Chapter 21

Additive Prime Number Theory

We now address additive questions involving prime numbers, particularly the problem of expressing an integer as a sum of k primes

$$n = p_1 + p_2 + \cdots + p_k \,. \tag{21.1}$$

The cases $k = 2$ and $k = 3$ of this were first enunciated by Goldbach in letters to Euler in 1742. We employ the 'circle method' of Hardy–Littlewood, as later modified and improved by Vinogradov. For sums of three primes our method is successful. For sums of two primes our method fails, but we can nevertheless show that almost all even numbers can be expressed as a sum of two primes.

Let $r_k(n)$ denote the number of solutions of (21.1) in prime numbers p_i, and let $r_k(n, X)$ denote the corresponding number with no p_i exceeding X. Thus $r_k(n) = r_k(n, X)$ for $n \le X$, and the identity

$$\sum_n r_k(n, X) e(n\alpha) = \Big(\sum_{p \le X} e(p\alpha) \Big)^k$$

is an immediate consequence of writing the product on the right as a k-fold sum over $p_1, \dots, p_k$, and then combining those terms for which $p_1 + \cdots + p_k = n$. Thus the generating function

$$S(\alpha) = \sum_{p \le X} e(p\alpha) \tag{21.2}$$

readily lends itself to the study of additive problems, and it is from its properties that we derive estimates for $r_k(n)$.

Since

$$\int_0^1 e(m\alpha)\, d\alpha = \begin{cases} 1 & \text{when } m = 0, \\ 0 & \text{otherwise,} \end{cases}$$

the functions $e(m\alpha)$ are orthonormal on the circle group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Therefore

$$r_k(n, X) = \int_0^1 S(\alpha)^k e(-n\alpha)\, d\alpha, \tag{21.3}$$

which is merely the formula for the n^{th} Fourier coefficient of S^k.

The size of $S(\alpha)$ at an arbitrary point α depends on the extent to which α can be approximated by a rational number a/q with q relatively small. The primes are uniformly distributed among the reduced residue classes modulo q, but the reduced residue classes are not equally distributed, so we expect that the numbers $S(a/q)$ are sometimes large. Indeed,

$$S(a/q) = \sum_{h=1}^{q} e(ha/q)\pi(X; q, h),$$

which is approximately

$$\frac{1}{\varphi(q)} \operatorname{li}(X) \sum_{h=1}^{q}{}^{*} e(ha/q)$$

if $q \le (\log X)^A$. The inner sum above is Ramanujan's sum $c_q(a)$, and by Theorem 4.1 we know that $c_q(a) = \mu(q)$ when $(a, q) = 1$. Thus the above is

$$= \frac{\mu(q)}{\varphi(q)} \operatorname{li}(X)$$

if $(a, q) = 1$. By partial summation we find that $S(\alpha)$ has a peak of width comparable to $1/X$ at a/q when q is squarefree. The principal of the Hardy–Littlewood circle method is to obtain an asymptotic formula for $r_k(n)$ by estimating the contributions to the integral (21.3) from those peaks when q is relatively small, and then showing that the remaining portions of $\mathbb{T}$ contribute *in toto* an amount of a smaller order of magnitude.

21.1 Sums of three primes

We now execute the approach outlined above in the case $k = 3$.

Theorem 21.1 (Vinogradov) *Let*

$$\mathfrak{S}_3(n) = \Big(\prod_{p \mid n} \big(1 - (p-1)^{-2}\big) \Big) \prod_{p \nmid n} \big(1 + (p-1)^{-3}\big)$$

and

$$\operatorname{ls}_3(n) = \sum_{\substack{m_1+m_2+m_3=n \\ m_i > 1}} \prod_{i=1}^{3} (\log m_i)^{-1}.$$

Then for any fixed positive number A,

$$r_3(n) = \mathfrak{S}_3(n) \operatorname{ls}_3(n) + O\big(n^2 (\log n)^{-A}\big),$$

and

$$\operatorname{ls}_3(n) = \frac{1}{2} n^3 (\log n)^{-3} \big(1 + O(1/\log n)\big).$$

The quantity $\mathfrak{S}_3$ is written above in the form of an Euler product, but we first encounter it below in expanded form, as an infinite series. In the parlance of Hardy–Littlewood, this is a *singular series*. Hence the use of the Gothic letter S to denote it.

It is readily seen that $\mathfrak{S}_3(n) = 0$ for even n and that $\mathfrak{S}_3(n) \asymp 1$ for odd n. Consequently, all sufficiently large odd numbers can be expressed as a sum of three primes.

We begin with two lemmas. In the first of these, we find that $S(\alpha)$ is relatively small when α is not near a rational number with small denominator. The second relates to a sum that is useful in describing the peaks of $S(\alpha)$.

Lemma 21.2 *Suppose that $|\alpha - a/q| \le 1/q^2$, that $(a,q) = 1$, and that $S(\alpha)$ is defined as in* (21.2)*. Then*

$$S(\alpha) \ll \big(Xq^{-1/2} + X^{4/5} + X^{1/2}q^{1/2}\big)(\log X)^3\,.$$

Proof Let $T(u) = \sum_{p\le u}(\log p)e(p\alpha)$. By Theorem 18.1 we see that

$$T(u) \ll \big(uq^{-1/2} + u^{4/5} + u^{1/2}q^{1/2}\big)(\log u)^4\,. \tag{21.4}$$

Then

$$S(\alpha) = \int_{2^-}^{X} (\log u)^{-1}\,dT(u) = \frac{T(X)}{\log X} + \int_2^X \frac{T(u)}{u(\log u)^2}\,du,$$

so the stated bound follows from (21.4). □

Lemma 21.3 *Let $U(\beta) = \sum_{1<m\le X} e(m\beta)/\log m$. Then*

$$U(\beta) \ll Y/\log Y$$

where $Y = \min\big(X, \|\beta\|^{-1}\big)$.

Proof When $\|\beta\| \le 1/X$, we argue that $|U(\beta)| \le U(0) \ll X/\log X$. When $\|\beta\| > 1/X$, we again bound the contribution of $m \le Y$ trivially. For the range $Y < m \le X$ we appeal to (16.5), and integrate by parts. □

Proof of Theorem 21.1 Let $P = (\log X)^B$ and $Q = X/P$ where B is to be selected later as a function of A. We now dissect $\mathbb{T}$ into appropriate arcs. For $q \le P$ and $(a,q) = 1$, let $\mathfrak{M}(q,a)$, called a *major arc*, denote the interval consisting of those α for which $|\alpha - a/q| \le 1/Q$. Further, let $\mathfrak{M}$ denote the union of these $\mathfrak{M}(q,a)$. If $\mathfrak{M}(q,a)$ and $\mathfrak{M}(q',a')$ are two major arcs with $a/q \ne a'/q'$, then

$$\Big|\frac{a}{q} - \frac{a'}{q'}\Big| \ge \frac{1}{qq'} \ge \frac{1}{P^2} > \frac{2}{Q},$$

so $\mathfrak{M}(q,a)$ and $\mathfrak{M}(q',a')$ are disjoint. We define $\mathfrak{m}$, the *minor arcs*, to be $\mathfrak{m} = \mathbb{T} \setminus \mathfrak{M}$.

