Chapter 27

Probabilistic Number Theory

We say that an arithmetic function f is *additive* if

$$f(mn) = f(m) + f(n) \tag{27.1}$$

whenever $(m, n) = 1$. The values of an additive function are determined by its values on prime-powers, since

$$f(n) = \sum_{p^k \| n} f\big(p^k\big)\,. \tag{27.2}$$

If the identity (27.1) holds for all pairs m, n, then we say that f is *totally additive*. If f is additive and $f\big(p^k\big) = f(p)$ for all p and all $k \geq 1$, then we say that f is *strongly additive*. For example, $\log n$ and $\Omega(n)$ are totally additive functions, $\log n/\varphi(n)$ and $\omega(n)$ are strongly additive, while $\Omega(n) - \omega(n)$ and $\log d(n)$ are additive but neither totally additive nor strongly additive.

In sieve theory we have seen that things do not always work out as one would expect on probabilistic grounds. However, we find that the distribution of the values of an additive function follow the natural probabilistic model very closely. Suppose that f is an additive function. The asymptotic density of integers n for which $p^k \| n$ is $p^{-k}(1 - 1/p)$. It is with this 'probability' that the term $f\big(p^k\big)$ is one of the terms in the sum (27.2). Accordingly, for each prime number p we define a random variable X_p that has the distribution

$$\begin{aligned} P\big(X_p = f\big(p^k\big)\big) &= \frac{1}{p^k}\Big(1 - \frac{1}{p}\Big) \qquad (k = 1, 2, \dots) \\ P\big(X_p = 0\big) &= 1 - \frac{1}{p}\,. \end{aligned} \tag{27.3}$$

If p and q are distinct primes, then by the Chinese remainder theorem we see that the asymptotic density of the integers n for which both $p^k \| n$ and $q^\ell \| n$ is $p^{-k}(1 - 1/p)q^{-\ell}(1 - 1/q)$. Hence the two events $p^k \| n$ and $q^\ell \| n$ are asymptotically independent. Thus it is natural to take the random variables X_p to be independent, and we set

$$X = \sum_p X_p\,. \tag{27.4}$$

This sum either converges almost always or almost nowhere. When it converges almost always, we find that the values of f have a limiting distribution that is the same as the distribution of X. When it converges almost nowhere, f does not have a limiting distribution.

We have already encountered a scattering of results concerning a few additive functions. In §2.3 we estimated the mean of $\omega(n)$, and also its variance about its mean. In §2.4 we determined the distribution of the additive function $\Omega(n)-\omega(n)$ by calculating the mean value of the multiplicative function $z^{\Omega(n)-\omega(n)}$. In §7.4 we put

$$\alpha_n = \frac{\Omega(n)-\log\log n}{\sqrt{\log\log n}},$$

and found that the distribution of α_n is asymptotically normal with mean 0 and variance 1. In this chapter we are more concerned with developing a general theory than with special examples.

27.1 The Turán–Kubilius inequality

Turán (1934) showed (cf Theorem 2.12) that

$$\sum_{n\le}(\omega(n)-\log\log x)^2 \ll x\log\log x,$$

and Kubilius (1955) generalized this to arbitrary additive functions.

Suppose that f is an additive function. Then

$$\begin{aligned}\sum_{n\le x} f(n) &= \sum_{n\le x}\sum_{p^k\|n} f\big(p^k\big)\\ &= \sum_{p^k\le x} f\big(p^k\big)\sum_{\substack{n\le x\\ p^k\|n}} 1\\ &= \sum_{p^k\le x} f\big(p^k\big)\Big(\Big[\frac{x}{p^k}\Big]-\Big[\frac{x}{p^{k+1}}\Big]\Big)\\ &= x\sum_{p^k\le x}\frac{f\big(p^k\big)}{p^k}\Big(1-\frac{1}{p}\Big)+O\Big(\sum_{p^k\le x}\big|f\big(p^k\big)\big|\Big). \qquad (27.5)\end{aligned}$$

For ease of reference, we set

$$A(x)=A(f,x)=\sum_{p^k\le x}\frac{f\big(p^k\big)}{p^k}\Big(1-\frac{1}{p}\Big). \qquad (27.6)$$

We anticipate that the variance of f about its mean should not be much more than

$$B(x)=B(f,x)=\sum_{p^k\le x}\frac{\big|f\big(p^k\big)\big|^2}{p^k}. \qquad (27.7)$$

By Cauchy's inequality,

$$\sum_{p^k\le x} |f(p^k)| \;\le\; B(x)^{1/2}\Big(\sum_{p^k\le x} p^k\Big)^{1/2} \;\ll\; B(x)^{1/2}\frac{x}{\sqrt{\log x}}\,.$$

Thus from (27.5) we see that

$$\sum_{n\le x} f(n) = A(x)x + O\big(B(x)^{1/2}x(\log x)^{-1/2}\big)\,. \tag{27.8}$$

As concerns the potential size of $A(x)$ relative to $B(x)$, we note by Cauchy's inequality that

$$\begin{aligned} |A(x)|^2 &\le \Big(\sum_{p^k\le x}\frac{|f(p^k)|^2}{p^k}\Big)\Big(\sum_{p^k\le x}\frac{1}{p^k}\Big) \\ &\ll B(x)\log\log x\,. \end{aligned} \tag{27.9}$$

We now show that $B(x)$ is within a constant factor of being an upper bound for the variance of the values of f about its mean.

Theorem 27.1 (The Turán–Kubilius inequality) *Suppose that f is an additive function, and let $A(x)$ and $B(x)$ be defined as in* (27.6) *and* (27.7)*. Then*

$$\sum_{n\le x}\big(f(n)-A(x)\big)^2 \;\ll\; xB(x)\,. \tag{27.10}$$

The implicit constant in the above is absolute—independent of both f and x. From (27.9) we see that

$$\sum_{n\le x}|A(x)|^2 \ll xB(x)\log\log x,$$

and by (27.10) it follows that also

$$\sum_{n\le x}|f(n)|^2 \ll xB(x)\log\log x\,.$$

Thus we see that (27.10) is never more than a factor of $\log\log x$ from being trivial. Despite this lack of quantitative depth, the Turán–Kubilius inequality turns out to be a quite useful result.

Proof We expand the sum on the left hand side, and obtain three terms. The simplest is

$$\begin{aligned} T_0 &= \sum_{n\le x}|A(x)|^2 = |A(x)|^2[x] = |A(x)|^2x + O\big(|A(x)|^2\big) \\ &= x|A(x)|^2 + O(B(x)\log\log x) \end{aligned} \tag{27.11}$$

by (27.9). The intermediate term is $T_1 = -2\Re\overline{A(x)}\sum_{n\le x} f(n)$. Thus by (27.8),

$$T_1 = -2|A(x)|^2 x + O\big(|A(x)|B(x)^{1/2}x(\log x)^{-1/2}\big),$$

and by (27.9) this is

$$= -2|A(x)|^2 x + O\big(B(x)x(\log x)^{-1/2}(\log\log x)^{1/2}\big)\,. \tag{27.12}$$

Finally,

$$T_2 = \sum_{n\le x}|f(n)|^2 = \sum_{n\le x}\sum_{p^k\|n} f\big(p^k\big)\sum_{q^\ell\|n}\overline{f}\big(q^\ell\big) = \sum_{p^k\le x}\sum_{q^\ell\le x} f\big(p^k\big)\overline{f}\big(q^\ell\big)\sum_{\substack{n\le x\\ p^k\|n\\ q^\ell\|n}} 1$$

where q denotes a prime number. The contribution of those terms for which $p = q$ is

$$T_2' = \sum_{p^k\le x}\big|f\big(p^k\big)\big|^2\Big(\Big[\frac{x}{p^k}\Big] - \Big[\frac{x}{p^{k+1}}\Big]\Big) \le xB(x)\,. \tag{27.13}$$

The remaining terms contribute

$$T_2'' = \sum_{\substack{p^kq^\ell\le x\\ p\ne q}} f\big(p^k\big)\overline{f}\big(q^\ell\big)\Big(\Big[\frac{x}{p^kq^\ell}\Big] - \Big[\frac{x}{p^{k+1}q^\ell}\Big] - \Big[\frac{x}{p^kq^{\ell+1}}\Big] + \Big[\frac{x}{p^{k+1}q^{\ell+1}}\Big]\Big)$$

$$= x\sum_{p^kq^\ell\le x}\frac{f\big(p^k\big)}{p^k}\,\frac{\overline{f}\big(q^\ell\big)}{q^\ell}\Big(1-\frac{1}{p}\Big)\Big(1-\frac{1}{q}\Big) \tag{27.14}$$

$$+ O\Big(\sum_{p^kq^\ell\le x}\big|f\big(p^k\big)\overline{f}\big(q^\ell\big)\big|\Big) + O\Big(x\sum_{p^{k+\ell}\le x}\frac{\big|f\big(p^k\big)\overline{f}\big(p^\ell\big)\big|}{p^{k+\ell}}\Big)\,. \tag{27.15}$$

By Cauchy's inequality the first error term is

$$\ll \Big(\sum_{p^kq^\ell\le x}\frac{\big|f\big(p^k\big)\big|^2\big|f\big(q^\ell\big)\big|^2}{p^kq^\ell}\Big)^{1/2}\Big(\sum_{p^kq^\ell\le x}p^kq^\ell\Big)^{1/2}.$$

Here the first sum is $\le B(x)^2$, and the second sum is

$$\le x\sum_{\substack{n\le x\\ \omega(n)\le 2}} 1 \ll x^2(\log x)^{-1}\log\log x$$

by (7.54). By Cauchy's inequality the second error term in (27.15) is

$$\ll x\Big(\sum_{p^{k+\ell}\le x}\frac{\big|f\big(p^k\big)\big|^2}{p^{k+\ell}}\Big)^{1/2}\Big(\sum_{p^{k+\ell}\le x}\frac{\big|f\big(p^\ell\big)\big|^2}{p^{k+\ell}}\Big)^{1/2} = x\sum_{p^k\le x}\frac{\big|f\big(p^k\big)\big|^2}{p^k}\sum_{\substack{\ell\\ p^\ell\le x/p^k}}\frac{1}{p^\ell} \ll xB(x)\,.$$

The expression (27.14) is

$$= x|A(x)|^2 - x \sum_{\substack{p^k \le x \\ q^\ell \le x \\ p^k q^\ell > x}} \frac{f(p^k)}{p^k} \frac{\overline{f}(q^\ell)}{q^\ell} \Big(1 - \frac{1}{p}\Big)\Big(1 - \frac{1}{q}\Big) .$$

By Cauchy's inequality the sum on the right is

$$\le \Big(\sum_{\substack{p^k \le x \\ q^\ell \le x \\ p^k q^\ell > x}} \frac{\big|f(p^k)\big|^2}{p^k} \frac{\big|f\big(q^\ell\big|^2\big)}{q^\ell} \Big)^{1/2} \Big(\sum_{\substack{p^k \le x \\ q^\ell \le x \\ p^k q^\ell > x}} \frac{1}{p^k q^\ell} \Big)^{1/2} .$$

