chi)|\ll\frac{1}{\log x}\max_{y\le x}|\psi(y,\chi)|+x^{1/2}\,.$$

   (f) Conclude that if $x\ge 2$, then

$$\sum_{q\le Q}\frac{q}{\varphi(q)}\sideset{}{^*}\sum_{\chi}\max_{y\le x}|\pi(y,\chi)| \ll \big(x+x^{5/6}Q+x^{1/2}Q^2\big)(\log x)^3\,.$$

2. Show that

$$\sum_{\chi}\Big|\sum_{n=M+1}^{M+N}c_n\chi(n)\Big|^2=\varphi(q)\sum_{h=1}^{q}\Big|\sum_{\substack{n=M+1\\ n\equiv h\,(\mathrm{mod}\,q)}}^{M+N}c_n\Big|^2$$

   where $\sum_\chi$ indicates a sum over all characters modulo q.

3. Show that

$$\sum_{\chi}\Big|\sum_{n=M+1}^{M+N}c_n\chi(n)\Big|^2\le(N+q)\sum_{n=M+1}^{M+N}|c_n|^2\,.$$

4. Show that

$$\sum_{\chi}\Big|\sum_{m=1}^{M}\sum_{n=1}^{N}a_mb_n\chi(mn)\Big|\le(M+q)^{1/2}(N+q)^{1/2}\Big(\sum_{m=1}^{M}|a_m|^2\Big)^{1/2}\Big(\sum_{n=1}^{N}|b_n|^2\Big)^{1/2}\,.$$

5. Show that

$$\sum_{\chi}\max_{y}\Big|\sum_{\substack{1\le m\le M\\ 1\le n\le N\\ mn\le y}} a_m b_n \chi(mn)\Big|$$
$$\ll (M+q)^{1/2}(N+q)^{1/2}\Big(\sum_{m=1}^{M}|a_m|^2\Big)^{1/2}\Big(\sum_{n=1}^{N}|b_n|^2\Big)^{1/2}\log 2MN\,.$$

6. Show that if $q \ge x$, then

$$\sum_{\chi}\max_{y\le x}|\psi(y,\chi)| \ll qx^{1/2}(\log 2x)^{3/2}\,.$$

7. (a) Show that

$$\sum_{\chi}\max_{y\le x}\Big|\sum_{\substack{M<m\le 2M\\ U<m\le y/V}}\mu(m)\sum_{V<k\le y/m}c(k)\chi(mk)\Big|$$
$$\ll \big(x+q^{1/2}xM^{-1/2}+q^{1/2}x^{1/2}M^{1/2}+qx^{1/2}\big)(\log x)^2\,.$$

(b) Deduce that

$$\sum_{\chi}\max_{y\le x}|S_4(y,\chi)| \ll \big(x+q^{1/2}xU^{-1/2}+q^{1/2}xV^{-1/2}+qx^{1/2}\big)(\log x)^3\,.$$

(c) Let S_2' and S_2'' be defined as in (20.9). Show that

$$\sum_{\chi}\max_{y\le x}|S_2''(y,\chi)| \ll \big(x+q^{1/2}xU^{-1/2}+q^{1/2}x^{1/2}U^{1/2}V^{1/2}+qx^{1/2}\big)(\log x)^3\,.$$

(d) Show that if $1 < q \le x$, then

$$\sum_{\chi}\max_{y\le x}|S_2'(y,\chi)| \ll q^{3/2}U(\log xU)^2\,.$$

(e) Show that if $1 < q \le x$, then

$$\sum_{\chi}\max_{y\le x}|S_3(y,\chi)| \ll q^{3/2}V(\log x)^2\,.$$

(f) Conclude that if $x \ge 2$ and $q > 1$, then

$$\sum_{\chi}\max_{y\le x}|\psi(y,\chi)| \ll \big(x+q^{1/6}x^{2/3}+qx^{1/2}\big)(\log 2x)^3\,.$$

2. Average distribution of primes

For $(a,q)=1$, let

$$E(x;q,a)=\psi(x;q,a)-\frac{x}{\varphi(q)}, \tag{20.14}$$

put

$$E(x,q)=\max_{\substack{a\\(a,q)=1}}|E(x;q,a)|, \tag{20.15}$$

and set

$$E^*(x,q)=\max_{y\le x}E(y,q)\,. \tag{20.16}$$

We show that $E^*(x,q)$ is considerably smaller than $x/\varphi(q)$ for most $q\le x^{1/2}(\log x)^{-A}$.

Theorem 20.2 (The Bombieri–Vinogradov theorem) *Let $A>0$ be fixed. Then*

$$\sum_{q\le Q}E^*(x,q)\ll x^{1/2}Q(\log x)^3 \tag{20.17}$$

for $x^{1/2}(\log x)^{-A}\le Q\le x^{1/2}$.

The implicit constant in (20.17) is non-effective, since our proof will involve an appeal to the Siegel–Walfisz theorem.

Let $\mathfrak{Q}$ be the set of those $q\le Q$ for which $E^*(x,q)>x/(\varphi(q)(\log x)^B)$. Since $\varphi(q)\le q$, we deduce that the number of members of $\mathfrak{Q}$ is

$$\ll Q^2x^{-1/2}(\log x)^{B+4}\,.$$

This is small compared with Q if $x^{1/2}(\log x)^{-A}\le Q$ and $Q=o\big(x^{1/2}(\log x)^{-B-4}\big)$.