From (21.3) we see that

$$r_3(n,X) \;=\; \int_{\mathfrak{m}} S(\alpha)^3 e(-n\alpha)\,d\alpha \;+\; \sum_{q\le P}\sideset{}{^*}\sum_{a=1}^{q} \int_{\mathfrak{M}(q,a)} S(\alpha)^3 e(-n\alpha)\,d\alpha\,.$$

We first estimate the integral over $\mathfrak{m}$. By Dirichlet's theorem, for any real number α and any $Q \geq 1$ there exist q and a with $q \leq Q$, $(a,q) = 1$, and $|\alpha - a/q| \leq 1/(qQ)$. If $q \leq P$, then $\alpha \in \mathfrak{M}(q,a)$. Thus if $\alpha \in \mathfrak{m}$, then $P < q \leq Q$, and hence

$$S(\alpha) \ll X(\log X)^{3-B/2}$$

by Lemma 21.2. By Parseval's identity and Chebyshev's estimate we have

$$\int_0^1 |S(\alpha)|^2 \, d\alpha = \pi(X) \ll X/\log X \,.$$

Therefore

$$\int_{\mathfrak{m}} |S(\alpha)|^3 \, d\alpha \leq \big(\max_{\mathfrak{m}} |S(\alpha)|\big) \int_0^1 |S(\alpha)|^2 \, d\alpha \ll X^2 (\log X)^{2-B/2} \,.$$

Thus

$$\int_{\mathfrak{m}} S(\alpha)^3 e(-n\alpha) \, d\alpha \ll X^2 (\log X)^{-A} \tag{21.5}$$

provided that

$$B \geq 2A + 4 \,. \tag{21.6}$$

When $\alpha \in \mathfrak{M}(q,a)$, we can approximate $S(\alpha)$ *via* the Siegel–Walfisz theorem. Let

$$S(a/q, x) = \sum_{p \leq x} e(pa/q) \,.$$

The number of primes p with $(p,q) > 1$ is $\ll \log q$. Thus

$$S(a/q, x) = \sum_{h=1}^{q}{}^{*} \pi(x; q, h) e(ha/q) \;+\; O(\log q) \,.$$

Let the *logarithmic sum* be $\mathrm{ls}(x) = \sum_{1<m\leq x} (\log m)^{-1} = \mathrm{li}(x) + O(1)$. By the Siegel–Walfisz theorem (Corollary 11.21),

$$\pi(x; q, h) = \frac{\mathrm{ls}(x)}{\varphi(q)} + O\big(X \exp(-c\sqrt{\log X})\big)$$

uniformly for $x \leq X$. Hence

$$S(a/q, x) = \frac{\mu(q)}{\varphi(q)} \mathrm{ls}(x) + O\big(X \exp(-c\sqrt{\log X})\big) \,. \tag{21.7}$$

Let $R(x) = S(a/q, x) - \mathrm{ls}(x)\mu(q)/\varphi(q)$, and set $\beta = \alpha - a/q$. Then

$$S(\alpha) = \int_1^X e(\beta x)\, dS(a/q, x) = \frac{\mu(q)}{\varphi(q)} \int_1^X e(\beta x)\, d\,\mathrm{ls}(x) + \int_1^X e(\beta x)\, dR(x)\,.$$

Here the first integral on the right is $U(\beta)$, in the notation of Lemma 21.3. We estimate the final integral by integrating by parts and applying (21.7). Thus we find that

$$S(\alpha) = \frac{\mu(q)}{\varphi(q)} U(\beta) + O\big((1 + |\beta|X) X \exp(-c\sqrt{\log X})\big)\,.$$

For $\alpha \in \mathfrak{M}(q, a)$ we have $|\beta| \le 1/Q$, and for arbitrary complex numbers u and v we have $|u^3 - v^3| \le 3|u - v| \max\big(|u|^2, |v|^2\big)$. Therefore

$$S(\alpha)^3 = \frac{\mu(q)}{\varphi(q)^3} U(\beta)^3 + O\big(X^3 \exp(-c\sqrt{\log X})\big)\,.$$

Thus

$$\int_{\mathfrak{M}} S(\alpha)^3 e(-n\alpha)\, d\alpha = I(n) \sum_{q \le P} J(q) \;+\; O\big(|\mathfrak{M}| X^3 \exp(-c\sqrt{\log X})\big) \tag{21.8}$$

where

$$I(n) = \int_{-1/Q}^{1/Q} U(\beta)^3 e(-n\beta)\, d\beta$$

and

$$J(q) = \frac{\mu(q)}{\varphi(q)^3} \sum_{a=1}^{q}{}^{*}\, e(-na/q)\,.$$

Here $I(n)$ is what Hardy & Littlewood would have called the *singular integral*, and $\sum J(q)$ will turn out to be the singular series.

The measure of $\mathfrak{M}$ is $\le 2P^2/Q \ll X^{-1}(\log X)^{3B}$. Moreover,

$$\sum_{q > P} J(q) \ll \sum_{q > P} \varphi(q)^{-2} \ll P^{-1} = (\log X)^{-B} \quad \text{and} \quad \sum_{q=1}^{\infty} J(q) \ll 1\,.$$

By Lemma 21.3,

$$\int_{1/Q}^{1-1/Q} |U(\beta)|^3\, d\beta \ll Q^2 (\log Q)^{-3} \ll X^2 (\log X)^{-2B-3}$$

and

$$\int_0^1 |U(\beta)|^3\, d\beta \ll X^2 (\log X)^{-3}\,.$$

Thus if (21.6) holds, then from (21.8) we deduce that

$$\int_{\mathfrak{M}} S(\alpha)^3 e(-n\alpha)\,d\alpha = \int_0^1 U(\beta)^3 e(-n\beta)\,d\beta \sum_{q=1}^{\infty} J(q) + O\big(X^2(\log X)^{-A}\big)\,.$$

Clearly

$$\int_0^1 U(\beta)^3 e(-n\beta)\,d\beta = \mathrm{ls}_3(n,X)$$

where

$$\mathrm{ls}_3(n,X) = \sum_{\substack{m_1+m_2+m+3=n\\ 1<m_i\le X}} \prod_{i=1}^{3} (\log m_i)^{-3}\,.$$

Moreover, the sum in the definition of $J(q)$ is Ramanujan's sum $c_q(-n)$, which is a multiplicative function of q (as we recall from Theorem 4.1). Since $c_p(-n) = p-1$ if $p|n$ and $c_p(-n) = -1$ otherwise, it follows that

$$\sum_{q=1}^{\infty} J(q) = \mathfrak{S}_3(n)\,.$$

We take $B = 2A+4$ so that (21.6) is satisfied, and conclude that

$$\int_{\mathfrak{M}} S(\alpha)^3 e(-n\alpha)\,d\alpha = \mathfrak{S}_3(n)\,\mathrm{ls}_3(n,X) + O\big(X^2(\log X)^{-A}\big)\,. \tag{21.9}$$

This and (21.5) with $X = n$ give the first part of the theorem.

To establish the asymptotic formula for ls_3, we first observe that

$$\begin{aligned}
\mathrm{ls}_3(n) &= \sum_{2\le m_1\le n-4} \frac{1}{\log m_1} \sum_{2\le m_2\le n-m_1-2} \frac{1}{(\log m_2)\log(n-m_1-m_2)}\\
&= \sum_{2\le m_1\le n-4} \frac{1}{\log m_1}\Big(\int_2^{n-m_1-2} \frac{dx}{(\log x)\log(n-m_1-x)} + O\Big(\frac{1}{\log n}\Big)\Big)\\
&= \int_2^{n-4}\int_2^{n-y-2} \frac{dx\,dy}{(\log x)(\log y)\log(n-x-y)} + O\Big(\frac{n}{(\log n)^2}\Big)\,. \qquad (21.10)
\end{aligned}$$

To estimate this integral, we first observe that if j and k are fixed integers, then

$$\int_2^{X/2} \frac{dx}{(\log x)^j(\log(X-x))^k} = \frac{X}{2(\log X)^{j+k}} + O\Big(\frac{X}{(\log X)^{j+k+1}}\Big)\,. \tag{21.11}$$

