D-modules, Hodge modules, and singularities
Notes for a course at the University of Tokyo, June-July 2024

Mircea Mustaţă

Department of Mathematics, University of Michigan, 530 Church Street, Ann Arbor, MI 48109, USA
Email address: mmustata@umich.edu

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CHAPTER 1

Preface

These lecture notes are based on a course taught at the University of Tokyo, in June-July 2024. The goal is to describe some applications of Saito's theory of Hodge modules to invariants of singularities. We begin with a quick overview of modules over the sheaf of differential operators on a smooth algebraic variety and then discuss two important notions in $\mathcal{D}$-module theory: the b-function (or Bernstein-Sato polynomial) and the V-filtration of Malgrange and Kashiwara, as well as the connection between them. We also give a quick introduction to the key technical tool we will be using: the theory of pure and mixed Hodge modules developed by Morihiko Saito. The second part of the course is devoted to the definition and study of several invariants of singularities that come out of Hodge module theory. In particular, we focus on one such invariant, the *minimal exponent* of a hypersurface, that provides a refinement of a familiar invariant in birational geometry, the *log canonical threshold*. In particular, we discuss how certain higher-order analogues of multiplier ideals, the *Hodge ideals*, can be used to prove the main properties of the minimal exponent. We also discuss the role of the minimal exponent in characterizing higher-order versions of the notions of rational and Du Bois hypersurface singularities. We end with a generalization of the theory of minimal exponents to the case of complete intersections of arbitrary codimension.

CHAPTER 2

An introduction to $\mathcal{D}$-modules

Our goal in this chapter is to give a quick introduction to the theory of $\mathcal{D}$-modules. Let X be a smooth, irreducible, n-dimensional algebraic variety over k. We will mainly be interested in the case $k = \mathbf{C}$, but unless explicitly mentioned otherwise, in this chapter k is an arbitrary algebraically closed field of characteristic 0. A good reference for the material in this chapter is [**HTT08**].

2.1. The sheaf of differential operators

The *sheaf of differential operators* on X is the subsheaf of k-algebras $\mathcal{D}_X \subseteq \mathcal{E}nd_k(\mathcal{O}_X)$ generated by $\mathcal{O}_X$ (with $\mathcal{O}_X$ acting on itself by multiplication) and the sheaf of k-derivations $\mathcal{T}_X = \mathcal{D}er_k(\mathcal{O}_X)$. If $x_1, \ldots, x_n$ form an algebraic system of coordinates on an open subset $U \subseteq X$ (that is, $dx_1, \ldots, dx_n$ trivialize Ω_U and we have the dual basis of $\mathcal{T}_U$ given by $\partial_1, \ldots, \partial_n$), then $\mathcal{D}_X|_U = \mathcal{D}_U$ is a free left (or right) $\mathcal{O}_U$-module, with basis given by $\partial^\alpha := \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$ for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbf{Z}^n_{\geq 0}$. An immediate consequence is that $\mathcal{D}_X$ is a quasi-coherent sheaf of left (or right) $\mathcal{O}_X$-modules. If X is affine, with $R = \mathcal{O}_X(X)$, then $D_R = \Gamma(X, \mathcal{D}_X)$ is the *ring of differential operators* of R.

EXAMPLE 2.1. The *Weyl algebra* $A_n = A_n(k)$ is the ring of differential operators of $R = k[x_1, \ldots, x_n]$. One can show that the kernel of the canonical morphism from the free associative k-algebra

$$k\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n \rangle$$

with generators $x_1, \ldots, x_n, \partial_1, \ldots, \partial_n$ to $A_n(k)$ is the 2-sided ideal generated by the following elements: $[x_i, x_j]$, $[\partial_i, \partial_j]$, and $[\partial_i, x_j] - \delta_{i,j}$ for $1 \leq i, j \leq n$, where $\delta_{i,j} = 1$ if $i = j$ and $\delta_{i,j} = 0$ otherwise.