Here the first sum is $\le B(x)^2$, and the second sum is

$$= 2 \sum_{p^k \le x^{1/2}} \frac{1}{p^k} \sum_{x/p^k < q^\ell \le x} \frac{1}{q^\ell} + \Big(\sum_{x^{1/2} < p^k \le x} \frac{1}{p^k} \Big)^2 \ll \sum_{p^k \le x^{1/2}} \frac{k \log p}{p^k \log x} + 1 \ll 1 .$$

On assumbling our estimates we deduce that

$$T_2'' = x|A(x)|^2 + O(x(B(x))) .$$

The stated result now follows by combining this with (27.11–13). □

27.1 Exercises

1. Show that almost all integers n have $(1/2 + o(1)) \log\log n$ prime factors $\equiv 1 \pmod 4$.

2. Let k be a fixed positive integer. Show that $d_k(n) = (\log n)^{(1+o(1))\log k}$ for almost all integers.

3. Show that
$$\sum_{n \le x} \Omega(n)\Omega(n+k) = x(\log\log x)^2 + cx \log\log x + O(x)$$
where
$$c =$$

4. Show that $\sum_{n \le x} \big(\omega\big(n^2+1\big) - \log\log n\big)^2 \ll x \log\log x$.

5. Show that $\sum_{p \le x} \big(\omega(p+1) - \log\log p\big)^2 \ll x \log\log x$.

6. Suppose that f is an additive function, and let $A(x)$ and $B(x)$ be defined as in (27.6) and (27.7).
(a) Show that if $n \le x$, then

$$|A(n) - A(x)|^2 \le B(x) \sum_{n<p^k\le x} \frac{1}{p^k}\,.$$

(b) Show that

$$\sum_{n\le x} |A(n) - A(x)|^2 \ll B(x)x/\log x\,.$$

(c) Conclude that

$$\sum_{n\le x} |f(n) - A(n)|^2 \ll xB(x)\,.$$

7. (a) Show that

$$\Big|\sum_{p^k\le x} \frac{f\big(p^k\big)}{p^{k+1}}\Big|^2 \ll B(x)\,.$$

(b) Show that

$$\Big|\sum_{\substack{p^k\le x\\ k>1}} \frac{f\big(p^k\big)}{p^{k}}\Big|^2 \ll B(x)\,.$$

(c) Put $A'(x) = \sum_{p\le x} f(p)/p$. Show that if f is an additive function, then

$$\sum_{n\le x} |f(n) - A'(x)|^2 \ll xB(x)\,.$$

8. The *Kubilius class* $\mathcal{H}$ consists of those additive functions f with the two properties

(i) $\quad B(f,x) \to \infty$ as $x \to \infty$;

(ii) $$\sum_{x^{1/2}<p^k\le x} \frac{\big|f\big(p^k\big)\big|^2}{p^k} = o(B(x)) \qquad (x\to\infty).$$

Show that if $f \in \mathcal{H}$, then

$$\sum_{n\le x} |f(n) - A(x)|^2 = (1+o(1))xB(x)$$

as $x \to \infty$.

9. Let $f(n) = \log n$.
(a) Show that $A(x) = \log x + O(1)$.
(b) Show that $B(x) = \frac{1}{2}(\log x)^2 + O(\log x)$.

(c) Deduce that $f \notin \mathcal{H}$.
(d) Show that $\sum_{n\le x} |f(n) - A(x)|^2 \ll x$.

10. Suppose that f is strongly additive, so that $f(n) = \sum_{p|n} f(p)$ for all n. Consider the bilinear form inequality

$$\sum_{n\le x} \Big| \sum_{p|n} f(p) - \sum_{p\le x} \frac{f(p)}{p} \Big|^2 \le \Delta \sum_{p\le x} \frac{|f(p)|^2}{p} \tag{27.16}$$

in the variables $f(p)$.
(a) By the change of variables $g(p) = f(p)/\sqrt{p}$, show that the above is equivalent to the bilinear form inequality

$$\sum_{n\le x} \Big| \sum_{p|n} g(p)p^{1/2} - \sum_{p\le x} \frac{g(p)}{p^{1/2}} \Big|^2 \le \Delta \sum_{p\le x} |g(p)|^2 . \tag{27.17}$$

(b) Use Theorem F.1 to show that the above is equivalent to the bilinear form inequality

$$\sum_{p\le x} p \Big| \sum_{\substack{n\le x \\ p|n}} h(n) - \frac{1}{p} \sum_{n\le x} h(n) \Big|^2 \le \Delta \sum_{n\le x} |h(n)|^2 \tag{27.18}$$

in the variables $h(n)$.
(c) Apply the large sieve, as discussed in §19.3 to show that

$$\sum_{p\le x^{1/2}} p \Big| \sum_{\substack{n\le x \\ p|n}} h(n) - \frac{1}{p} \sum_{n\le x} h(n) \Big|^2 \ll x \sum_{n\le x} |h(n)|^2 .$$

(d) Show that if $x^{1/2} < p \le x$, then

$$\Big| \sum_{\substack{n\le x \\ p|n}} h(n) \Big|^2 \ll \frac{x}{p} \sum_{\substack{n\le x \\ p|n}} |h(n)|^2 .$$

(e) Show that

$$\sum_{x^{1/2}<p\le x} p \Big| \sum_{\substack{n\le x \\ p|n}} h(n) \Big|^2 \ll x \sum_{n\le x} |h(n)|^2 .$$

(f) Show that

$$\sum_{x^{1/2}<p\le x} \frac{1}{p} \Big| \sum_{n\le x} h(n) \Big|^2 \ll x \sum_{n\le x} |h(n)|^2 .$$

(g) Deduce that (27.18) and hence also (27.16) holds with $\Delta \ll x$.

27.2 Mean values of multiplicative functions

Let f be a multiplicative function, which is to say that $f(mn) = f(m)f(n)$ whenever $(m,n) = 1$. Hence $f(n) = \prod_{p^k \| n} f\big(p^k\big)$. We let $\mathcal{M}_0$ denote the class of those multiplicative functions f for which $|f(n)| \le 1$ for all n. Our object is to characterize those members of $\mathcal{M}_0$ that have an asymptotic mean value. If f is a real-valued additive function, then $e(tf(n)) \in \mathcal{M}_0$, so an ability to compute mean values of multiplicative functions will help us to determine the Fourier transform of the distribution of additive functions. We begin with several simple results concerning (not necessarily multiplicative) arithmetic functions.

Theorem 27.2 *If $f(n) = \sum_{d|n} g(d)$, if the series $\sum_{d=1}^{\infty} g(d)/d$ converges, say to a, and if $\sum_{d\le x} |g(d)| = o(x)$ as $x \to \infty$, then*

$$S(x) = \sum_{n\le x} f(n) = ax + o(x)\,. \tag{27.19}$$

Proof Clearly

$$S(x) = \sum_{n\le x}\sum_{d|n} g(d) = \sum_{d\le x} g(d)[x/d] = x\sum_{d\le x} g(d)/d + O\Big(\sum_{d\le x}|g(d)|\Big).$$

Thus we have the stated result. □

Corollary 27.3 (Wintner) *If $f(n) = \sum_{d|n} g(d)$ and $\sum_{d=1}^{\infty} |g(d)|/d < \infty$, then* (27.19) *holds with $a = \sum_{d=1}^{\infty} g(d)/d$.*

Proof From the hypothesis that $\sum_{d=1}^{\infty} |g(d)|/d < \infty$, it follows by partial summation that $\sum_{d\le x} |g(d)| = o(x)$. □

Corollary 27.4 *If f is multiplicative, if*

$$\sum_{p} \frac{|1 - f(p)|}{p} < \infty, \tag{27.20}$$

and if

$$\sum_{\substack{p^k \\ k>1}} \frac{\big|f\big(p^k\big)\big|}{p^k} < \infty,$$

then (27.19) *holds with*

$$a = \prod_{p} \Big(1 - \frac{1}{p}\Big)\Big(1 + \frac{f(p)}{p} + \frac{f\big(p^2\big)}{p^2} + \cdots\Big)\,. \tag{27.21}$$

Proof Let g be the multiplicative function for which $g\big(p^k\big) = f\big(p^k\big) - f\big(p^{k-1}\big)$. Then $f(n) = \sum_{d|n} g(d)$, and

$$\sum_{d=1}^{\infty} \frac{|g(d)|}{d} = \prod_{p} \Big(1 + \frac{|f(p) - 1|}{p} + \frac{|f(p^2) - f(p)|}{p^2} + \cdots\Big) < \infty,$$

and thus

$$\sum_{d=1}^{\infty}\frac{g(d)}{d}=\prod_{p}\Big(1+\frac{f(p)-1}{p}+\frac{f(p^2)-f(p)}{p^2}+\cdots\Big)=a$$

where a is defined by (27.21). □

In the same vein we have

Theorem 27.5 *If $f(n)=\sum_{d|n}g(d)h(n/d)$, if $\sum_{m=1}^{\infty}|h(m)|/m<\infty$, and if $\sum_{d\le x}g(d)=bx+o(x)$, then we have* (27.19) *with $a=b\sum_{m=1}^{\infty}h(m)/m$.*

Here we see that a mean value for g yields one for f, provided that f is near g in the sense that $\sum|h(m)|/m<\infty$. If $h(1)=1$ and $h(m)=0$ for all $m>1$, then $f=g$.

Proof Put

$$r(x)=\sum_{d\le x}g(d)\ -\ bx\,.$$

Then

$$S(x)=\sum_{n\le x}\sum_{d|n}g(d)h(n/d)=\sum_{m\le x}h(m)\sum_{d\le x/m}g(d)=bx\sum_{m\le x}h(m)/m+\sum_{m\le x}h(m)r(x/m)\,.$$

There is a constant C (depending on g) such that $|r(x)|\le Cx$ for all $x\ge 1$, and for every $\varepsilon>0$ there is a δ such that $|r(x)|\le\varepsilon x$ for all $x\ge 1/\delta$. Thus the second sum above has absolute value not exceeding

$$\varepsilon x\sum_{m\le\delta x}|h(m)|/m\ +\ Cx\sum_{\delta x<m\le x}|h(m)|/m\,.$$

Here the first term is $\ll\varepsilon x$, and the second sum is small since it is part of the tail of a convergent series. Thus we have the stated result. □

In Theorem 27.2 we found a connection between the mean value of f and the convergence of the series $\sum g(d)/d$, but we find it more productive to pursue the line suggested by Corollary 27.4, which we now sharpen.

Theorem 27.6 (Delange) *Suppose that $f\in\mathcal{M}_0$, and that the series*

$$\sum_{p}\frac{1-f(p)}{p}\tag{27.22}$$

converges. Then (27.19) *holds with a given by* (27.21).