We recall (11.22), which is to say that

$$\psi(x;q,a)=\frac{1}{\varphi(q)}\sum_{\chi}\overline{\chi}(a)\psi(x,\chi)\,.$$

If we assume GRH, then we have a good estimate for $\psi(x,\chi)$, namely by (13.19),

$$\psi(x,\chi)=E_0(\chi)x+O\big(x^{1/2}(\log x)(\log qx)\big)$$

where

$$E_0(\chi)=\begin{cases}1 & (\chi=\chi_0),\\ 0 & (\text{otherwise}).\end{cases}$$

Put $\psi'(x,\chi) = \psi(x,\chi) - E_0(\chi)x$. Then

$$\psi(x;q,a) - x/\varphi(q) = \frac{1}{\varphi(q)}\sum_{\chi}\overline{\chi}(a)\psi'(x,\chi), \tag{20.17}$$

and so

$$E(x,q) \le \frac{1}{\varphi(q)}\sum_{\chi}|\psi'(x,\chi)| \tag{20.18}$$

by the triangle inequality. Thus on GRH,

$$E(x,q) \ll x^{1/2}(\log x)^2, \tag{20.19}$$

as was already noted in Corollary 13.8. In view of the Brun–Titchmarsh inequality (Theorem 3.9) we know that $E(x,q) \ll x/\varphi(q)$ for $q \le x^{1-\delta}$. Thus the estimate (20.19)—despite being a consequence of GRH—is worse than trivial when $q > x^{1/2}/\log x$. Here GRH is giving a weak result (when q is large) because we have thrown away any possible cancellation that might be presumed to occur in the sum over χ in (20.17). Indeed, by Corollary 13.10 we know that on GRH the root mean square size of $E(x;q,a)$ is $\ll x^{1/2}\varphi(q)^{-1/2}(\log x)^2$ when $q \le x$, and we expect that $E(x,q)$ is not much larger.

Conjecture 20.3 *If $(a,q)=1$ and $q \le x$, then*

$$\psi(x;q,a) = \frac{x}{\varphi(q)} + O\big(x^{1/2+\varepsilon}/q^{1/2}\big).$$

For many purposes, it would be enough to know this on average:

Conjecture 20.4 (The Elliott–Halberstam Hypothesis) *Let $A > 0$ and $\varepsilon > 0$ be fixed. In the notation of* (20.16),

$$\sum_{q\le Q} E^*(x,q) \ll x(\log x)^{-A}$$

provided that $Q \le x^{1-\varepsilon}$.

Proof of Theorem 20.2 From (20.18) we see that

$$E^*(x,q) \le \frac{1}{\varphi(q)}\sum_{\chi}\max_{y\le x}|\psi'(y,\chi)|.$$

Suppose that the character $\chi \pmod q$ is induced by the primitive character $\chi^\star \pmod d$. Then

$$\psi'(y,\chi^\star) - \psi'(y,\chi) \;=\; \sum_{p\mid q}\sum_{\substack{k\\ p^k\le y}}\chi^\star(p)^k\log p \;\ll\; \sum_{p\mid q}\log y \;=\; \omega(q)\log y \;\ll\; (\log qy)^2.$$

Hence

$$\sum_{q\le Q} E^*(x,q) \ll \sum_{d\le Q}\sum_{\chi^\star}\Big(\max_{y\le x}|\psi'(y,\chi^\star)| + O\big((\log Qx)^2\big)\Big)\sum_{\substack{q\le Q\\ d\mid q}}\frac{1}{\varphi(q)}\,.$$

Write $q = dm$. But $\varphi(dm)\ge\varphi(d)\varphi(m)$, so

$$\sum_{m\le Q/d}\frac{1}{\varphi(dm)} \le \frac{1}{\varphi(d)}\sum_{m\le Q/d}\frac{1}{\varphi(m)}\,.$$

Now

$$\begin{aligned}\sum_{m\le y}\frac{1}{\varphi(m)} &\le \sum_{\substack{m\\ p\mid m\Rightarrow p\le y}}\frac{1}{\varphi(m)} = \prod_{p\le y}\Big(1+\frac{1}{p-1}+\frac{1}{p(p-1)}+\cdots\Big)\\ &= \prod_{p\le y}\Big(1+\frac{p}{(p-1)^2}\Big) = \prod_{p\le y}\Big(1-\frac{1}{p}\Big)^{-1}\Big(1+\frac{1}{p(p-1)}\Big)\\ &\ll \log 2y\end{aligned}$$

by Mertens' formula (Theorem 2.7(e)). (Alternatively, we could appeal to (2.32) with $\kappa = 1$, and then integrate by parts. The asymptotic formula of Lemma 20.9 would be overkill at this point.) Hence

$$\sum_{q\le Q} E^*(x,q) \ll Q(\log Qx)^2 + \sum_{q\le Q}\frac{\log 2Q/q}{\varphi(q)}\sum_{\chi}^{*}\max_{y\le x}|\psi'(y,\chi)|\,. \tag{20.20}$$

Put $Q_1 = (\log x)^{A+1}$, and suppose that $Q_1\le U\le Q$. By Theorem 20.1 we see that

$$\begin{aligned}\sum_{U<q\le 2U}\frac{\log 2Q/q}{\varphi(q)}\sum_{\chi}^{*}\max_{y\le x}|\psi'(y,\chi)| &\ll \frac{\log 4Q/U}{U}\sum_{U<q\le 2U}\frac{q}{\varphi(q)}\sum_{\chi}^{*}\max_{y\le x}|\psi(y,\chi)|\\ &\ll \big(x/U + x^{5/6} + x^{1/2}U\big)(\log x)^3\log 4Q/U\,.\end{aligned}$$

On summing over $U = 2^kQ_1$, we deduce that

$$\begin{aligned}\sum_{Q_1<q\le Q}\frac{\log 2Q/q}{\varphi(q)}\sum_{\chi}^{*}\max_{y\le x}|\psi'(y,\chi)| &\ll xQ_1^{-1}(\log x)^4 + x^{5/6}(\log x)^5 + x^{1/2}Q(\log x)^3\\ &\ll x^{1/2}Q(\log x)^3 \qquad (20.21)\end{aligned}$$

Since $Q\ge x^{1/2}(\log x)^{-A}$. Suppose that χ is a primitive character modulo q with $q\le Q_1$. By the Siegel–Walfisz theorem (Corollary 11.18) we know that $\max_{y\le x}|\psi'(y,\chi)| \ll x\exp\big(-c_1\sqrt{\log x}\big)$. Hence

$$\sum_{q\le Q_1}\frac{\log 2Q/q}{\varphi(q)}\sum_{\chi}^{*}\max_{y\le x}|\psi'(y,\chi)| \ll x\exp\big(-c_2\sqrt{\log x}\big) \ll x^{1/2}Q(\log x)^3$$

since $Q\ge x^{1/2}(\log x)^{-A}$. We combine this with (20.21) in (20.20) to obtain the desired bound. □

Sometimes we want to work only with primes.