The point is that $(\log(X-x))^k = (\log X)^k(1+O(1/\log X))$ for $2\le x\le X/2$, and then the remaining integral can be estimated by integrating by parts. Similarly,

$$\int_2^{X/2} \frac{x\,dx}{(\log x)^j(\log(X-x))^k} = \frac{X^2}{8(\log X)^{j+k}} + O\Big(\frac{X^2}{(\log X)^{j+k+1}}\Big)\,. \tag{21.12}$$

From (21.11) we see that

$$\int_2^{X-2} \frac{dx}{(\log x)\log(X-x)} = 2\int_2^{X/2} \frac{dx}{(\log x)\log(X-x)} = \frac{X}{(\log X)^2} + O\Big(\frac{X}{(\log X)^3}\Big).$$

We take $X = n - y$, and insert this in (21.10) to see that

$$\mathrm{ls}_3(n) = \int_2^{n-4} \frac{n-y}{(\log y)(\log(n-y))^2}\, dy + O\Big(\frac{n^2}{(\log n)^4}\Big).$$

To estimate the contribution of the interval $2 \le y \le n/2$ we use both (21.11) and (21.12) with $j = 1$ and $k = 2$. To treat the interval $n/2 \le y \le n-4$ we replace y by $n-y$ and use (21.12) with $j = 2$ and $k = 1$. On assembling the various estimates we obtain the stated result.

21.1.1 Exercises

1. Let $r_k(n)$ denote the number of representations of n as a sum of k primes.
(a) Show that $r_k(n) = \sum_{p<n} r_{k-1}(n-p)$.
(b) Let

$$\mathfrak{S}_k(n) \;=\; \prod_{p\mid n}\Big(1 + \frac{(-1)^k}{(p-1)^{k-1}}\Big)\prod_{p\nmid n}\Big(1 - \frac{(-1)^k}{(p-1)^k}\Big) \tag{21.13}$$

and

$$\mathrm{ls}_k(n) \;=\; \sum_{\substack{m_1,m_2,\dots,m_k\\ \sum m_i = n\\ m_i>1}} \prod_{i=1}^{k} (\log m_i)^{-1}. \tag{21.14}$$

For each fixed $k \ge 3$ and each fixed $A > 0$, show that

$$r_k(n) = \mathfrak{S}_k(n)\,\mathrm{ls}_k(n) + O\big(n^{k-1}(\log n)^{-A}\big).$$

Do this by induction on k with the already completed case $k = 3$ as the basis of the induction. (Do not use the circle method.)

2. Show that

$$\int_2^{X-2} \frac{dx}{(\log x)\log(X-x)} \;=\; \frac{X}{(\log X)^2} + \frac{2X}{(\log X)^3} + O\Big(\frac{X}{(\log X)^4}\Big).$$

3. (a) Use (16.5) to show that $\sum_{n\le X} c_q(n) \ll q\log q$ for $q > 1$.
(b) Show that $\sum_{n\le X} \mathfrak{S}_3(n) = X + O(1)$.

21.2 Sums of two primes on average

Our minor arc treatment fails when we consider $r_2(n)$, but the major arc contributions suggest the conjecture that

$$r_2(n) \sim \mathfrak{S}_2(n)\,\mathrm{ls}_2(n)$$

as n tends to infinity through even values. Here $\mathfrak{S}_2(n)$ and $\mathrm{ls}_2(n)$ are defined as in (21.13) and (21.14). Although we are unable to prove the conjecture, we can prove that $r_2(n)$ is near $\mathfrak{S}_2(n)\,\mathrm{ls}_2(n)$ for most n. In order to display the flexibility of the circle method, we switch now to the von Mangoldt function $\Lambda(n)$ rather than count primes with weight 1.

Theorem 21.4 *Let*

$$\psi_2(n) = \sum_{m<n} \Lambda(m)\Lambda(n-m)\,.$$

Then for any fixed $A > 0$,

$$\sum_{n\le X} \big(\psi_2(n) - \mathfrak{S}_2(n)n\big)^2 \;\ll\; X^3(\log X)^{-A}\,.$$

Corollary 21.5 *Let $E(X)$ denote the number of even natural numbers $n \le X$ such that n is not the sum of two primes. Then $E(X) \ll X/(\log X)^A$ for any fixed $A > 0$.*

Proof If n is even but not the sum of two primes, then $\psi_2(n) \ll n^{1/2}\log n$. Let $E_1(X)$ denote the number of even n, $X/2 < n \le X$ such that $\psi_2(n) < \mathfrak{S}_2(n)n/2$. We observe that $\mathfrak{S}_2(n) \gg 1$ uniformly for even n. Thus if n is counted by $E_1(X)$, then $|\psi_2(n) - \mathfrak{S}_2(n)n| \gg n \gg X$. By Theorem 21.4 it follows that $E_1(X) \ll X/(\log X)^A$. But $E(X) \ll 1 + \sum_r E_1(X/2^r)$, so we have the stated result. □

From (21.13) we see that

$$\mathfrak{S}_2(n) = \prod_p \Big(1 + \frac{c_p(n)}{(p-1)^2}\Big)\,.$$

This product is absolutely convergent, since $c_p(n) = -1$ for all but finitely many primes (namely the primes dividing n). Hence we may expand the product, and find that

$$\mathfrak{S}_2(n) = \sum_{q=1}^{\infty} \frac{\mu(q)^2}{\varphi(q)^2} c_q(n)\,.$$

It is useful to be able to work with a truncation of this series. Thus, in preparation for the proof of Theorem 21.4, we establish

Lemma 21.6 *Let*

$$\mathfrak{S}_2(n,P) = \sum_{q\le P} \frac{\mu(q)^2}{\varphi(q)^2} c_q(n)\,.$$

Then

$$\sum_{n\le X}\big(\mathfrak{S}_2(n)-\mathfrak{S}_2(n,P)\big)^2 \ll P^{-2}X(\log X)^3$$

for $X\ge 2$.

Proof By (4.7) we see that

$$\mathfrak{S}_2(n)-\mathfrak{S}_2(n,P) \;=\sum_{q>Q}\frac{\mu(q)^2}{\varphi(q)^2}c_q(n) \;\ll\; \sum_{q>P}\frac{1}{\varphi(q)^2}\sum_{d\mid(q,n)} d$$

We write $q=dr$ and note that $\varphi(q)\ge\varphi(d)\varphi(r)$. Thus the above is

$$\ll\sum_{d\mid n}\frac{d}{\varphi(d)^2}\sum_{r>P/d}\frac{1}{\varphi(r)^2}\ll P^{-1}\sum_{d\mid n}\frac{d^2}{\varphi(d)^2}\,.$$

Put

$$f(n)=\Big(\sum_{d\mid n}\frac{d^2}{\varphi(d)^2}\Big)^2.$$

Then

$$\sum_{n\le X}\big(\mathfrak{S}_2(n)-\mathfrak{S}_2(n,P)\big)^2\ll P^{-2}\sum_{n\le X}f(n),$$

so to complete the proof it suffices to show that

$$\sum_{n\le X}f(n)\ll X(\log X)^3\,. \tag{21.15}$$

But this follows from Corollary 2.15, since f is a non-negative multiplicative function, $f(p)=\big(1+(p/(p-1))^2\big)^2=4+O(1/p)$ and $f(p^k)\ll k^2$, so that $\sum_{p\le x}f(p)\log p\ll x$,

$$\prod_{p\le X}\Big(1+\frac{f(p)}{p}+\frac{f(p^2)}{p^2}+\cdots\Big)\ll(\log X)^4,$$

and

$$\sum_{\substack{p^k\\ k\ge 2}}\frac{f\big(p^k\big)k\log p}{p^k}<\infty\,.$$

□

An alternative derivation of the estimate (21.15) that avoids the appeal to Corollary 2.15 is outlined in Exercise 21.2.1.4 below.