Since $\Re f(p)\le|f(p)|\le 1$, we see that the convergence of the series (27.22) implies that the sum of the real parts is absolutely convergent, just as it was in Corollary 27.4. Thus Theorem 27.6 is stronger by virtue of the fact that we are no longer assuming that the sum of the imaginary parts is absolutely convergent. Given the convergence of the product

(27.21), we see that $a \neq 0$ unless one of the individual factors vanishes. This happens only in the single case that

$$f\big(2^k\big) = -1 \qquad (k = 1, 2, 3, \dots)\,. \tag{27.23}$$

Proof We suppose first that in addition to the stated hypotheses, f has the further properties that

$$f\big(p^k\big) = f(p)^k \qquad (k = 1, 2, \dots), \tag{27.24}$$

and

$$\Re f(p) \geq 1/2 \tag{27.25}$$

for all p. Once we have established the theorem for such f, we extend the result to general f by an appeal to Theorem 27.5. Let P be a large parameter, let $\mathcal{P}_1$ denote the set of primes not exceeding P, and let $\mathcal{P}_2$ denote the primes larger than P. Let f_i be multiplicative, $f_i\big(p^k\big) = f(p)^k$ for $p \in \mathcal{P}_i$, and $f_i\big(p^k\big) = 1$ for $p \notin \mathcal{P}_i$. Thus $f = f_1 f_2$, and by Corollary 27.4,

$$\sum_{n \leq x} f_1(n) = a(P)x + o(x)$$

with

$$a(P) = \prod_{p \leq P} \Big(1 - \frac{1}{p}\Big)\Big(1 - \frac{f(p)}{p}\Big)^{-1}\,.$$

Since $|f\big(p^k\big)| \leq 1$, we see by (27.24) and (27.25) that we may write $f_2(n) = e^{g(n)}$ where g is an additive function such that $-\log 2 \leq \Re g(p) \leq 0$ and $|\Im g(p)| \leq \pi/3$. If $\Re z \leq 0$, then $\big|e^z - 1\big| = \big|\int_0^z e^w\,dw\big| \leq |z|$, so we see that

$$\Big|\sum_{n \leq x} f(n) - f_1(n)\Big| \leq \sum_{n \leq x} \big|f_2(n) - 1\big| \leq \sum_{n \leq x} |g(n)|\,.$$

Let $A(x) = A(g, x)$ be defined as in (27.6). Then by Cauchy's inequality the above is

$$\leq x|A(x)| + \sum_{n \leq x} |g(n) - A(x)| \leq x|A(x)| + x^{1/2}\Big(\sum_{n \leq x} |g(n) - A(x)|^2\Big)^{1/2},$$

and by the Turán–Kubilius inequality (Theorem 27.1), this is

$$\ll x|A(x)| + xB(x)^{1/2}\,.$$

We now relate $A(x)$ and $B(x)$ to $\sum_p (1 - f(p))/p$. While the imaginary part of this sum is not necessarily absolutely convergent, the real part of each term is non-negative, and so

the sum of the real parts is absolutely convergent. Also, $|1-f(p)| = 1-2\Re f(p)+|f(p)|^2 \le 2-2\Re f(p)$, $g\big(p^k\big) \ll k$, and $|g(p)| \asymp |1-f(p)|$, so that

$$B(x) = \sum_{p^k\le x} \frac{\big|g\big(p^k\big)\big|}{p^k} \ll \sum_{p>P} \frac{1}{p^2} + \sum_{P<p\le x} \frac{|1-f(p)|^2}{p} \ll \frac{1}{P} + \Re \sum_{P<p\le x} \frac{1-f(p)}{p}\,.$$

We also observe that $g(p) = f(p) - 1 + O\big(|1-f(p)|^2\big)$, so that

$$\begin{aligned} A(x) = \sum_{p^k\le x} \frac{g\big(p^k\big)}{p^k}\Big(1-\frac{1}{p}\Big) &= \sum_{P<p\le x} \frac{f(p)-1}{p} + O\Big(\sum_{P<p\le x} \frac{|1-f(p)|^2}{p}\Big) + O\Big(\sum_{P<p\le x} \frac{1}{p^2}\Big) \\ &\ll \frac{1}{P} + \Big|\sum_{P<p\le x} \frac{1-f(p)}{p}\Big|\,. \end{aligned}$$

On assembling our estimates, we find that

$$S(x) = a(P)x + o(x) + O\big(x/P^{1/2}\big) + O\Big(x\Big|\sum_{P<p\le x} \frac{1-f(p)}{p}\Big|^{1/2}\Big).$$

Since P can be arbitrarily large, this gives the desired result, subject to (27.24) and (27.25).

To complete the proof we now suppose only that $f \in \mathcal{M}_0$ and that $\sum_p (1-f(p))/p$ converges. Let $\mathcal{P}$ denote the set of primes p for which $\Re f(p) \le 1/2$. We note that

$$\sum_{p\in\mathcal{P}} \frac{1}{p} \;\le\; 2\Re \sum_p \frac{1-f(p)}{p} \;<\; \infty\,.$$

We define multiplicative functions g and h by the Euler products

$$\begin{aligned} G(s) \;&=\; \sum_{n=1}^{\infty} \frac{g(n)}{n^s} \;=\; \prod_{p\in\mathcal{P}} \Big(1-\frac{1}{p^s}\Big)^{-1} \prod_{p\notin\mathcal{P}} \Big(1-\frac{f(p)}{p^s}\Big)^{-1}, \\ H(s) \;&=\; \sum_{n=1}^{\infty} \frac{h(n)}{n^s} \;=\; \prod_{p\in\mathcal{P}} \Big(1-\frac{1}{p^s}\Big)\Big(1+\frac{f(p)}{p^s}+\frac{f(p^2)}{p^{2s}}+\cdots\Big) \\ &\qquad\qquad \times \prod_{p\notin\mathcal{P}} \Big(1-\frac{f(p)}{p^s}\Big)\Big(1+\frac{f(p)}{p^s}+\frac{f(p^2)}{p^{2s}}+\cdots\Big)\,. \end{aligned}$$

Thus

$$F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \prod_p \Big(1+\frac{f(p)}{p^s}+\frac{f(p^2)}{p^{2s}}+\cdots\Big) = G(s)H(s)\,.$$

We observe that $g \in \mathcal{M}_0$, $\sum_p (1-g(p))/p$ converges, and that g satisfies (27.24) and (27.25). Hence

$$\sum_{n\le x} g(n) = x \prod_{p\notin\mathcal{P}} \Big(1-\frac{1}{p}\Big)\Big(1-\frac{f(p)}{p}\Big)^{-1} + o(x),$$

and we obtain the desired result from Theorem 27.5, since

$$\sum_{m=1}^{\infty} \frac{|h(m)|}{m} \le \prod_{p\in\mathcal{P}} \Big(1+\frac{2}{p}+\frac{2}{p^2}+\cdots\Big) \prod_{p\notin\mathcal{P}} \Big(1+\frac{2}{p^2}+\frac{2}{p^3}+\cdots\Big) \ll \exp\Big(2\sum_{p\in\mathcal{P}}\frac{1}{p}\Big) < \infty\,.$$

□

In Theorem 27.6, the mean value is non-zero unless (27.23) holds. In §27.5 we characterize those $f \in \mathcal{M}_0$ with vanishing mean value in terms of the behaviour of the generating Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s} \qquad (\sigma > 1)\,. \tag{27.26}$$

To prepare for the proof of our next result we establish a variant of the Hardy–Littlewood tauberian theorem (Theorem 5.7).

Lemma 27.7 *Suppose that the numbers $c(p)$ are bounded, and that*

$$\lim_{\sigma\to 1^+} \sum_p \frac{c(p)}{p^\sigma}$$

exists and has the (finite) value c. Then $\sum_p c(p)/p$ converges, and has the value c.

Proof Put

$$a(u) = \sum_{e^{u-1}<p\le e^u} \frac{c(p)}{p}\,.$$

Then $a(u) \ll 1/u$ for $u \ge 1$, and

$$I(u) = \int_0^\infty a(u)e^{-\delta u}\,du = \frac{1-e^{-\delta}}{\delta} \sum_p \frac{c(p)}{p^{1+\delta}}$$

tends to c as $\delta \to 0^+$. Thus by the Hardy–Littlewood tauberian theorem (Theorem 5.7 with $\beta = 0$) it follows that $\int_0^U a(u)\,du$ tends to c as $U \to \infty$. But

$$\int_0^U a(u)\,du = \sum_{p\le e^U} \frac{c(p)}{p} - \sum_{e^{U-1}<p\le e^U} \frac{c(p)}{p}(U-\log p),$$

and the second sum is $\ll 1/U$, so $\sum_p c(p)/p$ converges to c □

For members of $\mathcal{M}_0$ with non-zero mean value, we have the following comprehensive result.

Theorem 27.8 (Delange) *Suppose that $f \in \mathcal{M}_0$, and let $S(x)$, the number a, and the function $F(s)$ be defined as in* (27.19), (27.21) *and* (27.26) *respectively. Then the following assertions are equivalent*:

(a) $S(x) \sim ax$ *and* $a \neq 0$;

(b) $\displaystyle\sum_{n\le x} \frac{f(n)}{n} \sim a\log x$ *and* $a \neq 0$;

(c) $\displaystyle F(\sigma) \sim \frac{a}{\sigma - 1}$ *as* $\sigma \to 1^+$ *and* $a \neq 0$;

(d) $\displaystyle\lim_{\sigma\to 1^+} \sum_p \frac{1 - f(p)}{p^\sigma}$ *exists and* (27.23) *fails*;

(e) $\displaystyle\sum_p \frac{1 - f(p)}{p}$ *converges and* (27.23) *fails.*

Proof We deduce (b) from (a) by partial summation, and similarly deduce (c) from (b). But (c) asserts that $\lim_{\sigma\to 1^+} F(\sigma)/\zeta(\sigma) = a$, which is to say that

$$\lim_{\sigma\to 1^+} \prod_p \Big(1 - \frac{1}{p^\sigma}\Big)\Big(1 + \frac{f(p)}{p^\sigma} + \frac{f(p^2)}{p^{2\sigma}} + \cdots\Big) = a \neq 0.$$

Each factor of the product has modulus not exceeding 1, so if (27.23) were to hold, then the limit would be 0. Thus (27.23) fails and the product is comparable to

$$\exp\Big(\sum_p \frac{1 - f(p)}{p^\sigma}\Big).$$

Hence we have (d). That (d) implies (e) is immediate from Lemma 27.7, and that (e) implies (a) follows from Theorem 27.6 □

27.2.1 Exercises

1. Suppose that $\sum_{d=1}^\infty g(d)/d$ converges, say to a.
(a) Show that $\sum_{d\le x} g(d) = o(x)$.
(b) Suppose also that $\sum_{d\le x} |g(d)| \ll x$. Use Axer's Theorem (Theorem 8.1) to show that $\sum_{d\le x} g(d\{x/d\} = o(x)$.
(c) Put $f(n) = \sum_{d|n} g(d)$. Under the above hypotheses, show that $\sum_{n\le x} f(n) = ax + o(x)$. (Note that this improves upon Theorem 27.2.)