Corollary 20.5 *For $(a,q)=1$ let*

$$E_\vartheta(x;q,a)=\vartheta(x;q,a)-\frac{x}{\varphi(q)},\ \ E_\vartheta(x,q)=\max_{\substack{a\\(a,q)=1}}|E_\vartheta(x;q,a)|,\ \ E_\vartheta^*(x,q)=\max_{y\le x}E(y,q),$$

$$E_\pi(x;q,a)=\pi(x;q,a)-\frac{\mathrm{li}(x)}{\varphi(q)},\ \ E_\pi(x,q)=\max_{\substack{a\\(a,q)=1}}|E_\pi(x;q,a)|,\ \ E_\pi^*(x,q)=\max_{y\le x}E(y,q),$$

and let $A>0$ be fixed. Then

$$\sum_{q\le Q}E_\vartheta^*(x,q)\ \ll\ x^{1/2}Q(\log x)^3 \tag{20.22}$$

and

$$\sum_{q\le Q}E_\pi^*(x,q)\ \ll\ x^{1/2}Q(\log x)^2 \tag{20.23}$$

provided that $x^{1/2}(\log x)^{-A}\le Q\le x^{1/2}$.

Proof We argue as in the proof of Corollary 11.20. We first observe that

$$\psi(y;q,a)-\vartheta(y;q,a)\le\psi(y)-\vartheta(y)\ll y^{1/2}\,.$$

Hence

$$|\vartheta(y;q,a)-y/\varphi(q)|\ll E(y;q,a)+y^{1/2}\,.$$

Thus

$$E_\vartheta^*(x,q)\ll E^*(x,q)+x^{1/2},$$

so (20.22) follows from Theorem 20.2. As for $\pi(y;q,a)$, we write

$$\pi(y;q,a)=\int_{2^-}^y\frac{1}{\log u}\,d\vartheta(u;q,a)=\frac{\mathrm{li}(y)}{\varphi(q)}+\int_{2^-}^y\frac{1}{\log u}\,d(\vartheta(u;q,a)-u/\varphi(u))\,.$$

By partial integration this last integral is

$$=\frac{\vartheta(u;q,a)-u/\varphi(q)}{\log u}\Big|_{2^-}^y-\int_2^y\frac{\vartheta(u;q,a)-u/\varphi(q)}{u(\log u)^2}\,du\,.$$

For $2\le u\le\sqrt{x}$ we use the trivial bound $\vartheta(u;q,a)\ll u(\log u)/q$, and for $\sqrt{x}\le u\le y$ we use the inequality $|E_\vartheta(u;q,a)|\le E_\vartheta^*(y,q)$. Thus

$$E_\pi^*(x;q,a)\ll x^{1/2}/q+E_\vartheta^*(x;q)/\log x\,.$$

Hence (20.23) follows from (20.22), and the proof is complete. □

The following variant of the Bombieri–Vinogradov theorem is convenient in some applications.

Corollary 20.6 *Let $A > 0$ be fixed. In the notation of* (20.16) *and of Corollary* 20.5*,*

$$\sum_{q\le Q} qE^*(x,q)^2 \ \ll\ x^{3/2}Q(\log x)^4, \tag{20.24}$$

$$\sum_{q\le Q} qE^*_\vartheta(x,q)^2 \ \ll\ x^{3/2}Q(\log x)^4, \tag{20.25}$$

and

$$\sum_{q\le Q} qE^*_\pi(x,q)^2 \ \ll\ x^{3/2}Q(\log x)^2 \tag{20.26}$$

provided that $x^{1/2}(\log x)^{-A} \le Q \le x^{1/2}$.

Proof If $q \le x$, then there are $\ll x/q$ integers $n \le x$ such that $n \equiv a \pmod q$. Thus it is trivial that $\psi(x;q,a) \ll x(\log x)/q$. (The Brun–Titchmarsh inequality gives a better bound.) Hence $qE^*(x,q)^2 \ll E^*(x,q)x\log x$, and so (20.24) follows from Theorem 20.2. Similarly, (20.25) follows from (20.22). For $\pi(x;q,a)$ the trivial bound is $\pi(x;q,a) \ll x/q$, so $qE^*_\pi(x,q)^2 \ll E^*_\pi(x,q)x$, and thus (20.26) follows from (20.23). □

The twin prime problem is to show that there are infinitely many prime numbers p such that $p+2$ is also prime. One way of attacking this problem would be to sieve the numbers $p+2$, and try to estimate the number of survivors. However, in order for a sieve to be applicable, we must know approximately how many multiples of d are in the set $\{p+2 : p \le x\}$. That is, we need to know that $\pi(x;d,-2)$ is approximately $\mathrm{li}(x)/\varphi(d)$ for most odd d up to a certain size. The Bombieri–Vinogradov theorem gives us precisely the sort of information we need for sifting up to $x^{1/2}(\log x)^{-A}$. By Selberg's lambda squared method we can show that the number of primes $p \le x$ for which $p+2$ is prime is $\le (4+o(1))cx/(\log x)^2$ where

$$c = 2\prod_{p>2}\Big(1-\frac{1}{(p-1)^2}\Big). \tag{20.27}$$

(The details are outlined in Exercises 20.2.1–3.) This is a factor of 2 better than the bound in Theorem 3.10, but is still a factor 4 larger than the conjectured truth. The Bombieri–Vinogradov theorem is also useful when a lower bound sieve is applied to the twin prime problem. This will be explored in Chapter 28.

To prepare for our next result, we establish some elementary asymptotics.