Proof of Theorem 21.4 The appropriate generating function is now

$$S(\alpha)=\sum_{n\le X}\Lambda(n)e(n\alpha).$$

Thus we define $\psi_2(n, X)$ by writing

$$S(\alpha)^2 = \sum_n \psi_2(n, X) e(n\alpha),$$

and we observe that $\psi_2(n, X) = \psi_2(n)$ for $n \le X$. In place of the auxiliary function U considered in §21.1, the appropriate function is

$$V(\beta) = \sum_{0 \le n \le X} e(n\beta)\,. \tag{21.16}$$

Let $w(n, X)$ denote the Fourier coefficients of $V(\beta)^2$, so that

$$V(\beta)^2 = \sum_n w(n, X) e(n\beta)\,.$$

Thus $w(n, X) = n + 1$ for $n \le X$. We retain without modification the definitions of P, Q, and the major and minor arcs given in the proof of Theorem 21.1, although the dependence of B on A may be different. The main idea is to apply Parseval's identity, but before we do so we truncate $\mathfrak{S}_2$. By Lemma 21.5 we see that

$$\sum_{n \le X} (n+1)^2 \big(\mathfrak{S}_2(n) - \mathfrak{S}_2(n, P)\big)^2 \ll X^3 (\log X)^{3-2B}\,.$$

Since also $\sum_{n \le X} \mathfrak{S}_2(n)^2 \ll X$, it suffices to show that

$$\sum_{0 \le n \le X} \big(\psi_2(n) - \mathfrak{S}_2(n, P)(n+1)\big)^2 \ll X^3 (\log X)^{-A} \tag{21.17}$$

if B is sufficiently large in terms of A. At this point, we require only that

$$B \ge (A+3)/2\,. \tag{21.18}$$

By Parseval's identity,

$$\sum_{n=0}^{\infty} \big(\psi_2(n, X) - \mathfrak{S}_2(n, P) w(n, X)\big)^2 = \int_0^1 \big|S(\alpha)^2 - T(\alpha)\big|^2 \, d\alpha \tag{21.19}$$

where

$$T(\alpha) \;= \sum_{n=0}^{\infty} \mathfrak{S}_2(n, P) w(n, X) e(n\alpha) \;=\; \sum_{q \le P} \frac{\mu(q)^2}{\varphi(q)^2} \sideset{}{^*}\sum_{a=1}^{q} V(\alpha - a/q)^2\,.$$

We first dispose of the minor arcs. By Cauchy's inequality,

$$|T(\alpha)|^2 \ll \Big(\sum_{q \le P} \frac{1}{\varphi(q)}\Big)\Big(\sum_{q \le P} \frac{1}{\varphi(q)^2} \sideset{}{^*}\sum_{a=1}^{q} |V(a/q)|^4\Big) \ll (\log P) \sum_{q \le P} \frac{1}{\varphi(q)^2} \sideset{}{^*}\sum_{a=1}^{q} \|\alpha - a/q\|^{-4}\,.$$

For $q \le P$ and $(a,q)=1$,

$$\int_{\mathfrak{m}} \|\alpha - a/q\|^{-4}\, d\alpha \ll \int_{1/Q}^{\infty} \beta^{-4}\, d\beta \ll Q^3 .$$

Hence

$$\int_{\mathfrak{m}} |T(\alpha)|^2\, d\alpha \ll Q^3 (\log P)^2 \ll X^3 (\log X)^{2-3B} .$$

From Theorem 18.1, as in the proof of Theorem 21.1, we find that

$$\max_{\mathfrak{m}} |S(\alpha)| \ll X (\log X)^{4-B/2} .$$

Hence

$$\int_{\mathfrak{m}} |S(\alpha)|^4\, d\alpha \le \big(\max_{\mathfrak{m}} |S(\alpha)|^2\big) \int_0^1 |S(\alpha)|^2\, d\alpha \ll X^3 (\log X)^{9-B} .$$

On combining these estimates we conclude that

$$\int_{\mathfrak{m}} |S(\alpha)^2 - T(\alpha)|^2\, d\alpha \ll X^3 (\log X)^{9-B} . \tag{21.20}$$

For $\alpha \in \mathfrak{M}(q,a)$, let $\beta = \alpha - a/q$. Then

$$T(\alpha) = \frac{\mu(q)^2}{\varphi(q)^2} V(q)^2 + O\Big(\sum_{r \le P} \frac{\mu(q)^2}{\varphi(q)^2} \sideset{}{^{**}}\sum_{b=1}^{r} \|\alpha - b/r\|^{-2} \Big)$$

where the sum over b is restricted to those b with $(b,r)=1$ and also $b \ne a$ if $r = q$. For such b we have $\|a/q - b/r\| \ge 1/(qr)$. Hence

$$\sideset{}{^{**}}\sum_{b=1}^{r} \|\alpha - b/r\|^{-2} \ll (qr)^2 + \sum_{m=1}^{r} (r/m)^2 \ll (qr)^2,$$

so that

$$T(\alpha) = \frac{\mu(q)^q}{\varphi(q)^2} V(q)^2 + O\big((\log X)^{3B}\big)$$

for $\alpha \in \mathfrak{M}(q,a)$. For such α, as in the proof of Theorem 21.1, we have

$$S(\alpha) = \frac{\mu(q)}{\varphi(q)} V(\beta) + O\big(X \exp\big(-c\sqrt{\log X}\big)\big),$$

whence

$$S(\alpha)^2 \;=\; \frac{\mu(q)^2}{\varphi(q)^2} V(\beta)^2 + O\big(X^2 \exp\big(-c\sqrt{\log X}\big)\big) .$$

By comparing our estimates for $S(\alpha)^2$ and $T(\alpha)$ we find that

$$\int_{\mathfrak{M}} |S(\alpha)^2 - T(\alpha)|^2\, d\alpha \;\ll\; |\mathfrak{M}| X^4 \exp\big(-c\sqrt{\log X}\big) \;\ll\; X^3 (\log X)^{-A} .$$

On combining this and (21.20) in (21.19), we deduce that

$$\sum_{n=0}^{\infty} \big(\psi_2(n,X) - \mathfrak{S}_2(n,P)w(n,X)\big)^2 \ \ll \ X^3(\log X)^{-A}$$

if $B \geq A+9$. Assuming that $A \geq 0$, we may take $B = A+9$, for then (21.18) is satisfied. Thus we have (21.17), and the proof is complete. □

21.2.1 Exercises

1. For positive integers k, let

$$T(X,k) = \sum_{k<n\leq X} \Lambda(n)\Lambda(n-k)\,.$$

Show that

$$\sum_{k\leq X} \big(T(X,k) - \mathfrak{S}_2(k)(X-k)\big)^2 \ll X^3(\log X)^{-A}$$

for any fixed A.

2. Show that there exist infinitely many pairs a, b such that a, $a+b$, and $a+2b$ are all prime. Do this in two ways:
(a) As a consequence of theorems already proved.
(b) By using the circle method to derive an asymptotic formula for the number of solutions of the equation $2p = p_1 + p_2$.