REMARK 2.2. It is important to note right away that $\mathcal{D}_X$ is a sheaf of *non-commutative* rings. In fact, for every $D \in \mathcal{D}er_k(\mathcal{O}_X)$ and every $f \in \mathcal{O}_X$, we have

$$[D, f] = D(f) \quad \text{in} \quad \mathcal{E}nd_k(\mathcal{O}_X), \tag{2.1}$$

where for $A, B \in \mathcal{E}nd_k(\mathcal{O}_X)$, we put $[A, B] = AB - BA$. Indeed, this assertion follows directly from the Leibniz formula.

2.1.1. The order filtration. In order to go from the non-commutative setting to the commutative one, the key tool is provided by the *order filtration* $F_\bullet\mathcal{D}_X$ on $\mathcal{D}_X$. This can be defined in terms of local coordinates $x_1, \ldots, x_n$ as follows: $F_p\mathcal{D}_X$ is the subsheaf of (left or right) $\mathcal{O}_X$-submodules of $\mathcal{D}_X$ generated by ∂^α, with $|\alpha| := \sum_i \alpha_i \leq p$. Note that $F_p\mathcal{D}_X = 0$ for $p < 0$, $F_0\mathcal{D}_X = \mathcal{O}_X$, and $F_1\mathcal{D}_X = \mathcal{D}er_k(\mathcal{O}_X)$.

REMARK 2.3. It is clear that we have $F_p\mathcal{D}_X \subseteq F_q\mathcal{D}_X$ for every $p \leq q$. Moreover, it is easy to show by induction on $p + q$ that the following hold for all $p, q \geq 0$:

i) $F_p\mathcal{D}_X \cdot F_q\mathcal{D}_X \subseteq F_{p+q}\mathcal{D}_X$;
ii) $[F_p\mathcal{D}_X \cdot F_q\mathcal{D}_X] \subseteq F_{p+q-1}\mathcal{D}_X$.

The key point is that for every $D_1, D_2 \in \mathcal{D}er_k(\mathcal{O}_X)$ and every $f \in \mathcal{O}_X$, we have $[D_1, f] = D_1(f)$ and $[D_1, D_2] \in \mathcal{D}er_k(\mathcal{O}_X)$.

Property i) in Remark 2.3 implies that

$$\mathrm{Gr}^F_\bullet(\mathcal{D}_X) := \bigoplus_p F_p\mathcal{D}_X/F_{p-1}\mathcal{D}_X$$

is a sheaf of graded rings and property ii) implies that it is a sheaf of commutative rings. Since $\mathrm{Gr}^F_0(\mathcal{D}_X) = \mathcal{O}_X$ and $\mathrm{Gr}^F_1(\mathcal{D}_X) = \mathcal{T}_X$, the universal property of the symmetric algebra induces a canonical morphism of sheaves of graded rings

$$\mathcal{S}ym^\bullet_{\mathcal{O}_X}(\mathcal{T}_X) \to \mathrm{Gr}^F_\bullet(\mathcal{D}_X). \tag{2.2}$$

By considering the description of both sides with respect to local algebraic coordinates, we see that (2.2) is an isomorphism. We will thus identify $\mathrm{Gr}^F_\bullet(\mathcal{D}_X)$ with $\pi_*\mathcal{O}_{T^*X}$, where $\pi\colon T^*X \to X$ is the canonical projection of the cotangent bundle of X. Using the isomorphism (2.2), it is easy to deduce that for every affine open subset $U \subseteq X$, the ring $\mathcal{D}_X(U)$ is both left and right Noetherian.