27.3 The distribution of additive functions

We now employ our understanding of the mean values of multiplicative functions to establish

Theorem 27.9 (Erdős–Wintner) *Let f be a real-valued additive function. The following are equivalent*:

(a) *Each of the following series is convergent*:

$$\sum_{\substack{p\\ |f(p)|\le 1}} \frac{f(p)}{p}, \qquad \sum_{\substack{p\\ |f(p)|\le 1}} \frac{|f(p)|^2}{p}, \qquad \sum_{\substack{p\\ |f(p)|>1}} \frac{1}{p}\,. \tag{27.27}$$

(b) *There is an increasing function $F(u)$ such that $\lim_{u\to-\infty} F(u) = 0$, $\lim_{n\to+\infty} = 1$, and such that*

$$\lim_{N\to\infty} \frac{1}{N} \operatorname{card}\{n \le N \,:\, f(n) \le u\} = F(u), \tag{27.28}$$

whenever u is not a point of discontinuity of F.

Since F is increasing, the set of its discontinuities is is at most countable. Later we shall see that if we define F to be right-continuous, so that $F(u) = F(u^+)$, then (27.28) holds for all values of u. When (b) holds we may say that F is the asymptotic distribution of f. Given F with the above properties, there is a unique probability measure μ such that $F(u) = \int_{-\infty}^{u} 1\, d\mu$. Moreover, we can construct a probability measure μ_N that attaches weight $1/N$ to each of the points $f(n)$ for $1 \le n \le N$. Then (27.28) asserts that

$$\lim_{N\to\infty} \int_{-\infty}^{u} 1\, d\mu_N = \int_{-\infty}^{u} 1\, d\mu,$$

which is to say that the measures μ_N tend weakly to μ. Our proof of Theorem 27.9 depends on our discussion in §16.5 of the weak convergence of measures.

Proof Suppose that (a) holds. By virtue of Theorem 16.18, in order to show that (b) holds it suffices to show that

$$\widehat{\mu}_N(t) = \int_{\mathbb{R}} e(-tu)\, d\mu_N(u) = \frac{1}{N} \sum_{n=1}^{N} e(-tf(n)) \tag{27.29}$$

has a limit $r(t)$ as $N \to \infty$, and that r is continuous at $t = 0$. Let $g(n) = g_t(n) = e(-tf(n))$. Then $g \in \mathcal{M}_0$, so by Theorem 27.6 the above tends to

$$r(t) = \prod_p \Big(1 - \frac{1}{p}\Big)\Big(1 + \frac{e(-tf(p))}{p} + \frac{e(-tf(p^2))}{p^2} + \cdots\Big) \tag{27.30}$$

provided that the sum

$$\sum_p \frac{1 - e(-tf(p))}{p} \tag{27.31}$$

converges. The above is

$$= 2\pi it \sum_{|f(p)|\le 1} \frac{f(p)}{p} + \sum_{|f(p)|\le 1} \frac{1 - 2\pi i t f(p) - e(-tf(p))}{p} + \sum_{|f(p)|>1} \frac{1 - e(-tf(p))}{p}. \tag{27.32}$$

By the hypotheses (a) we see that the first sum is a constant, and that the third sum is absolutely and uniformly convergent. Since $e(\theta) = 1 + 2\pi i\theta + O(\theta^2)$, the second sum is absolutely convergent, and uniformly so for t in a bounded set. Thus the sum (27.31) converges. Moreover, the expression (27.32) tends to 0 as $t \to 0$, so $r(t)$ tends to 1, and hence we have (b).

We now show that (b) implies (a). If the μ_N tend weakly to μ, then by Theorem 16.17 it follows that $\widehat{\mu}_N(t) \to \widehat{\mu}(t)$. Since $\widehat{\mu}(t)$ is continuous and $\widehat{\mu}(0) = 1$, it follows from (27.29) that the multiplicative function g_t has a non-zero mean value for all t near 0, and that this mean value tends to 1 as $t \to 0$. Hence by Theorem 27.8 we deduce that the sum (27.31) converges for all small t, and tends to 0 as $t \to 0$. Let $s(t)$ denote the real part of the series (27.31). Since each term has non-negative real part, the sum of the real parts is absolutely convergent, and uniformly bounded for $|t| \le \delta$. But then

$$\frac{1}{2\delta}\int_{-\delta}^{\delta} s(t)\,dt = \sum_p \frac{1}{p}\Big(1 - \frac{\sin 2\pi\delta f(p)}{2\pi\delta f(p)}\Big),$$

and hence this latter sum is finite. But

$$1 - \frac{\sin\theta}{\theta} \gg \min(1, \theta^2),$$

so the second and third sums in (27.28) are convergent. Hence the second and third sums in (27.32) and convergent, and since the sum (27.31) is convergent, it follows that the first sum in (27.32) is convergent. □

In the next section we shall find that much can be said about the distribution function F of a real-valued additive function f. At this point we content ourselves with the following simple result.

Theorem 27.10 *Let f be a real-valued additive function with limiting distribution function F. If the series*

$$\sum_{f(p)\ne 0} \frac{1}{p} \tag{27.33}$$

converges, then each value assumed by f is attained on a set of positive density, so that F has jump discontinuities but is otherwise constant (i.e., the associated measure μ is discrete). If the series diverges, then the distribution function F is continuous, and hence any value of f is assumed only on a set of density 0.

Proof Let $\mathcal{P}$ denote the set of primes p for which $f(p) \ne 0$, let $\mathcal{N}_1$ denote the set of positive integers composed entirely of primes $p \in \mathcal{P}$, let $\mathcal{N}_2$ denote the set of integers composed

entirely of primes $p \notin \mathcal{P}$ with each prime occurring with multiplicity > 1, and finally let $\mathcal{N}_3$ denote the set of squarefree integers composed entirely of primes $p \notin \mathcal{P}$. Each n can be written uniquely in the form $n = n_1 n_2 n_3$ with $n_i \in \mathcal{N}_i$ and $(n_2, n_3) = 1$, and $f(n)$ depends only on n_1 and n_2. The number of $n \le x$ with prescribed n_1 and n_2 is the number of squarefree $n_3 \le x/(n_1 n_2)$ such that $n_3 \in \mathcal{N}_3$ and $(n_2, n_3) = 1$. By Corollary 27.4, this is

$$\sim \frac{x}{n_1 n_2} \prod_{p\in\mathcal{P}} \Big(1-\frac{1}{p}\Big) \prod_{p\mid n_2} \Big(1-\frac{1}{p}\Big) \prod_{\substack{p\notin\mathcal{P}\\ p\nmid n_2}} \Big(1-\frac{1}{p^2}\Big) = \frac{x}{n_1 n_2} \prod_{p\in\mathcal{P}} \Big(1-\frac{1}{p}\Big) \prod_{p\mid n_2} \Big(1+\frac{1}{p}\Big)^{-1} \prod_{p\notin\mathcal{P}} \Big(1-\frac{1}{p^2}\Big).$$

Moreover, these densities sum to 1 as n_1 and n_2 range over $\mathcal{N}_1$ and $\mathcal{N}_2$.

Now suppose that the series (27.33) diverges. By Theorem 16.19 it suffices to show that

$$\int_{-T}^{T} |\widehat{\mu}(t)|^2 \, dt = o(T) \tag{27.34}$$

as $T \to \infty$. In the case at hand we know that

$$\widehat{\mu}(t) = \prod_p \Big(1-\frac{1}{p}\Big)\Big(1 + \frac{e(-tf(p))}{p} + \frac{e(-tf(p^2))}{p^2} + \cdots\Big). \tag{27.35}$$

Let $\mathcal{P}$ be a finite set of primes for which $f(p) \ne 0$, and put $s = \sum_{p\in\mathcal{P}} 1/p$. In the above product, each prime contributes a factor whose absolute value is ≤ 1. Thus

$$\begin{aligned} |\widehat{\mu}(t)| &\le \prod_{p\in\mathcal{P}} \Big|\Big(1-\frac{1}{p}\Big)\Big(1+\frac{e(-tf(p))}{p}+\cdots\Big)\Big| \\ &\ll \prod_{p\in\mathcal{P}} \Big|1-\frac{1}{p}+\frac{e(-tf(p))}{p}\Big| \ll \exp\Big(-2\sum_{p\in\mathcal{P}} \frac{\sin^2 \pi t f(p)}{p}\Big). \end{aligned}$$

Hence by Hölder's inequality

$$\begin{aligned} \frac{1}{2T}\int_{-T}^{T} |\widehat{\mu}(t)|^2 \, dt &\ll \frac{1}{2T}\int_{-T}^{T} \exp\Big(-4\sum_{p\in\mathcal{P}} \frac{\sin^2 \pi t f(p)}{p}\Big)\, dt \\ &\le \prod_{p\in\mathcal{P}} \Big(\frac{1}{2T}\int_{-T}^{T} \exp(-4s\sin^2 \pi t f(p))\, dt\Big)^{1/(sp)}. \end{aligned}$$

Suppose that $f(p) > 0$. The integrand has period $1/f(p)$, and

$$\begin{aligned} f(p)\int_0^{1/f(p)} \exp(-4s\sin^2 \pi t f(p))\, dt &= \int_0^1 \exp(-4s\sin^2 \pi t)\, dt \\ &\le \int_{-1/2}^{1/2} \exp(-16st^2)\, dt \le \int_{-\infty}^{\infty} \exp(-16st^2)\, dt = \frac{\sqrt{\pi}}{4\sqrt{s}}. \end{aligned}$$

Hence

$$\frac{1}{2T}\int_{-T}^{T}\exp(-4s\sin^2\pi t f(p))\,dt \le \frac{1}{\sqrt{s}}$$

for all sufficiently large T, and so

$$\limsup_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|\widehat{\mu}(t)|^2\,dt \le \frac{1}{\sqrt{s}}\,.$$

By choosing $\mathcal{P}$ suitably, we may make s as large as we please. Thus we have (27.34), and the proof is complete. □

27.3.1 Exercises

1. (a) Show that $\log\sigma(n)/n$ has a limiting distribution.
(b) Show that this limiting distribution is continuous.
(c) Deduce that the set of perfect numbers (i.e., those for which $\sigma(n) = 2n$ is a set of density 0.

2. Show that an integer-valued additive function f has a limiting distribution if and only if

$$\sum_{f(p)\neq 0}\frac{1}{p} < \infty\,.$$

3. Let f be a multiplicative function that takes only positive real values. Show that f has a limiting distribution if and only if each of the following four series converges:

$$\sum_{1/2\le|f(p)|\le 2}\frac{1-f(p)}{p},\qquad \sum_{1/2\le|f(p)|\le 2}\frac{|1-f(p)|^2}{p},\qquad \sum_{|f(p)|>2}\frac{1}{p},\qquad \sum_{|f(p)|<1/2}\frac{1}{p}\,.$$

4. Let $f_1,\dots,f_k$ be real-valued additive functions, and put $\boldsymbol{f}(n) = (f_1(n),\dots,f_k(n))$. Give necessary and sufficient conditions that $\boldsymbol{f}$ should have a limiting distribution in $\mathbb{R}^k$. Deduce a variant of Theorem 27.9 for complex-valued additive functions.