3. Let $S(\alpha) = \sum_{n\leq X} \Lambda(n)e(n\alpha)$.
(a) Show that

$$\sideset{}{^*}\sum_{a=1}^{q} S(a/q+\beta) = \mu(q)S(\beta) + O\big(q(\log qX)^2\big)\,.$$

(b) Let $\mathfrak{M}(q,a) = [a/q - 1/X, a/q + 1/X]$. Show that

$$\sideset{}{^*}\sum_{a=1}^{q} \int_{\mathfrak{M}(q,a)} |S(\alpha)|\,d\alpha \gg 1$$

provided that q is squarefree and $q \leq X/(\log X)^3$.
(c) Show that

$$\int_0^1 |S(\alpha)|\,d\alpha \gg X^{1/2}\,.$$

4. Let $f(n)$ be defined as in the proof of Lemma 21.5.
(a) Explain why

$$\sum_{n\leq X} f(n) \ll \sum_{\substack{c\leq X\\ d\leq X}} \frac{c^2d^2}{\varphi(c)^2\varphi(d)^2}\,\frac{X}{[c,d]}\,.$$

(b) Explain why the above is

$$= X \sum_{\substack{c\le X\\ d\le X}} \frac{cd}{\varphi(c)^2\varphi(d)^2} \sum_{\substack{r\mid c\\ r\mid d}} \varphi(r) = X \sum_{r\le X} \varphi(r)\Big(\sum_{\substack{d\le X\\ r\mid d}} \frac{d}{\varphi(d)^2}\Big)^2 .$$

(c) Write $d = rm$ and note that $\varphi(d) \ge \varphi(r)\varphi(m)$. Thus show that the above is

$$\le X \sum_{r\le X} \frac{r^2}{\varphi(r)^3}\Big(\sum_{m\le X/r} \frac{m}{\varphi(m)^2}\Big)^2 .$$

(d) Deduce (21.15).

21.3 Conditional estimates

The theorems that we have established thus far can be greatly sharpened if we assume the Generalized Riemann Hypothesis (GRH).

Theorem 21.7 *Assume GRH. Then*

$$\sum_{1\le n\le X} \big(\psi_2(n) - \mathfrak{S}_2(n)n\big)^2 \ll X^{5/2}(\log X)^5 .$$

In the same way that we derived Corollary 21.5 from Theorem 21.4, we have immediately

Corollary 21.8 *Assume GRH. Let $E(X)$ denote the number of even integers $n \le X$ such that n is not the sum of two primes. Then $E(X) \ll X^{1/2}(\log X)^5$.*

Corollary 21.9 *Assume GRH. Let*

$$\psi_3(n) = \sum_{m_1+m_2+m_3=n} \Lambda(m_1)\Lambda(m_2)\Lambda(m_3).$$

Then

$$\psi_3(n) = \frac{1}{2}\mathfrak{S}_3(n)n^2 + O\big(n^{7/4}(\log n)^3\big)$$

where $\mathfrak{S}_3(n)$ is defined as in Theorem 21.1.

Proof For even n this is trivial. Hence we may assume that n is odd. We note that

$$\psi_3(n) = \sum_{m<n} \Lambda(m)\mathfrak{S}_2(n-m)(n-m) + \sum_{m<n} \Lambda(m)\Delta(n-m) \tag{21.21}$$

where

$$\Delta(k) = \psi_2(k) - \mathfrak{S}_2(k)k .$$

By Cauchy's inequality and Theorem 21.6, the second sum in (21.21) is

$$\ll \Big(\sum_{m\le n} \Lambda(m)^2\Big)^{1/2} \big(n^{5/2}(\log n)^5\big)^{1/2} \ll n^{7/4}(\log n)^3 .$$

To estimate the first sum in (21.21), let f be the multiplicative function for which $f(2) = 0$, $f(p) = 1/(p-2)$ for $p > 2$, $f(p^r) = 0$ for $r > 1$, and put

$$C = 2\prod_{p>2}\Big(1 - \frac{1}{(p-1)^2}\Big).$$

Then

$$C\sum_{d|k} f(d) = 2\prod_{\substack{p|k\\ p>2}}\Big(1+\frac{1}{p-1}\Big)\prod_{\substack{p\nmid k\\ p>2}}\Big(1-\frac{1}{(p-1)^2}\Big). \tag{21.22}$$

This is $\mathfrak{S}_2(k)$ if k is even. Hence for odd n,

$$\sum_{m<n}\Lambda(m)\mathfrak{S}_2(n-m)(n-m) = C\sum_{d<n} f(d)\sum_{\substack{m<n\\ d|(n-m)}}\Lambda(m)(n-m) \;+\; O\big(n(\log n)^2\big).$$

Here the error term accounts for the contributions of those m that are powers of 2. If n is odd and m is a power of 2, then $\mathfrak{S}_2(n-m) = 0$, but the formula (21.22) returns a value that is $\ll \log n$.

If there is a prime p such that $p|n$ and $p|d$, then m must be a power of p. Thus the above is

$$= C\sum_{\substack{d<n\\ (d,n)=1}} f(d)\int_1^n \psi(x,d,n)\,dx + O\Big(n(\log n)^2\sum_{d<n} f(d)\Big).$$

On GRH, if χ is a character (mod q), then

$$\begin{aligned}\int_1^X \psi(x,\chi)\,dx &= \frac{-1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\frac{L'}{L}(s,\chi)\frac{X^{s+1}}{s(s+1)}\,ds\\ &= E_0(\chi)X^2/2 - \sum_\rho \frac{X^{\rho+1}}{\rho(\rho+1)} - \frac{1}{2\pi i}\int_{1/4-i\infty}^{1/4+i\infty}\frac{L'}{L}(s,\chi)\frac{X^{s+1}}{s(s+1)}\,ds.\end{aligned}$$

By Theorem 10.17, the sum over ρ is $\ll X^{3/2}\log 2q$. On GRH, the formula of Lemma 11.1 is valid when $\Re s = 1/4$, so the integral on the right above is $\ll X^{5/4}\log 2q$. Thus if $(d,n) = 1$, then

$$\int_1^n \psi(x;d,n)\,dx = \frac{n^2}{2\varphi(d)} + O\big(n^{3/2}\log 2d\big).$$

Therefore

$$\sum_{m<n}\Lambda(m)\mathfrak{S}_2(n-m)(n-m) = \frac{1}{2}Cn^2\sum_{\substack{d<n\\ (d,n)=1}}\frac{f(d)}{\varphi(d)} + O\Big(n^{3/2}(\log n)\sum_{d<n} f(d)\Big).$$

From the estimates

$$\sum_{d<n} f(d) \ll \log n, \qquad \sum_{d>n} \frac{f(d)}{\varphi(d)} \ll n^{-1}$$

we obtain the stated result, upon observing that

$$C \sum_{\substack{d=1 \\ (d,n)=1}}^{\infty} \frac{f(d)}{\varphi(d)} = \mathfrak{S}_3(n)\,.$$

□

We prove Theorem 21.7 by modifying our proof of Theorem 21.4. Corollary 21.9 could similarly be derived by modifying our proof of Theorem 21.1. Correspondingly, we could argue as above to derive Theorem 21.1 from Theorem 21.4, with the understanding that it would be necessary to derive a variant of Theorem 21.4 that counts primes with weight 1, rather than integers with weight $\Lambda(n)$. To prepare for the proof of Theorem 21.7, we first establish several useful estimates.