2.1.2. A presentation of the sheaf of differential operators. In order to describe sheaves of $\mathcal{D}_X$-modules, it is convenient to have a presentation of $\mathcal{D}_X$. For simplicity, we only formulate this when X is an affine variety, with $R = \mathcal{O}_X(X)$ and $T_R = \Gamma(X, \mathcal{T}_X)$. In this case, the ring D_R is isomorphic to the free associative k-algebra A generated by

$$\{\widetilde{a} \mid a \in R\} \quad \text{and} \quad \{\widetilde{D} \mid D \in T_R\},$$

modulo the following relations:

i) $\widetilde{a_1 a_2} = \widetilde{a_1} \cdot \widetilde{a_2}$ and $\widetilde{a_1 + a_2} = \widetilde{a_1} + \widetilde{a_2}$ for every $a_1, a_2 \in R$;
ii) $\widetilde{D_1 + D_2} = \widetilde{D_1} + \widetilde{D_2}$ for every $D_1, D_2 \in T_R$;
iii) $\widetilde{aD} = \widetilde{a} \cdot \widetilde{D}$ for every $a \in R$ and $D \in T_R$;
iv) $[\widetilde{D}, \widetilde{a}] = \widetilde{D(a)}$ for every $a \in R$ and $D \in T_R$;
v) $[\widetilde{D_1}, \widetilde{D_2}] = \widetilde{[D_1, D_2]}$ for every $D_1, D_2 \in T_R$.

2.2. Left and right $\mathcal{D}$-modules

A left (right) $\mathcal{D}$-module on X is a sheaf of left (respectively, right) $\mathcal{D}_X$-modules. Two basic examples of left $\mathcal{D}$-modules are $\mathcal{D}_X$ and $\mathcal{O}_X$ (note that we have a tautological action of $\mathcal{D}_X$ on $\mathcal{O}_X$ since $\mathcal{D}_X$ is a subsheaf of $\mathcal{E}nd_k(\mathcal{O}_X)$). Of course, $\mathcal{D}_X$ also has a natural right $\mathcal{D}_X$-module structure that makes it a bimodule.

The injective homomorphism $\mathcal{O}_X \hookrightarrow \mathcal{D}_X$ implies that every left $\mathcal{D}_X$-module has an induced structure of $\mathcal{O}_X$-module. For an $\mathcal{O}_X$-module $\mathcal{M}$, giving a left $\mathcal{D}_X$-module structure on $\mathcal{M}$ that induces the given $\mathcal{O}_X$-module structure is equivalent to giving an *integrable connection* on $\mathcal{M}$. Recall that a *connection* on $\mathcal{M}$ is a k-linear morphism

$$\nabla\colon \mathcal{M} \to \Omega_X \otimes_{\mathcal{O}_X} \mathcal{M}$$

that satisfies the *Leibniz condition*

$$\nabla(fu) = f\nabla(u) + df \otimes u \quad \text{for all} \quad f \in \mathcal{O}_X, u \in \mathcal{M}.$$

Given such a connection ∇ on $\mathcal{M}$, we define

$$\nabla\colon \Omega_X^p \otimes_{\mathcal{O}_X} \mathcal{M} \to \Omega_X^{p+1} \otimes_{\mathcal{O}_X} \mathcal{M}$$

for $p \geq 1$ by

$$\nabla(\eta \otimes u) = d\eta \otimes u + (-1)^p \eta \wedge \nabla(u).$$

The connection is *integrable* if $\nabla \circ \nabla = 0$ on $\mathcal{M}$; in this case, one can show that $\nabla \circ \nabla = 0$ on every $\Omega_X^p \otimes_{\mathcal{O}_X} \mathcal{M}$, hence we get a complex, the *de Rham complex* $\mathrm{DR}_X(\mathcal{M})$ of $\mathcal{M}$:

$$0 \to \mathcal{M} \xrightarrow{\nabla} \Omega_X^1 \otimes_{\mathcal{O}_X} \mathcal{M} \xrightarrow{\nabla} \dots \xrightarrow{\nabla} \Omega_X^n \otimes_{\mathcal{O}_X} \mathcal{M} \to 0.$$

We will always view this as placed in cohomological degrees $-n, \dots, 0$. We emphasize that the maps in this complex are not $\mathcal{O}_X$-linear.