Lemma 20.7 *For any real number $x > 0$ and any positive integer q,*

$$\sum_{\substack{m\le x\\(m,q)=1}}\frac{1}{m} = \Big(\log x + C_0 + \sum_{p\mid q}\frac{\log p}{p-1}\Big)\frac{\varphi(q)}{q} + O\big(2^{\omega(q)}/x\big).$$

Proof The sum in question is

$$= \sum_{m\le x}\frac{1}{m}\sum_{\substack{d\mid m\\ d\mid q}}\mu(d) = \sum_{d\mid q}\frac{\mu(d)}{d}\sum_{r\le x/d}\frac{1}{r}. \tag{20.28}$$

The asymptotic estimate

$$\sum_{r\le y}\frac{1}{r} = \log y + C_0 + O(1/y)$$

is familiar for $y \ge 1$ (cf Corollary 1.15), but actually holds uniformly for $y > 0$, although it is rather weak when y is small. Thus the right hand side of (20.28) is

$$= \sum_{d|q}\frac{\mu(d)}{d}\big(\log x/d + C_0 + O(d/x)\big) = (\log x + C_0)\varphi(q)/q - \sum_{d|q}\frac{\mu(d)\log d}{d} + O\big(2^{\omega(q)}/x\big).$$

We observe that

$$\begin{aligned}\sum_{d|q}\frac{\mu(d)\log d}{d} &= \sum_{d|q}\frac{\mu(d)}{d}\sum_{p|d}\log p = \sum_{p|q}\log p\sum_{\substack{d\\ p|d|q}}\frac{\mu(d)}{d}\\ &= -\sum_{p|q}\frac{\log p}{p}\sum_{\substack{r|q\\ p\nmid r}}\frac{\mu(r)}{r} = -\frac{\varphi(q)}{q}\sum_{p|q}\frac{\log p}{p-1},\end{aligned}$$

so the proof is complete. □

Lemma 20.8 *Let $f(q) = \sum_{p|q}(\log p)/(p-1)$. Then $f(q) \ll \log\log 3q$ uniformly for $q \ge 1$.*

Proof Let $1 = q_0 < q_1 < \cdots$ be the sequence of those q for which $f(q)$ is larger than for any smaller q. Since $(\log p)/(p-1)$ is monotinically decreasing, it follows that q_r is the product of the first r primes, $q_r = \prod_{p\le p_r} p$. Thus $f(q_r) \ll \log p_r$ by Theorem 2.7(b). But $\log q_r \asymp p_r$ by Chebyshev's estimates, so $f(q_r) \ll \log\log 3q_r$. Since $f(q) \le f(q_r)$ for $q_r \le q < q_{r+1}$ we have the desired bound for all q. □

Lemma 20.9 *For any $x \ge 2$ and any positive integer q,*

$$\begin{aligned}\sum_{\substack{n\le x\\ (n,q)=1}}\frac{1}{\varphi(q)} &= \frac{\zeta(2)\zeta(3)}{\zeta(6)}\prod_{p|q}\Big(1-\frac{p}{p^2-p+1}\Big)\Big(\log x + C_0 + \sum_{p|q}\frac{\log p}{p-1} - \sum_{p\nmid q}\frac{\log p}{p^2-p+1}\Big)\\ &\quad + O\big(2^{\omega(q)}(\log x)/x\big).\end{aligned}$$

Proof We note that $n/\varphi(n) = \sum_{d|n}\mu(d)^2/\varphi(d)$. Thus the sum in question is

$$= \sum_{\substack{n\le x\\ (n,q)=1}}\frac{1}{n}\sum_{d|n}\frac{\mu(d)^2}{\varphi(d)} = \sum_{\substack{d=1\\ (d,q)=1}}^{\infty}\frac{\mu(d)^2}{d\varphi(d)}\sum_{\substack{m\le x/d\\ (m,q)=1}}\frac{1}{m}.$$

By Lemma 20.7 this is

$$= \sum_{\substack{d=1\\ (d,q)=1}}^{\infty}\frac{\mu(d)^2}{d\varphi(d)}\Big(\log x/d + C_0 + \sum_{p|q}\frac{\log p}{p-1}\Big)\frac{\varphi(q)}{q} + O\Big(\frac{2^{\omega(q)}}{x}\sum_{d\le x}\frac{\mu(d)^2}{\varphi(q)}\Big). \tag{20.29}$$

Here the sum in the error term is

$$\leq \sum_{\substack{d\\ p\mid d\Rightarrow p\leq x}} \frac{\mu(d)^2}{\varphi(d)} = \prod_{p\leq x}\Big(1+\frac{1}{p-1}\Big) \ll \log x\,.$$

We observe that

$$\begin{aligned}\frac{\varphi(q)}{q}\sum_{\substack{d=1\\(d,q)=1}}^{\infty}\frac{\mu(d)^2}{d\varphi(d)} &= \prod_{p\mid q}\Big(1-\frac{1}{p}\Big)\prod_{p\nmid q}\Big(1+\frac{1}{p(p-1)}\Big)\\ &= \prod_{p\mid q}\Big(1-\frac{p}{p^2-p+1}\Big)\frac{\zeta(2)\zeta(3)}{\zeta(6)}\,. \qquad (20.30)\end{aligned}$$

Finally,

$$\sum_{\substack{d=1\\(d,q)=1}}^{\infty}\frac{\mu(d)^2\log d}{d\varphi(d)} = \sum_{\substack{d=1\\(d,q)=1}}^{\infty}\frac{\mu(d)^2}{d\varphi(d)}\sum_{p\mid d}\log p = -\sum_{p\nmid q}\frac{\log p}{p(p-1)}\sum_{\substack{r=1\\(r,pq)=1}}^{\infty}\frac{\mu(r)^2}{r\varphi(r)}\,.$$

By (20.30) with q replaced by pq, we deduce that

$$\begin{aligned}\frac{\varphi(q)}{q}\sum_{\substack{d=1\\(d,q)=1}}^{\infty}\frac{\mu(d)^2\log d}{d\varphi(d)} &= -\frac{\zeta(2)\zeta(3)}{\zeta(6)}\sum_{p\nmid q}\frac{\log p}{(p-1)^2}\prod_{p_1\mid pq}\frac{(p_1-1)^2}{p_1^2-p_1+1}\\ &= -\frac{\zeta(2)\zeta(3)}{\zeta(6)}\prod_{p\mid q}\Big(1-\frac{p}{p^2-p+1}\Big)\sum_{p\nmid q}\frac{\log p}{p^2-p+1}\,.\end{aligned}$$

We combine these results in (20.29) to complete the proof. □

Theorem 20.10 *The number of representations of a positive integer n as a sum of a prime and the product of two positive integers is*

$$\sum_{p<n} d(n-p) = \frac{\zeta(2)\zeta(3)}{\zeta(6)}\prod_{p\mid n}\Big(1-\frac{p}{p^2-p+1}\Big)n + O\Big(\frac{n\log\log 3n}{\log n}\Big)\,.$$

Here the main term is $\gg n/\log\log 3n$, so the main term is definitely of a larger order of magnitude than the error term.