Lemma 21.10 *Let*

$$\psi(X, \chi, \beta) = \sum_{n \le X} \Lambda(n)\chi(n)e(n\beta),$$

and set $\psi'(X, \chi, \beta) = \psi(X, \chi, \beta) - E_0(\chi)V(\beta)$ *where* $V(\beta)$ *is defined in* (21.13). *Assume GRH. Then for* $|\beta| \le 1$,

$$\psi'(X, \chi, \beta) = -\sum_{|\gamma| \le X^2} I(\beta, \gamma) + O\big((\log qX)^2\big)$$

where

$$I(\beta, \gamma) = \int_2^X e(\beta x) x^{-1/2+i\gamma}\, dx\,.$$

Proof By the explicit formula (12.14) with $T = X^2$ we see that

$$\begin{aligned}\psi'(X, \chi, \beta) &= \int_{2^-}^X e(\beta x)\, d\psi'(x, \chi) \\ &= -\sum_{|\gamma| \le X^2} I(\beta, \gamma) + \int_{2^-}^X e(\beta x)\, dE_1(x, \chi) + \int_{2^-}^X e(\beta x)\, dE_2(x, X^2, \chi)\,.\end{aligned}$$

By (12.15) we see that the integral with respect to E_1 is $\ll (\log qX)^2$. The integral with respect to E_2 is

$$= \Big[e(\beta x)E(x, X^2, \chi)\Big|_{2^-}^X - 2\pi i\beta \int_2^X e(\beta x)E_2(x, X^2, \chi)\, dx \ll (\log qX)^2$$

by (12.16) and (12.17). Thus we have the stated result. □

Lemma 21.11 *For real numbers β and γ, let $I(\beta,\gamma)$ be defined as in the preceding lemma. If $|\gamma| \le 1$, then $I(\beta,\gamma) \ll X^{1/2}$. If $1 \le |\gamma| \le 10|\beta|X$, then $I(\beta,\gamma) \ll |\beta|^{-1/2}$. If $|\gamma| \ge 1$ and $|\gamma| \ge 10|\beta|X$, then $I(\beta,\gamma) \ll X^{1/2}/|\gamma|$.*

Proof The first estimate is trivial, since by the triangle inequality, $|I(\beta,\gamma)| \le \int_2^X x^{-1/2}\,dx$. If $|\gamma| \ge 10|\beta|U$, then by Theorem 17.1 with $r(x) = x^{-1/2}$ and $\theta(x) = 2\pi\beta x + \gamma x$ we find that

$$\int_{U/2}^{U} e(\beta x) x^{-1/2+i\gamma}\,dx \ll \frac{U^{1/2}}{|\gamma|}\,. \tag{21.23}$$

If $|\gamma| \ge 10|\beta|X$, then we apply the above with $U = X2^{-r}$, and sum, to obtain the third estimate. Suppose now that $1 \le |\gamma| \le 10|\beta|X$. If β and γ have the same sign, then (by taking complex conjugates if necessary) we may assume that they are both positive, and in this case $\theta'(x) = 2\pi\beta + \gamma/x \ge \gamma/x$, and so (21.23) again holds. If β and γ have opposite signs, then put $x_0 = -\gamma/(2\pi\beta)$, and set $J_1 = [2,X] \cap [2,x_0/2]$, $J_2 = [2,X] \cap [x_0/2, 2x_0]$, and $J_3 = [2,X] \cap [2x_0,\infty)$. Thus $I(\beta,\gamma) = \int_{J_1} + \int_{J_2} + \int_{J_3} = I_1 + I_2 + I_3$, say. We cut J_1 into dyadic blocks and apply (21.23) to see that $I_1 \ll X/|\gamma|$. We apply Theorem 17.2 with $M \asymp |\beta|^{1/2}/|\gamma|^{1/2}$ and $\mu \asymp \beta^2/|\gamma|$ to see that $I_2 \ll |\beta|^{-1/2}$. If $U \ge 2x_0$, then by Theorem l7.1 we find that

$$\int_{U}^{2U} e(\beta x) x^{-1/2+i\gamma}\,dx \ll U^{-1/2}|\beta|^{-1}\,.$$

On summing over dyadic blocks, we deduce that $I_3 \ll |\beta|^{-1/2}|\gamma|^{-1/2}$. Thus we see that if $1 \le |\gamma| \le 10|\beta|X$, then $I(\beta,\gamma) \ll |\beta|^{-1/2}$, so the proof is complete. □

From Theorem 13.7 we know that the estimate $\psi'(x,\chi) \ll x^{1/2}(\log qx)^2$ is a consequence of GRH. By integrating by parts as in the proof of Theorem 21.1, we can deduce that

$$\psi'(X,\chi,\beta) \ll X^{1/2}(1+|\beta|X)(\log qX)^2\,. \tag{21.24}$$

However, by utilizing the more detailed information provided by Lemma 21.10, and the estimates in Lemma 21.11 for the integrals $I(\beta,\gamma)$, we obtain a better estimate, as follows.

Lemma 21.12 *Assume GRH, and let $\psi'(X,\chi,\beta)$ be defined as in Lemma* 21.10*. Then*

$$\psi'(X,\chi,\beta) \ll \big(X^{1/2} + |\beta|^{1/2}X\big)(\log qX)^2\,. \tag{21.25}$$

When $|\beta|X \le 1$, the bounds (21.24) and (21.25) are comparable, but when $|\beta|X > 1$, the bound of (21.25) is smaller than that of (21.24) by a factor of $(|\beta|X)^{1/2}$. Despite this improvement over (21.24) we expect that more is true, and conjecture that

$$\psi'(X,\chi,\beta) \ll X^{1/2+\varepsilon}(\log q)^2\,. \tag{21.26}$$

Proof We may assume that $|\beta| \le 1$, for otherwise the estimate is trivial. By Lemma 21.11 we see that

$$\sum_{|\gamma|\le X^2} I(\beta,\gamma) \ll \sum_{|\gamma|\le 1} X^{1/2} + \sum_{1<|\gamma|\le 10|\beta|X} |\beta|^{-1/2} + \sum_{1<|\gamma|\le X^2} X^{1/2}|\gamma|^{-1}\,.$$

By Theorem 10.17 we know that the number of zeros $1/2+i\gamma$ of $L(s,\chi)$ with $t<\gamma\le t+1$ is $\ll \log q\tau$. Thus the above is $\ll \big(X^{1/2}+|\beta|X\big)(\log qX)^2$, and the proof is complete. □

Proof of Theorem 21.7 Let $V(\beta)$, $w(n,X)$, $\mathfrak{S}_2(n,P)$, $\psi_2(n,X)$, $S(\alpha)$ and $T(\alpha)$ be defined as in the proof of Theorem 21.4, but we now take $P=Q=X^{1/2}$ and redefine the major arcs $\mathfrak{M}(q,a)$. Let $\mathcal{F}_Q$ denote the set of Farey fractions of order Q, which is to say the set of rational numbers a/q with $1\le a\le q$, $(a,q)=1$, and $q\le Q$. Let a'/q' and a''/q'' be the neighbors of $a/q\in\mathcal{F}_Q$ with $a'/q'<a/q<a''/q''$. Then we take $\mathfrak{M}(q,a)=\big[(a+a')/(q+q'),(a+a'')/(q+q'')\big)$. Since these intervals partition $\mathbb{T}$, we have no minor arcs.

By the method used to prove Theorem 21.4 we see that it suffices to show that

$$\int_0^1 |S(\alpha)^2-T(\alpha)|^2\,d\alpha \ll X^{5/2}(\log X)^5\,. \tag{21.27}$$

For $\alpha\in\mathfrak{M}(q,a)$, let

$$W(\alpha)=\frac{\mu(q)}{\varphi(q)}V(\alpha-a/q)\,.$$

Thus $W(\alpha)^2$ is one of the terms comprising $T(\alpha)$, and

$$T(\alpha)-W(\alpha)^2\ll\sum_{r\le Q}\frac{1}{\varphi(r)^2}\sideset{}{^{**}}\sum_{b=1}^{r}\|\alpha-b/r\|^{-2}$$

where the sum over b is restricted to $(b,r)=1$ and with the further restriction that $b\ne a$ if $r=q$. Suppose that $R/2<r\le R\le Q$. Since $\alpha\notin\mathfrak{M}(r,b)$, we have $\|\alpha-b/r\|\ge 1/(r(r+r'))\ge 1/(2rQ)\ge 1/(2RQ)$ where $(b+b')/(r+r')$ is the endpoint of $\mathfrak{M}(r,b)$ lying between b/r and α. Also, if $b_1/r_1\ne b_2/r_2$, then $\|b_1/r_1-b_2/r_2\|\ge 1/(r_1r_2)\ge 1/R^2$. Therefore

$$\sum_{R/2<r\le R}\sideset{}{^{**}}\sum_{b=1}^{r}\|\alpha-b/r\|^{-2}\ll R^2Q^2+\sum_{k=1}^{\infty}(k/R^2)^{-2}\ll R^2Q^2\,.$$