The relation between the $\mathcal{D}_X$-module structure on a left $\mathcal{D}_X$-module $\mathcal{M}$ and the corresponding connection is as follows: if $D \in \mathrm{Der}_k(\mathcal{O}_X)$ and ∇ is the connection on $\mathcal{M}$, then the composition

$$\nabla_D\colon \mathcal{M} \xrightarrow{\nabla} \Omega_X \otimes_{\mathcal{O}_X} \mathcal{M} \xrightarrow{D\otimes \mathrm{Id}} \mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{M} = \mathcal{M}$$

corresponds to left multiplication by D. The fact that this gives a bijection between left $\mathcal{D}_X$-module structures on $\mathcal{M}$ that induce the given $\mathcal{O}_X$-module structure and integrable connections on $\mathcal{M}$ follows from the local description of $\mathcal{D}_X$ in Section 2.1.2: the Leibniz rule corresponds to the fact that the relations in iv) vanish, while the integrability of the connection corresponds to the fact that the relations in v) vanish.

EXAMPLE 2.4. The "smooth" left $\mathcal{D}_X$-modules are those that are coherent as $\mathcal{O}_X$-modules ($\mathcal{O}_X$ is one such example). In fact, every such $\mathcal{D}_X$-module is a free $\mathcal{O}_X$-module of finite rank; in other words, it is a vector bundle with an integrable connection. This can be shown as follows: suppose that $\mathcal{M}$ is an $\mathcal{O}_X$-coherent left $\mathcal{D}_X$-module and $x \in X$ is such that $\mathcal{M}_x$ is not a free $\mathcal{O}_{X,x}$-module. If $u_1, \dots, u_d \in \mathcal{M}_x$ give a minimal system of generators as $\mathcal{O}_{X,x}$-module and if $a_1, \dots, a_d \in \mathcal{O}_{X,x}$ are not all 0 and $\sum_i a_i u_i = 0$, then we argue by induction on the smallest order[1] $q = \min_i \mathrm{ord}_x(a_i) > 0$ to get a contradiction. The key point is to choose i and a derivation D such that $\mathrm{ord}_x\big(D(a_i)\big) = q - 1$ to get another relation for which q is replaced by $q - 1$. For details, see [**HTT08**, Theorem 1.4.10].

EXAMPLE 2.5. If $f \in \mathcal{O}_X(X)$ is nonzero, then we have a natural left $\mathcal{D}_X$-module structure on the $\mathcal{O}_X$-module $\mathcal{O}_X[1/f]$. This corresponds to the connection

$$\nabla\colon \mathcal{O}_X[1/f] \to \Omega_X \otimes_{\mathcal{O}_X} \mathcal{O}_X[1/f]$$

given by the quotient rule:

$$\nabla\big(\tfrac{g}{f^q}\big) = dg \otimes \tfrac{1}{f^q} - df \otimes \tfrac{qg}{f^{q+1}}.$$

More generally, if Z is a hypersurface in X, then by considering local equations defining Z, we obtain a natural left $\mathcal{D}_X$-module structure on

$$\mathcal{O}_X(*Z) := \bigcup_{q \geq 1} \mathcal{O}_X(qZ).$$

This example will play an important role in the later chapters.

[1] The order at x of $g \in \mathcal{O}_{X,x}$ is the largest q such that $g \in \mathfrak{m}_x^q$, where $\mathfrak{m}_x \subset \mathcal{O}_{X,x}$ is the maximal ideal, with the convention that this is ∞ if $g = 0$.

EXAMPLE 2.6. For every closed subset Z of X, we have the q^{th} local cohomology functor $\mathcal{M} \mapsto \mathcal{H}^q_Z(\mathcal{M})$. This is the q^{th} derived functor of the left exact functor that maps $\mathcal{M}$ to its subsheaf of sections supported on Z. For every q, the sheaf $\mathcal{H}^q_Z(\mathcal{O}_X)$ has a natural left $\mathcal{D}_X$-module structure.