5. Let f be a real-valued additive function with limiting distribution F, and let μ denote the associated limiting measure. Show that either $\widehat{\mu}(t)$ is never 0, or that its zeros form an arithmetic progression of the form $c(2k+1)$ for $k\in\mathbb{Z}$.

27.4 Applications of probability theory

Let f be a real-valued additive function, and for each prime p let X_p denote the random variable defined in (27.3). We take the X_p to be independent, and ask whether the random variable X defined in (27.4) exists. In this connection we quote without proof

Kolmogorov's Three Series Theorem *Let Y_n be independent random variables. If each of the three series*

$$\sum_n \int_{|Y_n|\le 1} Y_n, \qquad \sum_n \int_{|Y_n|\le 1} \int |Y_n|^2, \qquad \sum_n \int_{|Y_n|>1} 1$$

converges, then the sum $Y = \sum_n Y_n$ converges almost everywhere. If any one of these series diverges, then the sum $\sum_n Y_n$ diverges almost everywhere.

For our sum (27.4), the conditions of Kolmogorov's theorem are precisely the conditions of Theorem 27.9(a). Hence the random variable X exists precisely when f has a limiting distribution F. In the context of Kolmogorov's Three Series Theorem, when Y exists its Fourier transform is

$$\widehat{Y}(t) = \int e(-tY) = \int \prod_n e(-tY_n) = \prod_n \int e(-tY_n) = \prod_n \widehat{Y}_n(t)$$

by the independence of the Y_n. In the case of the variable X, we find that

$$\begin{aligned}\widehat{X}_p(t) = \int e(-tX_p) &= 1 - \frac{1}{p} + e(-tf(p))\frac{1}{p}\Big(1-\frac{1}{p}\Big) + e(-tf(p^2))\frac{1}{p^2}\Big(1-\frac{1}{p}\Big) + \cdots \\ &= \Big(1-\frac{1}{p}\Big)\Big(1 + \frac{e(-tf(p))}{p} + \frac{e(-tf(p^2))}{p^2} + \cdots\Big),\end{aligned}$$

and hence

$$\widehat{X}(t) = \prod_p \Big(1-\frac{1}{p}\Big)\Big(1 + \frac{e(-tf(p))}{p} + \frac{e(-tf(p^2))}{p^2} + \cdots\Big).$$

But this is the same as $\widehat{\mu}(t)$ given in (27.35), so by the uniqueness of the Fourier transform (Corollary 16.16) it follows that F is the distribution function of X. A great deal is known concerning the distribution function of a sum of random variables, so by appealing to this theory we obtain further information concerning F. In particular, we note the

Law of Pure Types (Jessen–Wintner) *Let Y_n be independent random variables such that $Y = \sum_n Y_n$ converges almost everywhere, and suppose that there is a countable set $\mathcal{C}$ such that $P(Y_n \in \mathcal{C}) = 1$ for all n. Then the distribution of Y is of pure type: Either it is discrete, singular, or absolutely continuous.*

Hence we see that the distribution function F of a real-valued additive function is of pure type. In Theorem 27.10 we characterized the situation in which the distribution is discrete; this can also be obtained by applying a general theorem of Lévy concerning sums of independent random variables. We have no similar criterion to distinguish between singular and absolutely continuous distributions, although all three types do occur. In particular, the distribution of $\log \varphi(n)n$ is singular, as we now show.

Theorem 27.11 *Let μ denote the probability measure such that*

$$\lim_{N\to\infty}\frac{1}{N}\operatorname{card}\{n\le N\,:\,\varphi(n)/n\le c\}=\mu((-\infty,c])\,.$$

Let α be fixed, $1<\alpha<e-1$, put $I_k=[\varphi(k)/k-1/k^\alpha,\varphi(k)/k]$, and set

$$\mathcal{S}=\{x\in[0,1]\,:\,x\in I_k \text{ for infinitely many } k\}.$$

Then $m(\mathcal{S})=0$, and $\mu(\mathcal{S})=1$.

Here $m(\mathcal{S})$ denotes the Lebesgue measure of $\mathcal{S}$.

Proof The first assertion is clear, since

$$\mathcal{S}\subseteq\bigcup_{k>K}I_k$$

for any K, so that

$$m(\mathcal{S})\le\sum_{k>K}k^{-\alpha}\,.$$

As for the second assertion, we show that for any $\varepsilon>0$ and any K there is an L such that

$$\mu\Big(\bigcup_{K<k\le L}I_k\Big)\ge 1-\varepsilon\,. \tag{27.36}$$

The advantage of this finite form of the assertion is that we can estimate the left hand side by considering densities of set of integers: If $\mathcal{N}=\big\{n\,:\,\varphi(n)/n\in\bigcup_{K<k\le L}I_k\big\}$, then $d(\mathcal{N})=\mu\big(\bigcup_{K<k\le L}I_k\big)$. In establishing (27.36), we may assume that K is large, for if the above holds for one value of K, then it also holds for all smaller values of K. For a given number n, write $n=\prod_{i=1}^{\Omega(n)}p_i$ with $p_1\le p_2\le\cdots\le p_{\Omega(n)}$, and set $d_r=\prod_{i\le r}p_i$. We shall show that if L is sufficiently large, then most integers n have a divisor d_r, $K<d_r\le L$, such that

$$\frac{\varphi(n/d_r)}{n/d_r}\ge 1-d_r^{-\alpha}.$$

In this case $\varphi(n)/n\in I_{d_r}$, since

$$\frac{\varphi(d_r)}{d_r}\ge\frac{\varphi(n)}{n}\ge\frac{\varphi(d_r)}{d_r}\,\frac{\varphi(n/d_r)}{n/d_r}\ge\frac{\varphi(d_r)}{d_r}\big(1-d_r^{-\alpha}\big)\ge\frac{\varphi(d_r)}{d_r}-d_r^{-\alpha}\,.$$

Let $\mathcal{N}_0$ denote the complementary set of numbers, i.e., the n for which $\varphi(n/d_r)/(n/d_r)<1-d_r^{-\alpha}$ for all $d_r\in(K,L]$. To estimate the size of $\mathcal{N}_0$ we consider various possibilities. Let $\mathcal{N}_1$ be the set of n such that the interval $(K,\log L]$ contains none of the special divisors d_r. Let $\beta=\alpha/2+(e-1)/2$, so that $1<\alpha<\beta<e-1$, and let $\mathcal{N}_2$ be the set of numbers $n$such that $p_{r+1}<d_r^\beta$ whenever $d_r\in(K,L]$. Finally, let $\mathcal{N}_3$ be the set of those n such that there is a $d_r\in(K,L]$ for which $p_{r+1}>d_r^\beta$ and $\varphi(n/d_r)/(n/d_r)<1-d_r^{-\alpha}$. The sets $\mathcal{N}_i$

possess asymptotic densities, but for our present purpose it suffices to bound their upper asymptotic densities where the upper asymptotic density of a set $\mathcal{A}$ is

$$\overline{d}(\mathcal{A}) = \limsup_{x\to\infty} \frac{1}{x} \operatorname{card}\{n \le x \,:\, n \in \mathcal{A}\}\,.$$

The main estimates to be established are that

$$\overline{d}\mathcal{N}_1 \ll \frac{\log K}{\log\log L}, \tag{27.38}$$

$$\overline{d}(\mathcal{N}_1^c\mathcal{N}_2) \ll \frac{1}{\log\log L}, \tag{27.39}$$

$$\overline{(}\mathcal{N}_3) \ll K^{\alpha-e+1}\,. \tag{27.40}$$

Once these estimates are in place, we argue that $\mathcal{N}_2 = \mathcal{N}_1\mathcal{N}_2 \cup \mathcal{N}_1\mathcal{N}_2^c \subseteq \mathcal{N}_1 \cup \mathcal{N}_1^c\mathcal{N}_2$. Since $\overline{d}(\mathcal{A}\cup\mathcal{B}) \le \overline{d}(\mathcal{A}) + \overline{d}(\mathcal{B})$, it follows from (27.38) and (27.39) that

$$\overline{d}(\mathcal{N}_2) \ll \frac{\log K}{\log\log L}\,. \tag{27.41}$$

We also observe that $\mathcal{N}_0\mathcal{N}_2^c \subseteq \mathcal{N}_3$. Thus $\mathcal{N}_0 = \mathcal{N}_0\mathcal{N}_2 \cup \mathcal{N}_0\mathcal{N}_2^c \subseteq \mathcal{N}_2 \cup \mathcal{N}_3$, so from (27.40) and (27.41) we deduce that

$$\overline{d}(\mathcal{N}_0) \le \overline{d}(\mathcal{N}_2) + \overline{(}\mathcal{N}_3) \ll K^{\alpha-e+1} + \frac{\log K}{\log\log L}\,.$$

Thus $\overline{d}(\mathcal{N}_0) < \varepsilon$ if K is sufficiently large and if L is sufficiently large compared with K.

To prove (27.38), we suppose, as we may, that $L > \exp(K^2)$. For $n \in \mathcal{N}_1$, choose r so that $d_r < K$ and $d_{r+1} > \log L$. Thus $n = d_r m$ with m composed entirely of primes $> (\log L)/d_r$. This decomposition is unique, since d_r is composed entirely of primes $< K$, and m is composed entirely of primes $> (\log L)/d_r > K^2/d_r \ge K$. Hence

$$\operatorname{card}\{n \le x \,:\, n \in \mathcal{N}_1\} = \sum_{d<K} \operatorname{card}\{m \le x/d \,:\, p|m \implies p > (\log L)/d\}\,.$$

By the theorem of Eratosthenes–Legendre (Theorem 3.1), this is

$$\sim x \sum_{d<K} \frac{1}{d} \prod_{p<(\log L)/d} \Big(1-\frac{1}{p}\Big) \le x\Big(\sum_{d<K}\frac{1}{d}\Big) \prod_{p<(\log L)/K} \Big(1-\frac{1}{p}\Big) \ll x\frac{\log K}{\log((\log L)/K)},$$

so we have (27.38).