Proof Let $\mathcal{P}$ denote the set of prime numbers, and put $Q = n^{1/2}(\log n)^{-A}$. Then by the method of the hyperbola,

$$\begin{aligned}\sum_{p<n} d(n-p) &= \sum_{\substack{d,e\\ de\leq n\\ n-de\in\mathcal{P}}} 1 = \sum_{d\leq Q}\pi(n;d,n) + \sum_{Q<d\leq n/Q}\pi(n;d,n) + \sum_{e\leq Q}\pi(n;e,n) - \sum_{\substack{d\leq n/Q\\ e\leq Q\\ n-de\in\mathcal{P}}} 1\\ &= \Sigma_1+\Sigma_2+\Sigma_3-\Sigma_4, \qquad (20.31)\end{aligned}$$

say. If $(d,n)>1$, then $\pi(n;d,n)\le 1$. Thus

$$\Sigma_1=\sum_{\substack{d\le Q\\(d,n)=1}}\pi(n;d,n)+O(Q)=\mathrm{li}(n)\sum_{\substack{d\le Q\\(d,n)=1}}\frac{1}{\varphi(d)}+\sum_{\substack{d\le Q\\(d,n)=1}}E_\pi(n;d,n)+O(Q)\,.$$

By Lemma 20.8, Lemma 20.9, and Corollary 20.5 we deduce that

$$\Sigma_1=\frac{\zeta(2)\zeta(3)}{\zeta(6)}\prod_{p\mid n}\Big(1-\frac{p}{p^2-p+1}\Big)n\frac{\log Q}{\log n}+O\Big(\frac{n\log\log n}{\log n}\Big)+O\big(n(\log n)^{-A+2}\big)\,. \tag{20.32}$$

By the Brun–Titchmarsh inequality (Theorem 3.9),

$$\Sigma_2\ll\frac{n}{\log n}\sum_{Q<d\le n/Q}\frac{1}{\varphi(d)}\ll\frac{n\log\log n}{\log n}\,. \tag{20.33}$$

Clearly $\Sigma_3=\Sigma_1$. We note that

$$\Sigma_4=\sum_{e\le Q}\sum_{\substack{n-ne/Q\le p\le n\\ p\equiv n\pmod e}}1\,.$$

Thus by the Brun–Titchmarsh inequality,

$$\Sigma_4\ll\frac{n}{Q\log n}\sum_{e\le Q}\frac{e}{\varphi(e)}\ll\frac{n}{\log n}\,. \tag{20.34}$$

We take $A=3$, and note that $\log Q=\frac12\log n+O(\log\log n)$. Thus the stated result follows on combining (20.32)–(20.34) in (20.31). □

20.2.1 Exercises

1. Let $f(n)$ and $g(n)$ be multiplicative functions defined as follows:

$$f(n)=\prod_{p^\alpha\|n}\frac{1}{(p-1)^\alpha},\qquad g(n)=\prod_{p^\alpha\|n}\frac{p^{\alpha-1}}{(p-1)^\alpha}\,.$$

   (a) Show that $nf(n)=\sum_{d\mid n}g(n)$
   (b) Show that

$$\sum_{\substack{n\le z\\2\nmid n}}f(n)=\sum_{\substack{d\le z\\2\nmid d}}\frac{g(d)}{d}\sum_{\substack{m\le z/d\\2\nmid m}}\frac{1}{m}\,.$$

(c) Show that

$$\sum_{\substack{m\le w\\ 2\nmid m}} \frac{1}{m} = \frac{1}{2}\log w + C_1 + O(1/w)$$

where $C_1 = (C_0 + \log 2)/2$.
(d) Show that

$$\sum_{\substack{d=1\\ 2\nmid d}}^{\infty} \frac{g(d)}{d} = \frac{2}{c}$$

where c is defined as in (20.27).
(e) Show that

$$\sum_{\substack{n\le z\\ 2\nmid n}} f(n) = \frac{\log z}{c} + C_2 + O((\log z)/z)$$

where

$$C_2 = \frac{C_0 + \log 2}{c} - \frac{1}{2}\sum_{\substack{d=1\\ 2\nmid d}}^{\infty} \frac{g(d)\log d}{d}\,.$$

2. Let

$$\varphi_2(n) = n\prod_{p\mid n}\Big(1 - \frac{2}{p}\Big)\,.$$

(a) Show that

$$\sum_{\substack{n\le z\\ 2\nmid n}} \frac{\mu(n)^2}{\varphi_2(n)} = \sum_{\substack{n\le z\\ 2\nmid n}} \frac{\mu(n)^2}{\varphi(n)} \prod_{p\mid n}\Big(1 + \frac{1}{p-1} + \frac{1}{(p-1)^2} + \cdots\Big)\,.$$

(b) Let $f(n)$ be defined as in Exercise 1. Explain why the right hand side above is

$$\ge \sum_{\substack{n\le z\\ 2\nmid n}} f(n)\,.$$

(c) Conclude that

$$\sum_{\substack{n\le z\\ 2\nmid n}} \frac{\mu(n)^2}{\varphi_2(n)} \ge \frac{\log z}{c} + O(1)$$

where c is defined as in (20.27).