In order to describe this, we argue locally and assume that X is affine, with $R = \mathcal{O}_X(X)$, and Z is defined by the ideal $I = (f_1, \ldots, f_r)$. In this case, it is known that $H^q_I(R) = \Gamma\big(X, \mathcal{H}^q_Z(\mathcal{O}_X)\big)$ is the q^{th} cohomology of the Čech type complex

$$0 \to R \to \bigoplus_i R_{f_i} \to \bigoplus_{i<j} R_{f_i f_j} \to \ldots \to R_{f_1 \cdots f_r} \to 0,$$

placed in cohomological degrees $0, \ldots, r$. By the previous example, each term in this complex has a natural left D_R-module structure and the maps in the complex (whose component are given, up to signs, by localization maps) are D_R-linear. This implies that each cohomology group has an induced structure of a left D_R-module. It is not hard to see that this structure is independent of the choice of ideal I and system of generators $f_1, \ldots, f_r$. Moreover, when X is not-necessarily-affine, the $\mathcal{D}_{U_i}$-modules $\mathcal{H}^q_{Z\cap U_i}(\mathcal{O}_{U_i})$ glue to a left $\mathcal{D}_X$-module structure on $\mathcal{H}^q_Z(\mathcal{O}_X)$. A special case is that when Z is a hypersurface in X, in which case $\mathcal{H}^1_Z(\mathcal{O}_X) = \mathcal{O}_X(*Z)/\mathcal{O}_X$ (and all other local cohomology sheaves vanish). The theory of $\mathcal{D}$-modules was used to prove interesting finiteness properties of local cohomology modules, see [**Lyu93**].

EXAMPLE 2.7. The previous two examples admit obvious extensions to the case when $\mathcal{O}_X$ is replaced by any left $\mathcal{D}_X$-module $\mathcal{M}$. More precisely, if $f \in \mathcal{O}_X(X)$, then $\mathcal{M}[1/f] = \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X[1/f]$ has a natural structure of left $\mathcal{D}_X$-module. Similarly, if Z is any closed subset of X, then each local cohomology sheaf $\mathcal{H}^q_Z(\mathcal{M})$ has a natural left $\mathcal{D}_X$-module structure. However, we will not use this general framework in these notes.

2.2.1. Left vs. right $\mathcal{D}$-modules. In what follows we will mostly consider left $\mathcal{D}_X$-modules, but right $\mathcal{D}_X$-modules sometimes naturally enter the picture (for example, when considering duality and the direct image of $\mathcal{D}$-modules). An important fact is that there is a canonical equivalence of categories between left and right $\mathcal{D}_X$-modules, as follows. If we denote by $\mathcal{M}^r$ the right $\mathcal{D}_X$-module corresponding to the left $\mathcal{D}_X$-module $\mathcal{M}$, then as $\mathcal{O}_X$-modules we have $\mathcal{M}^r = \mathcal{M} \otimes_{\mathcal{O}_X} \omega_X$, where $\omega_X = \Omega^n_X$, so $\mathcal{M} \simeq \mathcal{H}om_{\mathcal{O}_X}(\omega_X, \mathcal{M}^r)$. The right action of $\mathcal{D}_X$ on $\mathcal{M} \otimes_{\mathcal{O}_X} \omega_X$ is easy to describe in a system of coordinates $x_1, \ldots, x_n$, as follows. The map given by the *classical adjoint*

$$P = \sum_{\alpha \in \mathbf{Z}^n_{\geq 0}} h_\alpha \partial_x^\alpha \mapsto {}^t P := \sum_{\alpha \in \mathbf{Z}^n_{\geq 0}} (-\partial_x)^\alpha h_\alpha \tag{2.3}$$

(which depends on the choice of coordinates) is an automorphism $\mathcal{D}_X \simeq \mathcal{D}_X^{\text{op}}$ and if $dx = dx_1 \wedge \ldots \wedge dx_n$, then

$$(m \otimes dx) \cdot P = ({}^t P \cdot m) \otimes dx.$$

For the fact that, in general, we get a global equivalence of categories, see [**HTT08**, Section 1.2].