As for (27.39), suppose that $n \in \mathcal{N}_1^c\mathcal{N}_2$ and that $d_r \in (K, L]$. Then $d_{r+1} = d_r p_{r+1} \le d_r^{1+b}$. Thus by induction, if r_0 is the least r for which $d_r > K$, then

$$d_r \le d_{r_0}^{(1+b)^{r-r_0}} < (\log L)^{(1+b)^{r-r_0}}$$

provided that this bound is $\le L$. Set

$$R = \Big[\frac{\log\log L - \log\log\log L}{\log(1+b)}\Big] .$$

Then $d_r \le L$ for $r \le r_0 + R$. Let

$$\Omega_L(n) = \sum_{\substack{p^k \| n \\ p \le L}} 1 .$$

Thus $\Omega_L(n) \ge R$ if $n \in \mathcal{N}_2$. By the Turán–Kubilius inequality (Theorem 27.1),

$$\sum_{n \le x} (\Omega_L(n) - \log\log L)^2 \ll x \log\log L .$$

Let $c = 1/\log(1+\beta)$. Here $c > 1$, since $1+\beta < e$, and $R > (c-\varepsilon)\log\log L$ if L is sufficiently large. Thus $\Omega_L(n) - \log\log L \gg \log\log L$ when $n \in \mathcal{N}_2$, and so we have (27.39).

If $n \in \mathcal{N}_3$, then we may write $n = dm$ where $K < d \le L$, $p|m$ implies $p > d^\beta$, and $\varphi(m)/m < 1 - d^{-\alpha}$. This decomposition may not be unique, but

(27.41)
$$\operatorname{card}\{n \le x : n \in \mathcal{N}_3\} \le \sum_{K < d \le L} \operatorname{card}\{m \le x/d : p|m \implies p > d^\beta,\ \varphi(m)/m < 1 - d^{-\alpha}\} .$$

Let

$$f_y(m) = \sum_{\substack{p|n \\ p>y}} \log(1 - 1/p)^{-1} .$$

This is an additive function with

$$A(f_y, z) \le (1 + o(1))(y \log y)^{-1}, \qquad B(f_y, z) \ll y^{-2}(\log y)^{-1} .$$

Thus if $V \ge 2/(y \log y)$, then by the Turán–Kubilius inequality we see that

$$\operatorname{card}\{m \le z : f_y(n) > V\} \ll \frac{z}{V^2 y^2 \log y} .$$

On taking $z = x/d$, $y = d^\beta$, $V = \log(1 - d^{-\alpha})^{-1} \asymp d^\alpha$, we see that the $m \le x/d$ for which $f_y(m) > V$ includes the m in (27.41), and hence

$$\operatorname{card}\{n \le x : n \in \mathcal{N}_3\} \ll x \sum_{K < d \le L} d^{-1+2\alpha-2\beta}(\log d)^{-1} \ll x K^{2\alpha-2\beta}(\log K)^{-1}.$$

This gives (27.40), in view of the definition of β. Thus the proof is complete. □

27.4.1 Exercises

27.5 Multiplicative functions with vanishing mean value

Suppose that

$$S(x) = \sum_{n\le x} f(n)\,. \tag{27.41}$$

If $S(x) \ll x$, then by Theorem 1.3 the Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} f(n)n^{-s} \tag{27.42}$$

converges for $\sigma > 1$, and

$$F(s) = s\int_1^{\infty} S(x)x^{-s-1}\,dx$$

for $\sigma > 1$. From this formula it is immediate that if $S(x) = ax + o(x)$, then

$$F(s) = \frac{a}{s-1} + O\Big(\frac{\tau}{\sigma-1}\Big)$$

as $\sigma \to 1^+$. This is a simple abelian theorem. In prior discussions of tauberian converses, such as in §5.2, we imposed a bound on the size of $f(n)$ so that $S(x)$ could not change to quickly. In the present context, the hypothesis that $|f(n)| \le 1$ for all n does not yield a converse (cf Exercise 27.5.1.1), but we find that the hypothesis that $f \in \mathcal{M}_0$ *is* sufficient. The lesson is that for $f \in \mathcal{M}_0$, the quantity $|S(x)|$ changes less slowly on average than it might under the weaker assumption that $|f(n)| \le 1$.

Theorem 27.12 *Suppose that $f \in \mathcal{M}_0$, let $S(x)$ and $F(s)$ be defined by* (27.41) *and* (27.42)*, and for $\alpha > 0$ put*

$$M(\alpha) = \Big(\sum_{k=-\infty}^{\infty} \frac{1}{k^2+1} \max_{\substack{\sigma\ge 1+\alpha\\ |t-k|\le 1/2}} |F(s)|^2\Big)^{1/2}.$$

Then

$$S(x) \ll \frac{x}{\log x}\int_{1/\log x}^{1} \frac{M(\alpha)}{\alpha}\,d\alpha\,. \tag{27.43}$$

From the trivial bound $F(s) \ll 1/(\sigma-1)$ it follows that $M(\alpha) \ll 1/\alpha$, and when this is inserted in (27.43) we find that $S(x) \ll x$, which is also trivial. However, if for every T we have $F(s) = o(1/(\sigma-1))$ uniformly for $|t| \le T$, then $M(\alpha) = o(1/\alpha)$, and hence $S(x) = o(x)$.

We show below that

$$M(\alpha) \gg 1 \tag{27.44}$$

uniformly for $f \in \mathcal{M}_0$. Thus the right hand side of (27.43) is $\gg x(\log x)^{-1}\log\log x$. That this should be the limit of the method is not surprising, in view of the example considered in Exercise 27.5.1.2, for which $f \in \mathcal{M}_0$, $M(\alpha) \asymp 1$, and yet there is a large x for which $|S(x)| \gg x(\log x)^{-1}\log\log x$.

To establish (27.44), we write

$$F(s) = (1 + D(s))G(s)H(s) \tag{27.45}$$

where

$$D(s) = \sum_{k=1}^{\infty} \frac{f(2^k)}{2^{ks}},\ G(s) = \prod_{p>2}\Big(1-\frac{f(p)}{p^s}\Big)^{-1},\ H(s) = \prod_{p>2}\Big(1-\frac{f(p)}{p^s}\Big)\Big(1+\frac{f(p)}{p^s}+\frac{f(p^2)}{p^{2s}}+\cdots\Big).$$

Here

$$\Big(1-\frac{f(p)}{p^s}\Big)\Big(1+\frac{f(p)}{p^s}+\frac{f(p^2)}{p^{2s}}+\cdots\Big) = 1+\frac{f(p^2)-f(p)^2}{p^{2s}}+\frac{f(p^3)-f(p)f(p^2)}{p^{3s}}+\cdots,$$

which is 1 plus a quantity whose absolute value does not exceed $2p^{-2\sigma}(1-p^{-\sigma})^{-1}$. Thus the product $H(s)$ is absolutely and uniformly convergent for $\sigma \ge 2/3$, and so

$$\log H(s) \ll 1 \qquad (\sigma \ge 2/3). \tag{27.46}$$

Choose t_0, $|t_0| \le \pi/\log 2$, so that $f(2)/2^{it_0}$ is positive real. Then $\Re f(2)/2^{it} \ge 0$ for $|t-t_0| \le \pi/(2\log 2)$, and so $|1+D(s)| \ge 1/2$ for $\sigma \ge 1$, $|t-t_0| \le \pi/(2\log 2)$. Also,

$$\int_{t_0-1}^{t_0+1} \log G(\sigma+it)\,dt \;=\; \sum_{p>2}\sum_{k=1}^{\infty}\frac{f(p)^k}{kp^{k(1+\alpha)}}\int_{t_0-1}^{t_0+1} p^{-ikt}\,dt \;\ll\; \sum_{p}\sum_{k=1}^{\infty}\frac{1}{k^2p^k\log p} \;\ll\; 1\,.$$

Thus there is an absolute constant $c > 0$, and a t_1, $|t_0 - t_1| \le 1$, such that $|1 + D(1+\alpha+it_1)| \ge 1/2$ and $|G(1+\alpha+it_1)| \ge c$, so we have (27.44).

In order to prepare for the proof of Theorem 27.12, we establish

Lemma 27.13 *For positive integers n let λ_n be real, suppose that $|a_n| \le b_n$ for all n, and that $\sum_{n=1}^{\infty} b_n < \infty$. Then for any real T_0 and any $T > 0$,*

$$\int_{T_0-T}^{T_0+T}\Big|\sum_{n=1}^{\infty} a_n e(\lambda_n t)\Big|^2\,dt \le 3\int_{-T}^{T}\Big|\sum_{n=1}^{\infty} b_n e(\lambda_n t)\Big|^2\,dt\,.$$

Proof It suffices to prove the inequality when $T_0 = 0$, for once this is done, the general case follows by replacing a_n by $a_n e(\lambda_n T_0)$. Let $K(t) = \max(1 - |t|/T, 0)$. Then

$$\widehat{K}(u) = \frac{1}{T}\Big(\frac{\sin \pi T u}{\pi u}\Big)^2 \geq 0,$$

so

$$\begin{aligned}\int_{-\infty}^{\infty} K(t)\Big|\sum_{n=1}^{\infty} a_n e(\lambda_n t)\Big|^2 dt &= \sum_{m,n} a_m \overline{a_n}\widehat{K}(\lambda_n - \lambda_m)\\ &\leq \sum_{m,n} b_m b_n \widehat{K}(\lambda_n - \lambda_m) = \int_{-\infty}^{\infty} K(t)\Big|\sum_{n=1}^{\infty} b_n e(\lambda_n t)\Big|^2 dt\,.\end{aligned}$$

By replacing a_n by $a_n e(\lambda_n T_0)$, we see more generally that

$$\int_{-\infty}^{\infty} K(t - T_0)\Big|\sum_{n=1}^{\infty} a_n e(\lambda_n t)\Big|^2 dt \leq \int_{-\infty}^{\infty} K(t)\Big|\sum_{n=1}^{\infty} b_n e(\lambda_n t)\Big|^2 dt\,. \tag{27.47}$$

But $\chi_{[-T,T]}(t) \leq K(t+T) + K(t) + K(t-T)$, so we apply (27.47) three times, with $T_0 = -T$, $T_0 = 0$, and $T_0 = T$, and sum to find that

$$\int_{-T}^{T} \Big|\sum_{n=1}^{\infty} a_n e(\lambda_n t)\Big|^2 dt \leq 3\int_{-\infty}^{\infty} K(t)\Big|\sum_{n=1}^{\infty} b_n e(\lambda_n t)\Big|^2 dt\,.$$

This gives the stated result, since $K(t) \leq \chi_{[-T,T]}(t)$. □

Proof of Theorem 27.12 We shall establish the two main estimates

$$S(x) \ll \frac{x}{\log x}\int_1^x \frac{|S(u)|}{u^2}\,du + \frac{x\log\log x}{\log x}, \tag{27.47}$$

$$\int_1^x \frac{|S(u)|\log u}{u^2}\,du \ll M(2/\log x)\log x\,. \tag{27.48}$$

These suffice to give the stated result, since from (27.48) it is evident that

$$\int_{x^{1/2}}^x \frac{|S(u)|}{u^2}\,du \ll M(2/\log x) \ll \int_{1/\log x}^{2/\log x} \frac{M(\alpha)}{\alpha}\,d\alpha\,.$$

We replace x by $x^{1/2^k}$ and sum over k to show that

$$\int_1^x \frac{|S(u)|}{u^2}\,du \ll \int_{1/\log x}^1 \frac{M(\alpha)}{\alpha}\,d\alpha\,.$$

We insert this in (27.47) to obtain the stated result. The second term in (27.47) can be neglected, in view of (27.44).