3. Put $P = \prod_{2<p\le z} p$, and let Λ_d be real numbers such that $\Lambda_1 = 1$ and $\Lambda_d = 0$ for $d > z$.

(a) Explain why the number of primes $p \le x$ for which $p+2$ is prime does not exceed

$$\pi(z) + \sum_{p \le x} \Big(\sum_{d \mid (p+2)} \Lambda_d \Big)^2 .$$

(b) Show that the sum above is

$$= \sum_{\substack{d \mid P \\ e \mid P}} \Lambda_d \Lambda_e \pi(x; [d,e], -2) .$$

(c) Write the above as

$$\mathrm{li}(x) \sum_{\substack{d \mid P \\ e \mid P}} \frac{\Lambda_d \Lambda_e}{\varphi([d,e])} \; + \; \sum_{\substack{d \mid P \\ e \mid P}} \Lambda_d \Lambda_e E_\pi(x; [d,e], -2) . \tag{20.35}$$

(d) Show that if f is a multiplicative function, then $f((d,e))f([d,e]) = f(d)f(e)$.
(e) Let $\varphi_2(n)$ be defined as in Exercise 2. Show that if n is squarefree, then

$$\varphi(n) = \sum_{d \mid n} \varphi_2(d) .$$

(f) Show that the first sum in (20.35) is $\sum_{\delta \mid P} \varphi_2(\delta) y_\delta^2$ where

$$y_\delta = \sum_{\substack{d \mid P \\ \delta \mid d}} \frac{\Lambda_d}{\varphi(d)} .$$

(g) Show that if $\Lambda_d = 0$ for $d > z$, then $y_\delta = 0$ for $\delta > z$.
(h) Show that

$$\Lambda_d = \varphi(d) \sum_{\substack{\delta \mid P \\ d \mid \delta}} \mu(\delta/d) y_\delta .$$

(i) Show that if $y_\delta = 0$ for $\delta > z$, then $\Lambda_d = 0$ for $d > z$.
(j) Explain why $\sum_{\delta \mid P} \mu(\delta) y_\delta = 1$.
(k) Put

$$L = \sum_{\substack{\delta \le z \\ 2 \nmid \delta}} \frac{\mu(\delta)^2}{\varphi_2(\delta)} .$$

(l) Show that

$$\sum_{\substack{\delta \mid P \\ \delta \le z}} \varphi_2(\delta) y_\delta^2 \;=\; \frac{1}{L} + \sum_{\substack{\delta \mid P \\ \delta \le z}} \varphi_2(\delta) \big(y_\delta - \mu(\delta)/(L\varphi_2(\delta)) \big)^2 .$$

(m) Take $y_\delta = \mu(\delta)/(L\varphi_2(\delta))$ for $\delta|P$, $\delta \le z$. Show that the first term in (20.35) is

$$\le \frac{c\,\mathrm{li}(x)}{\log z} + O\big(x/((\log x)(\log z)^2)\big)\,.$$

(n) Show that

$$\Lambda_d = \frac{\mu(d)\varphi(d)}{L\varphi_2(d)} \sum_{\substack{r\le z/d\\(r,2d)=1}} \frac{\mu(r)^2}{\varphi_2(r)}\,.$$

(o) Explain why

$$\frac{\varphi(d)}{\varphi_2(d)} \sum_{\substack{r\le z/d\\(r,2d)=1}} \frac{\mu(r)^2}{\varphi_2(r)} \le L,$$

and hence deduce that $|\Lambda_d| \le 1$ for all d.
(p) Show that if $q|P$, then

$$\sum_{\substack{d,e\\[d,e]=q}} |\Lambda_d\Lambda_e| \le 3^{\omega(q)}\,.$$

(q) Show that the second term in (20.35) has absolute value not exceeding

$$\sum_{\substack{q\le z^2\\2\nmid q}} \mu(q)^2 3^{\omega(q)} E_\pi(x,q)\,.$$

(r) Show that

$$\sum_{\substack{q\le z^2\\2\nmid q}} \frac{\mu(q)^2 9^{\omega(q)}}{q} \le \prod_{2<p\le z^2} \Big(1+\frac{9}{p}\Big) \ll (\log z)^9\,.$$

(s) Deduce by (20.26) that the second term in (20.35) is $\ll x^{3/4}z(\log xz)^6$.
(t) Take $z = x^{1/4}(\log x)^{-9}$. Conclude that the numper of primes $p \le x$ for which $p+2$ is prime does not exceed

$$\frac{4cx}{(\log x)^2}\Big(1+O\Big(\frac{\log\log x}{\log x}\Big)\Big)$$

where c is defined as in (20.27).

3. Mean square distribution

We begin with an upper bound for the mean square error in the prime number theorem for arithmetic progressions, which we then use to derive an asymptotic estimate for the same quantity.

Theorem 20.11 *Let A be fixed. If $x/(\log x)^A \le Q \le x$, then*

$$\sum_{q\le Q}\sideset{}{^*}\sum_{a=1}^{q}(\psi(x;q,a)-x/\varphi(q))^2 \ll Qx\log x\,. \tag{20.36}$$

Proof Put $\psi'(x,\chi)=\psi(x,\chi)-E_0(\chi)x$ where $E_0(\chi)=1$ if $\chi=\chi_0$, and $E_0(\chi)=0$ otherwise. Then

$$\psi(x;q,a)-\frac{x}{\varphi(q)} = \frac{1}{\varphi(q)}\sum_{\chi}\overline{\chi}(a)\psi'(x,\chi),$$

so that by orthogonality (as in (4.12) or Exercise 4.2.2),

$$\sideset{}{^*}\sum_{a=1}^{q}(\psi(x;q,a)-x/\varphi(q))^2=\frac{1}{\varphi(q)}\sum_{\chi}|\psi'(x,\chi)|^2\,.$$

If $\chi^\star$ is the primitive character that induces χ, then

$$\psi'(x,\chi^\star)-\psi'(x,\chi) = \sum_{\substack{n\le x\\(n,q)>1}}\chi^\star(n)\Lambda(n)\ll\sum_{\substack{p^k\le x\\p\mid q}}\log p=\sum_{p\mid q}\Big[\frac{\log x}{\log p}\Big]\log p\ll(\log qx)^2\,.$$