REMARK 2.8. The classical adjoint globalizes to a ring isomorphism $\mathcal{D}_X^{\text{op}} \to \omega_X \otimes_{\mathcal{O}_X} \mathcal{D}_X \otimes_{\mathcal{O}_X} \omega_X^{-1}$ that in the presence of a local system of coordinates $x_1, \ldots, x_n$

maps P to $dx \otimes {}^tP \otimes dx^{-1}$. Using this, we see that if $\mathcal{F}$ is an $\mathcal{O}_X$-module and $\mathcal{M} = \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{F}$, viewed as a left $\mathcal{D}_X$-module via the left $\mathcal{D}_X$-module structure on $\mathcal{D}_X$, then the corresponding right $\mathcal{D}_X$-module is

$$\begin{aligned}\mathcal{M}^r = \omega_X \otimes_{\mathcal{O}_X} \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{F} &\simeq (\omega_X \otimes_{\mathcal{O}_X} \mathcal{D}_X \otimes_{\mathcal{O}_X} \omega_X^{-1}) \otimes_{\mathcal{O}_X} (\omega_X \otimes_{\mathcal{O}_X} \mathcal{F}) \\ &\simeq (\omega_X \otimes_{\mathcal{O}_X} \mathcal{F}) \otimes_{\mathcal{O}_X} \mathcal{D}_X,\end{aligned}$$

where now the right $\mathcal{D}_X$-module structure is induced by the right $\mathcal{D}_X$-module structure on $\mathcal{D}_X$.

2.3. Filtrations on coherent $\mathcal{D}$-modules

From now on, unless explicitly mentioned otherwise, all our $\mathcal{D}_X$-modules are left $\mathcal{D}_X$-modules. Let $\mathcal{M}$ be a *coherent* such $\mathcal{D}_X$-module: this means that $\mathcal{M}$ is quasi-coherent as an $\mathcal{O}_X$-module and it is locally finitely generated over $\mathcal{D}_X$. In this case one can show that there is a *good filtration* on $\mathcal{M}$, that is, an increasing filtration $(F_p\mathcal{M})_{p\in\mathbf{Z}}$ with the following properties:

i) $F_p\mathcal{M} = 0$ for $p \ll 0$ and $\mathcal{M} = \bigcup_{p\in\mathbf{Z}} F_p\mathcal{M}$.
ii) $F_p\mathcal{D}_X \cdot F_q\mathcal{M} \subseteq F_{p+q}\mathcal{M}$ for all $p, q \in \mathbf{Z}$.
iii) $\mathrm{Gr}_\bullet^F(\mathcal{M}) := \bigoplus_{p\in\mathbf{Z}} F_p\mathcal{M}/F_{p-1}\mathcal{M}$ is locally finitely generated over $\mathrm{Gr}_\bullet^F(\mathcal{D}_X)$.

It is easy to see that the condition in iii) is equivalent to the fact that each $F_p\mathcal{M}$ is a coherent $\mathcal{O}_X$-module and there is q such that the inclusion in ii) is an equality for all $p \geq 0$ (in this case, we say that the filtration $F_\bullet\mathcal{M}$ is *generated at level* q).

Suppose now that $F_\bullet\mathcal{M}$ is a good filtration on $\mathcal{M}$. Note that if $\pi\colon T^*X \to X$ is the cotangent bundle of X, then it follows from condition iii) above that we have a coherent sheaf $\mathcal{F}$ on T^*X such that $\pi_*(\mathcal{F}) \simeq \mathrm{Gr}_\bullet^F(\mathcal{M})$. While $\mathcal{F}$ depends on the choice of filtration, its support $\mathrm{Supp}(\mathcal{F})$ (viewed as a subset of T^*X) is independent of the filtration. This is the *characteristic variety* (or *singular support*) $\mathrm{SS}(\mathcal{M})$. The *dimension* $\dim(\mathcal{M})$ of $\mathcal{M}$ is, by definition, the dimension of $\mathrm{SS}(\mathcal{M})$. Note that $\dim(\mathcal{M}) \leq \dim(T^*X) = 2n$.