To establish (27.47) we first observe that

$$(\log x)\sum_{n\le x} f(n) - \sum_{n\le x} f(n)\log n = \sum_{n\le x} f(n)\log x/n \ll \sum_{n\le x}\log x/n \ll x\,. \tag{27.49}$$

Furthermore,

$$\begin{aligned}\sum_{n\le x} f(n)\log n &= \sum_{n\le x} f(n)\sum_{d\mid n}\Lambda(d)\\ &= \sum_{d\le x}\Lambda(d)\sum_{m\le x/d} f(md)\\ &= \sum_{p\le x}(\log p)\sum_{m\le x/p} f(mp) + O\Big(x\sum_{\substack{p^k\le x\\ k>1}}\frac{\log p}{p^k}\Big)\\ &= \sum_{p\le x}(\log p)f(p)S(x/p) + O\Big(\sum_{p\le x}(\log p)\sum_{m\le x/p}|f(mp)-f(m)f(p)|\Big) + O(x)\,.\end{aligned}$$

Since $f(mp) = f(m)f(p)$ unless $p|m$, we see that the sum over $m \le x/p$ is $\ll x/p^2$, and so the first error term above is $\ll x$. On combining this with (27.49), we deduce that

$$S(x)\log x \ll x + \sum_{p\le x}|S(x/p)|\log p\,. \tag{27.50}$$

Here we have a bound for $|S(x)|$ in terms of S at smaller arguments. The trivial bound for either side is $x\log x$. Thus if it were the case that $S(x)$ were of the order of x, then $S(x/p)$ would have to be of the order of x/p for many primes p. If the primes were exactly uniformly distributed, then the sum over p would be

$$\int_1^x |S(x/v)|\,dv = \int_1^x \frac{|S(u)|}{u^2}\,du\,.$$

Of course the primes are rather irregularly distributed, but as x varies the points x/p also move, so by averaging over x we can pass to a smoother average of $|S|$. Suppose that $X > Y > Z \ge 2$. Since $|S(X) - S(x)| \le |X - x| + 1$, we see that

$$|S(X)|\log X \ll \frac{1}{Y}\int_X^{X+Y} |S(x)|\log x\,dx + Y\log X\,.$$

By (27.50) this is

$$\ll X + Y\log X + \frac{1}{Y}\int_X^{X+Y}\sum_{p\le x}|S(x/p)|\log p\;dx\,.$$

We bound the contribution of the smaller primes trivially:

$$\sum_{p\le X/Z} |S(x/p)| \log p \ll x \sum_{p\le X/Z} \frac{\log p}{p} \ll X \log X/Z\,.$$

As for the contribution of the larger primes, we note that

$$\begin{aligned}\int_X^{X+Y} \sum_{X/Z<p\le 2X} |S(x/p)| \log p \; dx &= \sum_{X/Z<p\le 2X} \int_X^{X+Y} |S(x/p)|\, dx \;\log p \\ &= \sum_{X/Z<p\le 2X} \int_{X/p}^{(X+Y)/p} |S(u)|\, du \; p \log p \\ &= \int_1^{2Z} |S(u)| \sum_{\substack{X/Z<p\le 2X \\ X/u<p\le (X+Y)/u}} p \log p \; du\,.\end{aligned}$$

Here we can restrict to $u \le 2Z$ because the two intervals that p must lie in are disjoint if $u > 2Z$. In estimating the above, we now drop the condition $X/Z < p \le 2X$. The remaining condition stipulates that p must lie in an interval whose length is $Y/u \ge Y/(2Z) \ge 2$ if $Z \le Y/4$. Thus by the Brun–Titchmarsh inequality (Corollary 3.4) the number of primes in the interval is bounded by the length of the interval divided by the logarithm of its length. Hence the above sum over primes is

$$\ll \frac{XY \log X/u}{u^2 \log Y/u}\,.$$

Here the quotient of logarithms is an increasing function of u, so the above is uniformly

$$\ll \frac{XY \log X/(2Z)}{u^2 \log Y/(2Z)}\,.$$

We take $Y = X/\log X$ and $Z = X/(\log X)^2$, and on assembling our estimates discover that

$$S(X) \log X \ll X \int_1^X \frac{|S(u)|}{u^2}\, du + X \log\log X\,.$$

That is, we have (27.47).

Finally we prove (27.48). Let $S_1(x) = \sum_{n\le x} f(n) \log n$. By (27.44) and (27.49) it suffices to show that

$$\int_1^x \frac{|S_1(u)|}{u^2}\, du \ll M(2/\log x) \log x\,. \tag{27.51}$$

By the Cauchy–Schwarz inequality,

$$\int_1^x \frac{|S_1(u)|}{u^2}\, du \le \Big(\int_1^x \frac{|S_1(u)|^2}{u^3}\, du \Big)^{1/2} \Big(\int_1^x \frac{1}{u}\, du \Big)^{1/2}\,.$$

It now suffices to show that

$$\int_1^\infty \frac{|S_1(u)|^2}{u^{3+2\alpha}}\,du \ll \frac{M(\alpha)^2}{\alpha} \tag{27.52}$$

for $0<\alpha\le 1$, since we obtain (27.51) by taking $\alpha = 2/\log x$. By Plancherel's formula as in (5.26), we see that

$$\begin{aligned}\int_1^\infty \frac{|S_1(u)|^2}{u^{3+2\alpha}}\,du &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\Big|\frac{F'(1+\alpha+it)}{1+\alpha+it}\Big|\,dt\\ &\ll \sum_{k=-\infty}^{\infty}\frac{1}{k^2+1}\int_{k-1/2}^{k+1/2}|F'(1+\alpha+it)|^2\,dt\,.\end{aligned}$$

We multiply and divide by $|F(1+\alpha+it)|^2$ to see that the above is

$$\le \sum_{k=-\infty}^{\infty}\frac{1}{k^2+1}\int_{k-1/2}^{k+1/2}\Big|\frac{F'}{F}(1+\alpha+it)\Big|^2\,dt \max_{|t-k|\le 1/2}|F(1+\alpha+it)|^2\,.$$

Thus to obtain (27.52) it suffices to show that

$$\int_{k-1/2}^{k+1/2}\Big|\frac{F'}{F}(1+\alpha+it)\Big|^2\,dt \ll \frac{1}{\alpha}\,. \tag{27.53}$$

By (27.45) we see that

$$\frac{F'}{F}(s) = \frac{D'(s)}{1+D(s)} + \frac{G'}{G}(s) + \frac{H'}{H}(s)\,.$$

From (27.46) it is evident that $\frac{H'}{H}(s)\ll 1$ uniformly for $\sigma\ge 1$. By Lemma 27.13 we see that

$$\int_{k-1/2}^{k+1/2}\Big|\frac{G'}{G}(1+\alpha+it)\Big|^2\,dt \le 3\int_{-1/2}^{1/2}\Big|\frac{\zeta'}{\zeta}(1+\alpha+it)\Big|^2\,dt\,.$$

If $0<\alpha\le 1$, then by Theorem 6.7 this latter integral is

$$\ll \int_{-1/2}^{1/2}|\alpha+it|^{-2}\,dt \ll \frac{1}{\alpha}\,.$$

Clearly $D'(s)\ll 1$ for $\sigma\ge 1$. For $\sigma>1$ we write

$$\frac{1}{1+D(s)} = \sum_{j=0}^{\infty}(-D(s))^j\,.$$

This is a Dirichlet series whose coefficients do not exceed those of

$$1+\sum_{j=1}^{\infty}\Big(\sum_{k=1}^{\infty}2^{-ks}\Big)^j=1+\sum_{j=1}^{\infty}(2^s-1)^{-j}=\frac{2^s-1}{2^s-2}\,.$$

Hence by Lemma 27.13,

$$\int_{k-1/2}^{k+1/2}|1+D(1+\alpha+it)|^{-2}\,dt\le 3\int_{-1/2}^{1/2}\Big|\frac{2^{1+\alpha+it}-1}{2^{1+\alpha+it}-2}\Big|^2\,dt\,. \tag{27.54}$$

Here $|2^{1+\alpha+it}-1|\asymp 1$ uniformly for $0\le\alpha\le 1$, and

$$2^{1+\alpha+it}-2=2(\log 2)\int_0^{\alpha+it}2^s\,ds\,.$$

This integrand has real part $\ge 1/2$ for $\sigma\ge 0$ and $|t|\le\pi/(3\log 2)=1.5107867\ldots$, so $|2^{1+\alpha+it}-2|\gg|\alpha+it|$ for $\alpha\ge 0$ and $|t|\le 1/2$. Thus the right hand side of (27.54) is

$$\ll\int_{-1/2}^{1/2}|\alpha+it|^{-2}\,dt\ll\frac{1}{\alpha},$$

so we have (27.53), and the proof is complete. □

We comment that the first part of our proof is reminiscent of the elementary proof of the Prime Number Theorem, as found in §8.2. Moreover, the identity on the left hand side of (27.42) is equivalent to integrating by parts in Perron's formula, as

$$\frac{\log x}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{F(s)x^s}{s}\,ds=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{F(s)x^s}{s^2}\,ds-\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{F'(s)x^s}{s}\,ds\,.$$

This is expected to produce a gain, since we expect that $F(s)/s$ is not generally very rapidly changing, while x^s is spinning fairly rapidly. Indeed, suppose we tried to do something as simple as using Perron's formula to show that $[x]\ll x$. Since

$$\int_{-1}^{1}|\zeta(1+\alpha+it)|x^{1+\alpha}\,dt\asymp x^{1+\alpha}\log\frac{1}{\alpha},$$

we are unable to obtain a bound better than $x\log\log x$. On the other hand, if we were to use Perron's formula to show that $\sum_{n\le x}\log n\ll x\log x$, we fare better, since

$$\int_{-1}^{1}|\zeta'(1+\alpha+it)|x^{1+\alpha}\,dt\asymp\frac{x^{1+\alpha}}{\alpha},$$

and we can take $\alpha=1/\log x$. In both of these approaches, we would still have the problem that the kernel in Perron's formula decays only like an inverse first power. This could be overcome by smoothing, but in the argument just completed we avoided that problem by averaging over x, which allows us to appeal to Plancherel's identity.

The proof just completed depends only on properties of the zeta function in a neighbourhood of $s=1$, but if we take $f(n)=\mu(n)$, so that $F(s)=1/\zeta(s)$, then the further information that $\zeta(1+it)\ne 0$ implies that $M(x)=o(x)$.

We now relate the behaviour of $F(s)$ to the values of $f(p)$.