Hence the left hand of (20.36) is

$$\ll\sum_{q\le Q}\frac{1}{\varphi(q)}\sum_{\chi}\big(|\psi'(x,\chi^\star)|^2+(\log qx)^4\big)\ll\sum_{q\le Q}\Big(\sideset{}{^*}\sum_{\chi}|\psi'(x,\chi)|^2\Big)\Big(\sum_{r\le Q/q}\frac{1}{\varphi(qr)}\Big)+Q(\log Qx)^4\,.$$

Now $\varphi(qr)\ge\varphi(q)\varphi(r)$, and

$$\begin{aligned}\sum_{r\le y}\frac{1}{\varphi(r)} &\le \sum_{p\mid r\Rightarrow p\le y}\frac{1}{\varphi(r)}=\prod_{p\le y}\Big(1+\frac{1}{p-1}+\cdots\Big)=\prod_{p\le y}\Big(1-\frac{1}{p}\Big)^{-1}\Big(1+\frac{1}{p(p-1)}\Big)\\ &\ll \log 2y,\end{aligned}$$

so it suffices to show that

$$\sum_{q\le Q}\frac{\log\frac{2Q}{q}}{\varphi(q)}\sideset{}{^*}\sum_{\chi}|\psi'(x,\chi)|^2 \ll Qx\log x. \tag{20.37}$$

By the Siegel–Walfisz theorem (Corollary 11.18) we know that

$$\psi'(x,\chi)\ll x\exp\big(-c\sqrt{\log x}\big)$$

for $q \le (\log x)^{A+2}$. The contribution of such q is

$$\ll x^2(\log x)^{A+3}\exp\big(-c\sqrt{\log x}\big) \ll x^2(\log x)^{-A} \ll Qx\,.$$

Consider now a range $Q_1 < q \le 2Q_1$ with $1 < Q_1 \le Q$. Then $\psi'(x,\chi) = \psi(x,\chi)$, and the contribution is

$$\ll \frac{\log \frac{2Q}{Q_1}}{Q_1} \sum_{Q_1<q\le 2Q_1} \frac{q}{\varphi(q)} \sideset{}{^*}\sum_{\chi} |\psi(x,\chi)|^2\,.$$

By the large sieve this is

$$\ll \frac{\log \frac{2Q}{Q_1}}{Q_1}(x+Q_1^2)\sum_{n\le x}\Lambda(n)^2 \ll (x^2Q_1^{-1}+xQ_1)(\log x)\log\frac{2Q}{Q_1}\,.$$

We cover the interval $(\log x)^{A+2} \le q \le Q$ with ranges of the above sort, and sum, to obtain (20.37). Thus the proof is complete. □

For many applications the estimate of Theorem 20.11 is sufficient, but it is interesting to note that with a little more work we can obtain not just an upper bound but an asymptotic estimate. To prepare for the main argument we first establish a lemma.

Lemma 20.12 *There exist absolute constants a and b such that*

$$\sum_{n\le y}\frac{(1-n/y)^2}{\varphi(n)} = \frac{\zeta(2)\zeta(3)}{\zeta(6)}\log y + a + \frac{\log y}{y} + \frac{b}{y} + O_\varepsilon\big(y^{-3/2+\varepsilon}\big) \tag{20.38}$$

for $y \ge 1$.

Proof By manipulating Euler products we see that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{\varphi(n)n^s} &= \zeta(s+1)\prod_p\Big(1+\frac{1}{(p-1)p^{s+1}}\Big)\\ &= \zeta(s+1)\zeta(s+2)\prod_p\Big(1+\frac{1}{(p-1)p^{s+2}}-\frac{1}{(p-1)p^{2s+3}}\Big)\\ &= \zeta(s+1)\zeta(s+2)F(s),\end{aligned}$$

say. By taking $k=2$ in (5.19), we see that the left hand side in (20.38) is

$$= \frac{2}{2\pi i}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty}\zeta(s+1)\zeta(s+2)F(s)\frac{y^s}{s(s+1)}\,ds$$

where $\sigma_0 > 0$. The Euler product $F(s)$ is absolutely convergent for $\sigma > -3/2$, and is uniformly bounded for $\sigma \ge -3/2+\delta$. We let σ_1 be slightly larger than $-3/2$, and apply Cauchy's theorem with a path from $\sigma_0 - iT$ to σ_0+iT to σ_1+iT to σ_1-iT to σ_0-iT. By Corollaries 1.17 and 10.5 we see that $\zeta(s+1)\zeta(s+2) \ll \tau^{3/2}$ on this contour. Thus the

integral from $\sigma_1 + iT$ to $\sigma_1 - iT$ is $\ll y^{\sigma_1}$. Within the contour the integrand has double poles at $s = 0$ and at $s = -1$. At $s = 0$, the residue is

$$\zeta(2)G(0)\Big(C_0 + \frac{\zeta'}{\zeta}(2) + \frac{G'}{G}(0) - \frac{3}{2}\Big)\,.$$

This gives the first two main terms, since $G(0) = \zeta(3)/\zeta(6)$. At $s = -1$, the residue is

$$-2\zeta(0)G(0)y^{-1}\Big(\frac{\zeta'}{\zeta}(0) + C_0 + \frac{G'}{G}(-1) + \log y\Big)\,.$$

We recall (10.11), which asserts that $\zeta(0) = -1/2$. Since $G(-1) = 1$, we have the remaining main terms. □

Theorem 20.13 *Let $A > 0$ be fixed. If $x/(\log x)^A \le Q \le x$, then*

$$\sum_{q\le Q}\sideset{}{^*}\sum_{a=1}^{q}(\psi(x;q,a) - x/\varphi(q))^2 \;=\; Qx\log Q + O(Qx)\,. \tag{20.39}$$

Proof Let $Q_1 = x^2(\log x)^{-A-1}$. By Theorem 20.11, the contribution of $q \le Q_1$ to the above is $\ll x^2(\log x)^{-A} \ll Qx$. So we may restrict our attention to the range $Q_1 \le q \le Q$. The inner sum on the left hand side is

$$= \sideset{}{^*}\sum_{a=1}^{q}\psi(x;q,q)^2 \;-\; 2\frac{x}{\varphi(q)}\sideset{}{^*}\sum_{a=1}^{q}\psi(x;q,a) \;+\; \frac{x^2}{\varphi(q)}\,. \tag{20.40}$$