Thus

$$\sum_{R/2<r\le R}\frac{1}{\varphi(r)^2}\sideset{}{^{**}}\sum_{b=1}^{r}\|\alpha-b/r\|^{-2}\ll Q^2(\log\log Q)^2,$$

whence

$$T(\alpha)-W(\alpha)^2\ll X(\log X)^2\,.$$

Thus to prove (21.27) it suffices to show that

$$\int_0^1 |S(\alpha)^2-W(\alpha)^2|^2\,d\alpha\ll X^{5/2}(\log X)^5\,. \tag{21.28}$$

To this end we first estimate $S(\alpha)-W(\alpha)$. Suppose that $\alpha \in \mathfrak{M}(q,a)$ and that $\beta = \alpha - a/q$. By the definition (9.3) of a Gauss sum and the basic orothogonality (4.15) of characters we see that

$$\frac{1}{\varphi(q)}\sum_{\chi}\tau(\overline{\chi})\chi(b) = \begin{cases} e(b/q) & \text{if } (b,q)=1, \\ 0 & \text{otherwise.} \end{cases} \tag{21.29}$$

We set $b = an$, multiply by $\Lambda(n)e(n\beta)$, and sum over $n \le X$ to see that

$$S(\alpha) = \frac{1}{\varphi(q)}\sum_{\chi}\tau(\overline{\chi})\chi(a)\psi(X,\chi,\beta) + O\Big(\sum_{\substack{n\le X\\(n,q)>1}}\Lambda(n)\Big)$$

where $\psi(X,\chi,\beta)$ is defined as in Lemma 21.10. Thus

$$S(\alpha) - \frac{\mu(q)}{\varphi(q)}V(\beta) = \frac{1}{\varphi(q)}\sum_{\chi}\tau(\overline{\chi})\chi(a)\psi'(X,\chi,\beta) + O\big((\log qX)^2\big)\,. \tag{21.30}$$

By Lemma 21.12 this is

$$\ll q^{1/2}\big(X^{1/2} + |\beta|^{1/2}X\big)(\log X)^2 \ll \big(q^{1/2}X^{1/2} + Q^{-1/2}\big)$$

since $|\beta| \le 1/(qQ)$. Thus

$$S(\alpha) - W(\alpha) \ll X^{3/4}(\log X)^2 \tag{21.31}$$

uniformly in α

By Parseval's identity,

$$\int_0^1 |S(\alpha)|^2\,d\alpha = \sum_{n\le X}\Lambda(n)^2 \ll X\log X,$$

while

$$\int_0^1 |W(\alpha)|^2\,d\alpha \ll \sum_{q\le Q}\varphi(q)^{-2}\sideset{}{^*}\sum_{a=1}^{q}\int_0^1 \min(X^2, \|\beta\|^{-2})\,d\beta \ll X\sum_{q\le Q}\varphi(q)^{-1} \ll X\log X\,.$$

Thus from (21.31) we deduce that

$$\int_0^1 |S(\alpha)^2 - W(\alpha)^2|^2\,d\alpha \ll X^{3/2}(\log X)^4\int_0^1 |S(\alpha)+W(\alpha)|^2\,d\alpha \ll X^{5/2}(\log X)^5\,.$$

Thus we have (21.28), and the proof is complete. □

The argument just completed may be expected to be inefficient in two respects. Some considerable cancellation should occur in the sum over χ in (21.30), and we also expect that

the bound in Lemma 21.12 is weaker than the truth. Indeed, we expect that $S(\alpha)-W(\alpha) \ll X^{1/2+\varepsilon}$ for all α. It would then follow that

$$\sum_{n\le X} \big(\psi_2(n) - \mathfrak{S}_2(n)n\big)^2 \ll X^{2+\varepsilon}\,. \tag{21.32}$$

While we are unable to establish that the sum in (21.30) cancels uniformly, we can at least demonstrate the cancellation in mean square. By orthogonality,

$$\sum_{a=1}^{q}{}^{*} \Big|S(a/q+\beta) - \frac{\mu(q)}{\varphi(q)}V(\beta)\Big|^2 \ll q(\log xQ)^4 + \frac{1}{\varphi(q)}\sum_{\chi}|\tau(\overline{\chi})|^2|\psi'(X,\chi,\beta)|^2, \tag{21.33}$$

which by Lemma 21.12 is

$$\ll (qX + q|\beta|X^2)(\log qX)^4\,. \tag{21.34}$$

For $|\beta| \le q^{-1}X^{-1/2}$, $q \le X^{1/2}$, this is uniformly $\ll X^{3/2}(\log X)^4$. By comparison, if were to estimate the left hand side by applying (21.31) for each a, then the bound we would obtain would be much worse, namely $\ll X^2(\log X)^4$.

If $(a,q)=1$, then $a/q+\beta \in \mathfrak{M}(q,a)$ precisely when

$$\frac{-1}{q(q+q')} \le \beta \le \frac{1}{q(q+q'')}\,.$$

Since the dimensions of this interval depend on a, we are not immediately able to apply (21.34). To circumvent this difficulty, we replace $\mathfrak{M}(q,a)$ by the slightly larger interval $\mathfrak{M}^*(q,a) = (a/q - 1/(qQ), a/q + 1/(qQ))$. Hence

$$\begin{aligned}\int_0^1 |W(\alpha)(S(\alpha)-W(\alpha))|^2\,d\alpha \\ &\le \sum_{q\le Q}\varphi(q)^{-2}\sum_{a=1}^{q}{}^{*}\int_{-1/(qQ)}^{1/(qQ)} |V(\beta)|^2\Big|S(a/q+\beta) - \frac{\mu(q)}{\varphi(q)}V(\beta)\Big|^2\,d\beta\,.\end{aligned}$$

By (21.34) this is

$$\begin{aligned}&\ll \sum_{q\le Q}\varphi(q)^{-2}\int_0^1 \min\big(X^2,\beta^{-2}\big)\big(qX+q\beta X^2\big)(\log X)^4\,d\beta \\ &\ll X^2(\log X)^5\sum_{q\le Q} q\varphi(q)^{-2} \ll X^2(\log X)^6\,. \end{aligned}\tag{21.35}$$

Thus to sharpen Theorem 21.7 it suffices to improve our bound for

$$\int_0^1 |S(\alpha)(S(\alpha)-W(\alpha))|^2\,d\alpha,$$

or equivalently, for $\int_0^1 |S(\alpha)-W(\alpha)|^4\,d\alpha$.

We next show that Lemma 21.12 can similarly be improved in mean square with respect to β.