Theorem 27.14 (Halász) *Suppose that $f \in \mathcal{M}_0$, and let $S(x)$ and $F(s)$ be defined as in* (27.41) *and* (27.42)*. Then the following are equivalent:*

(a) $S(x) = o(x)$ *as* $x \to \infty$;

(b) *For each* $T > 0$, $F(s) = o(1/(\sigma - 1))$ *as* $\sigma \to 1^+$ *uniformly for* $|t| \le T$;

(c) *For each fixed* t, $F(\sigma + it) = o(1/(\sigma - 1))$ *as* $\sigma \to 1^+$;

(d) *For each* t, *at least one of the following holds:*

$$\text{(i)} \quad \sum_p \frac{1 - \Re(f(p)p^{-it})}{p} = +\infty, \tag{27.55}$$

$$\text{(ii)} \quad f(2^k) = -2^{ikt} \text{ for } k = 1, 2, \dots .$$

Moreover, condition (27.55i) *holds for all but at most one value of* t.

The combined effect of Theorem 27.8 and Theorem 27.14 is rather comprehensive, for if (27.55) fails and

$$\sum_p \frac{\Im(f(p)p^{-it})}{p} \tag{27.56}$$

converges, then by Theorem 27.8 we have

$$\sum_{n \le x} f(n)n^{-it} = ax + o(x)$$

where

$$a = \prod_p \Big(1 - \frac{1}{p}\Big)\Big(1 + \frac{f(p)}{p^{1+it}} + \frac{f(p^2)}{p^{2+2it}} + \cdots\Big)$$

is non-zero, and by partial summation,

$$S(x) = \frac{a}{1+it}x^{1+it} + o(x)\,.$$

In the one remaining case, in which (27.55) fails for some t, and (27.56) does not hold for that t, then with more work it can be shown that

$$S(x) = \frac{x^{1+it}}{1+it}\prod_{p \le x}\Big(1 - \frac{1}{p}\Big)\Big(1 + \frac{f(p)}{p^{1+it}} + \frac{f(p^2)}{p^{2+2it}} + \cdots\Big) + o(x)\,. \tag{27.57}$$

Proof That (a) implies (b) was the subject of the opening remarks of this section. That (b) implies (a) was established in a strong quantitative form in Theorem 27.12. To see that (c) and (d) are equivalent, we recall the decomposition (27.45). By (27.46) we know that $H(s) \asymp 1$ uniformly for $\sigma > 1$. Also, $\lim_{\sigma \to 1} D(s) = D(1 + it)$, and $1 + D(1 + it) = 0$ if and only if (27.55ii) holds. Finally,

$$|G(s)(\sigma - 1)| \asymp \Big|\frac{G(s)}{\zeta(\sigma)}\Big| \asymp \exp\Big(-\sum_p \frac{\Re(1 - f(p)p^{-it})}{p}\Big)\,.$$

Here the summands are nonnegative, so the expression is bounded, and tends to 0 if and only if the sum tends to infinity. But since the summands are nonnegative, this is equivalent to (27.55i). Thus (c) and (d) are equivalent. Next we show that (d) implies (b). Suppose first that (27.55i) holds for all t in an interval $[T_1, T_2]$. We observe that

$$F(s)(\sigma-1) \ll \exp\Big(-\sum_p \frac{\Re(1-f(p)p^{-it})}{p^\sigma}\Big).$$

The function on the right hand side decreases to 0 as $\sigma \to 1^+$. Thus we obtain (b) for the interval $[T_1, T_2]$ by appealing to the following elementary consequence of compactness: If $r(\sigma, t)$ is continuous in t for each fixed $\sigma > 1$, and if for each fixed $t \in [T_1, T_2]$ the function $r(\sigma, t)$ is monotonically decreasing to 0 as $\sigma \to 1^+$, then $r(\sigma, t)$ tends to 0 as $\sigma \to 1^+$ uniformly for $t \in [T_1, T_2]$. Now suppose that (27.55i) fails for $t = t_0$, but that (27.55ii) holds for $t = t_0$. Then for $|t - t_0| \le 1$ we have

$$F(s)(\sigma-1) \ll |s-1-it_0| \exp\Big(-\sum_p \frac{\Re(1-f(p)p^{-it})}{p^\sigma}\Big).$$

Again the right hand side decreases monotonically to 0 as $\sigma \to 1^+$, since (27.55i) holds for $0 < |t - t_0| \le 1$. Thus by the compactness principle again, we have (b) uniformly for $|t-t_0| \le 1$. Thus (d) implies (b). Since (b) clearly implies (c), we have shown that (a)–(d) are equivalent.

As for the last assertion, let $t_1 < t_2$ be fixed real numbers, and let $\mathcal{P}$ be the set of primes p for which $\arg p^{i(t_2-t_1)} \in [2\pi/3, 4\pi/3] \pmod{2\pi}$. That is, if

$$I_k = [\exp(2\pi(k+1/3)/(t_2-t_1)), \exp(2\pi(k+2/3)/(t_2-t_1))],$$

then $\mathcal{P}$ consists of those primes such that $p \in I_k$ for some k. By the Prime Number Theorem we see that $\sum_{p\in I_k} 1/p \asymp 1/k$ for all large k. Hence $\sum_{p\in\mathcal{P}} 1/p = +\infty$. If $\Re(1-f(p)p^{-it}) \le 1/2$, then $|\arg f(p)p^{-it}| \le \pi/3$. If this holds for both t_1 and t_2, then $|\arg p^{i(t_1-t_2)}| \le 2\pi/3$. Thus if $p \in \mathcal{P}$, then the inequality $\Re(1-f(p)p^{-it_j}) \le 1/2$ fails for at least one value of j, and so

$$\sum_{p\in\mathcal{P}} \frac{1}{p} \le 2\sum_p \frac{\Re(1-f(p)p^{-it_1})}{p} + 2\sum_p \frac{\Re(1-f(p)p^{-it_2})}{p}.$$

Consequently at most one of the sums on the right is convergent, and the proof is complete. □

Suppose that $f \in \mathcal{M}_0$. If there is a point of the unit circle $|z| = 1$ that is not a limit point of the numbers $f(p)$, then for any $t \ne 0$ there is a delta such that

$$\sum_{\substack{p \\ \Re(1-f(p)p^{-it})>\delta}} \frac{1}{p} = +\infty,$$

so (27.55i) holds for all $t \ne 0$. In closing we mention a commonly occurring situation.

Corollary 27.15 *Suppose that $f \in \mathcal{M}_0$, and that there is a constant $c > 0$ such that $|\Im f(p)| \le c\,\Re(1 - f(p))$. Then*

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n\le x} f(n)$$

exists.

Proof In view of the remark made prior to this Corollary, the condition (27.55i) holds for all $t \neq 0$. If (27.55) holds when $t = 0$, then by Theorem 27.14 the mean value tends to 0. Otherwise,

$$\sum_p \frac{|1 - f(p)|}{p} \le \sum_p \frac{\Re(1 - f(p))| + \Im f(p)|}{p} \le (c+1) \sum_p \frac{\Re(1 - f(p))}{p} < \infty,$$

so the mean value exists and is non-zero, by Corollary 27.4. □

27.5.1 Exercises

1. Put $f(n) = 1$ for $N < n \le 2N$, $f(n) = 0$ otherwise.
(a) By Theorem 1.12, or otherwise, show that $F(s) \ll 1 + \tau/N$ uniformly for $\sigma \ge 1$.
(b) Note that $S(x) \asymp x$ when $x = 2N$.

2. (Montgomery 1978) (a) Let $f_0(n) = i^{\Omega(n)}$. By Theorem 7.18, or otherwise, show that

$$S_0(x) = \sum_{n\le x} f_0(n) = cx(\log x)^{i-1} + O(x(\log x)^{-2})$$

where

$$c = \frac{1}{\Gamma(i)} \prod_p \Big(1 - \frac{i}{p}\Big)^{-1} \Big(1 - \frac{1}{p}\Big)^{i}.$$

(b) Let $F_0(s) = \sum_{n=1}^{\infty} f_0(n) n^{-s}$. By means of Theorem 6.7, or otherwise, show that $F_0(s) \ll \log \tau$ uniformly for $\sigma > 1$.
(c) Let f be a totally multiplicative function with

$$f(p) = \begin{cases} i & (\text{if } \le x^{1/2} \text{ or } p > x), \\ e(\theta_p) & (\text{for } x^{1/2} < p \le x) \end{cases}$$

where the θ_p are to be determined.
(d) Explain why

$$S(x) = \sum_{\substack{n\le x \\ p|n \implies p\le x^{1/2}}} f_0(n) \; + \sum_{x^{1/2}<p\le x} e(\theta_p) S_0(x/p).$$

(e) Deduce that there is a choice of the θ_p such that

$$|S(x)| = \Big| \sum_{\substack{n\le x \\ p|n \Longrightarrow p\le x^{1/2}}} f_0(n)\Big| + \sum_{x^{1/2}<p\le x} |S_0(x/p)|.$$

(f) Show that

$$\sum_{x^{1/2}<p\le x} |S_0(x/p)| \asymp \frac{x\log\log x}{\log x}\,.$$

(g) Show that $M(\alpha) \ll 1$ uniformly for $\alpha > 0$.

3. Recall that the 'negative binomial theorem' asserts that

$$(1-z)^{-r-1} = \sum_{n=0}^{\infty} \binom{n+r}{n} z^n$$

for $|z| < 1$. Here r is any complex number.
(a) Show that if $r > -1$, then $\binom{n+r}{n} \ge 0$ for all n.
(b) Suppose that f is totally multiplicative, that $|f(n)| \le 1$ for all n, and let F be defined as in (27.42). Show that if q is a positive real number, then

$$\int_{T_0-T}^{T_0+T} |F(\sigma+it)|^q\,dt \le 3\int_{-T}^{T} |\zeta(\sigma+it)|^q\,dt\,.$$

(c) Use (27.45) to show that if $f \in \mathcal{M}_0$ and $q > 0$, then

$$\int_{T_0-T}^{T_0+T} |F(\sigma+it)|^q\,dt \ll_q 3\int_{-T}^{T} |\zeta(\sigma+it)|^q\,dt\,.$$

4. (Turán) Let $f(n)$ be an integer-valued additive function, and let $N(x;q,a)$ denote the number of $n \le x$ such that $f(n) \equiv a \pmod q$.
(a) Show that $\lim_{x\to\infty} N(x;,q,a)/x = n(q,a)$ exists for all a and q.
(b) Show that $n(q,a) = 1/q$ for all a if and only if both the following hold:
(i) For each odd prime $p_1|q$,

$$\sum_{\substack{p \\ p_1|f(p)}} \frac{1}{p} = +\infty;$$

(ii) If $2|q$, then

$$\sum_{\substack{p \\ 2|f(p)}} \frac{1}{p} = +\infty$$

or both of the following hold: $f(2^k)$ is odd for all $k > 0$, and if $4|q$, then

$$\sum_{\substack{p \\ 4|f(p)}} \frac{1}{p} = +\infty.$$