Here the second sum is

$$= \sum_{\substack{n\le x\\(n,q)=1}}\Lambda(n) \;=\; \psi(x) - \sum_{p\mid q}\Big[\frac{\log x}{\log p}\Big]\log p$$
$$= x + O\big((\log qx)^2\big) + O\big(x\exp\big(-c\sqrt{\log x}\big)\big)\,.$$

The first sum in (20.40) is

$$\sum_{\substack{m,n\le x\\ m\equiv n\ (q)\\ (mn,q)=1}}\Lambda(m)\Lambda(n)\,.$$

If the condition $(mn, q) = 1$ is omitted, then the value of the above is changed by not more than

$$\sum_{p\mid q}\Big[\frac{\log x}{\log p}\Big](\log p)^2 \;\ll\; (\log qx)^3\,.$$

Thus by Lemma 20.9 with $q = 1$ we deduce that

$$\sum_{Q_1<q\le Q} \sideset{}{^*}\sum_{a=1}^{q} (\psi(x;q,a) - x/\varphi(q))^2$$
$$= \sum_{Q_1<q\le Q} \sum_{\substack{m,n\le x\\ m\equiv n\ (q)}} \Lambda(m)\Lambda(n) \ - \ \frac{\zeta(2)\zeta(3)}{\zeta(6)} x^2 \log\frac{Q}{Q_1} \ + \ O(Qx)\,.$$

The terms with $m = n$ contribute an amount

$$(Q - Q_1 + O(1)) \sum_{n\le x} \Lambda(n)^2 \ = \ Qx\log x + O(Qx)\,.$$

Hence to obtain the stated result it suffices to show that

$$\sum_{Q<q\le Q} \sum_{\substack{m<n\le x\\ m\equiv n\ (q)}} \Lambda(m)\Lambda(n) \ = \ \frac{\zeta(2)\zeta(3)}{2\zeta(6)} x^2 \log\frac{Q}{Q_1} \ - \ \frac{1}{2} Qx \log\frac{x}{Q} \ + \ O(Qx)\,. \tag{20.41}$$

To this end we show that

$$\sum_{y<q\le x} \sum_{\substack{m<n\le x\\ m\equiv n\ (q)}} \Lambda(m)\Lambda(n) \ = \ \frac{\zeta(2)\zeta(3)}{2\zeta(6)} x^2 \log\frac{x}{y} \ + \ \frac{a}{2}x^2 \ + \ \frac{1}{2}xy\log\frac{x}{y} \ + \ O(xy) \tag{20.42}$$

for $x(\log x)^{-A-1} \le y \le x$, where a is the constant in Lemma 20.12. This suffices, for on taking $y = Q_1$ and $y = Q$, and differencing, we obtain (20.41).

The left hand side of (20.42) is

$$\sum_{y<q\le x} \sum_{0<k\le x/q} \sum_{0<m\le x-kq} \Lambda(m)\Lambda(m+kq)$$
$$= \sum_{0<k\le x/y} \sum_{y<q\le x/k} \sum_{0<m\le x-kq} \Lambda(m)\Lambda(m+kq)$$
$$= \sum_{0<k\le x/y} \sum_{0<m\le x-ky} \Lambda(m) \sum_{y<q\le (x-m)/k} \Lambda(m+kq)$$
$$= \sum_{0<k\le x/y} \sum_{0<m\le x-ky} \Lambda(m)(\psi(x;k,m) - \psi(m+ky;k,m))\,.$$

If m is a prime-power and $(m,k) > 1$, then $m = p^r$, say, where $p|k$, and the prime-powers congruent to m modulo k are powers of the same prime p. Thus the pairs $m,\ k$ for which $(m,k) > 1$ contribute to the above an amount

$$\ll \sum_{k\le x/y} \sum_{p|k} \Big[\frac{\log x}{\log p}\Big] (\log p)^2 \ \ll \ \sum_{k\le x/y} (\log kx)^3 \ \ll \ (\log x)^{A+4}\,.$$

On the other hand, by the Siegel–Walfisz theorem (Corollary 11.19), the pairs $k,\ m$ for which $(k,m)=1$ contribute the amount

$$\sum_{0<k\le x/y}\frac{1}{\varphi(k)}\sum_{\substack{0<m\le x-ky\\(m,k)=1}}\Lambda(m)(x-m-ky)+O\Big(\sum_{0<k\le x/y}\sum_{m\le x-ky}\Lambda(m)x\exp\big(-c\sqrt{\log x}\big)\Big).$$

The error term here is $\ll(\log x)^{A+1}x^2\exp\big(-c\sqrt{\log x}\big)\ll x^2(\log x)^{-A}$, so can be ignored. In the main term, if the condition that $(m,k)=1$ is dropped, then the expression is altered my an amount that is

$$\ll\ x\sum_{0<k\le x/y}\sum_{p\mid k}\Big[\frac{\log x}{\log p}\Big]\log p\ \ll\ x(\log x)^3\ \ll\ x^2(\log x)^{-A}.$$

By the Prime Number Theorem we know that

$$\sum_{m\le z}\Lambda(m)(z-m)=\frac{1}{2}z^2\ +O\big(z^2\exp\big(-c\sqrt{\log z}\big)\big).$$

On taking $z=x-ky$, we see that the remaining main term is

$$\frac{1}{2}\sum_{0<k\le x/y}\frac{(x-ky)^2}{\varphi(k)}\ +\ O\Big(x^2\exp\big(-c\sqrt{\log x}\big)\sum_{0<k\le x/y}\frac{1}{\varphi(k)}\Big).$$

By Lemma 20.12 this is

$$=\ \frac{\zeta(2)\zeta(3)}{2\zeta(6)}x^2\log\frac{x}{y}\ +\ \frac{a}{2}x^2\ +\ \frac{1}{2}xy\log\frac{x}{y}\ +\ O(xy).$$

Thus we have (20.42), and the proof is complete. □