Theorem 21.13 *Assume GRH. Let $\delta > 0$, and let χ be any character modulo q. Then*

$$\int_{-\delta}^{\delta} |\psi'(X,\chi,\beta)|^2\, d\beta \ll \delta X (\log qX)^4 .$$

This with (21.33) gives

Corollary 21.14 *Assume GRH. Then*

$$\sum_{a=1}^{q}{}^{*} \int_{-\delta}^{\delta} \Big| S(a/q+\beta) - \frac{\mu(q)}{\varphi(q)} V(\beta)\Big|^2 d\beta \ll \delta q X (\log qX)^4 .$$

Proof of Theorem 21.13 If $\delta X \le 1$, then it suffices to appeal to Lemma 21.12. Thus we assume that $\delta X \ge 1$. We may also assume that $\delta \le 1$, since it is trivial that

$$\int_{0}^{1} |\psi'(X,\chi,\beta)|^2\, d\beta \ll X \log X .$$

By Lemma 21.10 we see that

$$\int_{-\delta}^{\delta} |\psi'(X,\chi,\beta)|^2\, d\beta \ll \int_{-\delta}^{\delta} \Big| \sum_{|\gamma|\le X^2} I(\beta,\gamma)\Big|^2 d\beta + \delta (\log qX)^4 .$$

By Lemma 21.11 we know that $I(\beta,\gamma) \ll X^{1/2}|\gamma|^{-1}$ when $|\gamma| \ge 10\delta X$. Thus

$$\int_{-\delta}^{\delta} \Big| \sum_{10\delta X<|\gamma|\le X^2} I(\beta,\gamma)\Big|^2 d\beta \ll \delta X (\log qX)^4 .$$

On the other hand,

$$\int_{-\delta}^{\delta} \Big| \sum_{|\gamma|\le 10\delta X} I(\beta,\gamma)\Big|^2 d\beta \le \int_{-\infty}^{\infty} \Big| \sum_{|\gamma|\le 10\delta X} \int_2^X e(\beta x) x^{-1/2+i\gamma}\, dx\Big|^2 d\beta,$$

which by Plancherel's formula is

$$= \int_2^X \Big| \sum_{|\gamma|\le 10\delta X} x^{i\gamma}\Big|^2 \frac{dx}{x} .$$

We make the change of variable $x = e^y$, and note that

$$\int_Y^{Y+1} \Big| \sum_{|\gamma|\le T} e^{i\gamma y}\Big|^2 dy \ll \sum_{\substack{|\gamma|\le T\\ |\gamma'|\le T}} \min(1, 1/|\gamma-\gamma'|) .$$

For any given γ, the sum over γ' is $\ll (\log qT)^2$, as we see by using the bound of Theorem 10.17 in the same way that we did in the proof of Corollary 21.9. The number of γ is $\ll T \log qT$, so the above is $\ll T(\log qT)^3$. On taking $T = 10\delta X$, and sum over $\ll \log X$ values of Y to see that

$$\int_{-\delta}^{\delta} \Big| \sum_{|\gamma|\le 10\delta X} I(\beta,\gamma)\Big|^2 d\beta \ll \delta X (\log qX)^4 .$$

Thus the proof is complete. □

21.3.1 Exercises

1. Let $\mathfrak{M}(q,a)$ and $\mathfrak{M}^*(q,a)$ be defined as in the proof of (21.35).
(a) Show that if a/q and a'/q' are neighbouring members of $\mathcal{F}_Q$, then $\mathfrak{M}^*(q,a)$ and $\mathfrak{M}^*(q',a')$ overlap.
(b) Show that if a/q and a'/q' are neighbouring members of $\mathcal{F}_Q$, then $a/q \notin \mathfrak{M}^*(q',a')$.
(c) Conclude that every α is in at least one, but not more than two of the arcs $\mathfrak{M}^*(q,a)$.

2. (a) By introducing appropriate weights before expanding and integrating, show that

$$\int_Y^{Y+1} \Big| \sum_{|\gamma|\le T} e^{i\gamma} \Big|^2 dy \ll T(\log qT)^2 .$$

Here the γ's are the imaginary parts of zeros of $L(s,\chi)$ and χ is a character modulo q.
(b) In the context of the proof of Theorem 21.13, show that

$$\int_{-\delta}^{\delta} \Big| \sum_{|\gamma|\le 10\delta X} I(\beta,\gamma) \Big|^2 d\beta \ll \delta X(\log qX)^3 .$$

3. Let f and C be defined as in the proof of Corollary 21.9.
(a) Show that

$$\sum_{n\le X} \mathfrak{S}_2(n) = C \sum_{m\le X/2} \sum_{d\mid 2m} f(d) .$$

(b) Deduce that

$$\sum_{n\le X} \mathfrak{S}_2(n) = \frac{1}{2} CX \sum_{d\le X/2} f(d)/d + O\Big(\sum_{d\le X/2} f(d) \Big) .$$

(c) Show that

$$\sum_{d=1}^{\infty} \frac{f(d)}{d} = \frac{2}{C} .$$

(d) Conclude that

$$\sum_{n\le X} \mathfrak{S}_2(n) = X + O(\log X) .$$

21.4 A lower bound for the error term

We have estimated the mean square error in the Goldbach problem, and discussed the plausibility of sharper estimates such as (21.32). We now establish a bound in the opposite direction.

Theorem 21.15 *Suppose that* $1/2 < r < 1$, *and let* $R = 1/(1-r)$. *Then*

$$\sum_{n=1}^{\infty} \big(\psi_2(n) - \mathfrak{S}_2(n)n\big)^2 r^{2n} \gg R^2(\log R)^2 .$$

Corollary 21.16 *As* X *tends to infinity,*

$$\sum_{n\le X} \big(\psi_2(n) - \mathfrak{S}_2(n)n\big)^2 = \Omega\big(X^2(\log X)^2\big),$$

and

$$\psi_2(n) - \mathfrak{S}_2(n)n = \Omega\big(n^{1/2}\log n\big) .$$

Proof of Theorem 21.15 By Lemma 21.5 we see that

$$\sum_{N<n\le 2N} (n+1)^2\big(\mathfrak{S}_2(n) - \mathfrak{S}_2(n,Q)\big)^2 r^{2n} \ll Q^{-2}r^{2N}N^3(\log N)^3 .$$

On setting $N = 2^k R$ and summing over k, we deduce that

$$\sum_{n=1}^{\infty}(n+1)^2\big(\mathfrak{S}_2(n) - \mathfrak{S}_2(n,Q)\big)^2 r^{2n} \ll Q^{-2}R^3(\log R)^3 .$$

Since

$$\sum_{n=1}^{\infty} \mathfrak{S}_2(n)^2 r^{2n} \ll R,$$

it follows that

$$\sum_{n=1}^{\infty} \big(\mathfrak{S}_2(n)n - \mathfrak{S}_2(n,Q)(n+1)\big)^2 r^{2n} \ll Q^{-2}R^3(\log R)^3 .$$

We take $Q = R^\kappa$ with $1/2 < \kappa < 1$. Thus it suffices to show that

$$\sum_{n=1}^{\infty} \big(\psi_2(n) - \mathfrak{S}_2(n,Q)(n+1)\big)^2 \gg R^2(\log R)^2 .$$

By Parseval's identity the left hand side is T_2 where

$$T_k = \int_0^1 \Big|S(\alpha)^2 - \sum_{q\le Q} \frac{\mu(q)^2}{\varphi(q)^2} \sideset{}{^*}\sum_{a=1}^{q} (1 - re(\alpha - a/q))^{-2}\Big|^k \, d\alpha$$

and

$$S(\alpha) = \sum_{n=1}^{\infty} \Lambda(n) r^n e(n\alpha) .$$

By Cauchy's inequality, $T_2 \geq T_1^2$. But

$$\begin{aligned} T_1 &\geq \int_0^1 \Big| \sum_{n=1}^{\infty} \Lambda(n) r^n e(n\alpha) \Big| \, d\alpha - \sum_{q \leq Q} \frac{\mu(q)^2}{\varphi(q)^2} \sideset{}{^*}\sum_{a=1}^{q} \int_0^1 |1 - re(\alpha - a/q)|^{-2} \, d\alpha \\ &= \sum_{n=1}^{\infty} \Lambda(n)^2 r^{2n} - \sum_{q \leq Q} \frac{\mu(q)^2}{\varphi(q)^2} (1 - r^2)^{-1}. \end{aligned}$$

Here the first sum is $\frac{1}{2} R \log R + O(R)$, and the sum over q is $\frac{1}{2} R \log Q + O(R)$, in view of Exercise 1.2.1.17. Thus the above is $\frac{1}{2} R \log R/Q \gg R \log R$, so the proof is complete. □

21.4.1 Exercises