Example 2.9. If we take on $\mathcal{D}_X$ the order filtration, then we see that $\mathrm{SS}(\mathcal{D}_X) = T^*X$, and thus $\dim(\mathcal{D}_X) = 2n$.

Example 2.10. Suppose now that $\mathcal{E}$ is an $\mathcal{O}_X$-coherent $\mathcal{D}_X$-module (for example, we could take $\mathcal{E} = \mathcal{O}_X$). In this case we can take $F_p\mathcal{E} = 0$ for $p < 0$ and $F_p\mathcal{E} = \mathcal{E}$ for $p \geq 0$, so $\mathrm{Gr}_\bullet^F(\mathcal{E}) = \mathcal{E}$ and $\mathrm{SS}(\mathcal{M})$ is the zero-section of T^*X.

Remark 2.11. Recall that if $\mathcal{M}$ is a quasi-coherent $\mathcal{O}_X$-module, the *support* $\mathrm{Supp}(\mathcal{M})$ is the subset $\{x \in X \mid \mathcal{M}_x \neq 0\}$ of X. If $\mathcal{M}$ is a coherent module, then $\mathrm{Ann}_{\mathcal{O}_X}(\mathcal{M})$ is a coherent ideal of $\mathcal{O}_X$ and $\mathrm{Supp}(\mathcal{M})$ is the closed subset defined by this ideal. However, in general, $\mathrm{Supp}(\mathcal{M})$ is not a closed subset of X. On the other hand, if $\mathcal{M}$ is a coherent $\mathcal{D}_X$-module, $F_\bullet\mathcal{M}$ is a good filtration on $\mathcal{M}$, and $\mathcal{F}$ is the corresponding sheaf on T^*X, then one can easily check that

$$\mathrm{Supp}(\mathcal{M}) = \pi\big(\mathrm{Supp}(\mathcal{F})\big) = i_0^{-1}\big(\mathrm{Supp}(\mathcal{F})\big),$$

where $i_0\colon X \hookrightarrow T^*X$ is the zero-section. In particular, we see that in this case $\mathrm{Supp}(\mathcal{M})$ is a closed subset of X.

Remark 2.12. We may and will consider good filtrations also on right $\mathcal{D}_X$-modules. The convention regarding filtrations is that if $F_\bullet\mathcal{M}$ is a filtration on the

left $\mathcal{D}_X$-module $\mathcal{M}$ and $\mathcal{M}^r = \omega_X \otimes_{\mathcal{O}_X} \mathcal{M}$ is the corresponding right $\mathcal{D}_X$-module, then the corresponding filtration on $\mathcal{M}^r$ is given by

$$F_{p-n}\mathcal{M}^r = \omega_X \otimes_{\mathcal{O}_X} F_p\mathcal{M}. \tag{2.4}$$

2.3.1. Holonomic $\mathcal{D}$-modules. We have the following fundamental result regarding the dimension of coherent $\mathcal{D}_X$-modules (*Bernstein's inequality*):

THEOREM 2.13. *If X is a smooth, irreducible, n-dimensional algebraic variety and $\mathcal{M}$ is a coherent $\mathcal{D}_X$-module, then every irreducible component of* $\mathrm{SS}(\mathcal{M})$ *has dimension $\geq n$.*

This result was first proved by Bernstein [**Ber71**] in the case $X = \mathbf{A}^n$ and then, in general and in a stronger form, via analytic methods, by Sato, Kashiwara, and Kawai [**SKK73**]. A purely algebraic proof was given by Gabber in [**Gab81**]. In Section 2.4 below, we give Bernstein's proof.

Motivated by the above theorem, we now introduce the most important class of coherent $\mathcal{D}$-modules:

DEFINITION 2.14. A coherent $\mathcal{D}_X$-module $\mathcal{M}$ is *holonomic* if $\dim(\mathcal{M}) = n$ or $\mathcal{M} = 0$.

As a consequence of Bernstein's inequality, holonomic $\mathcal{D}_X$-modules satisfy some formal properties similar to those of 0-dimensional coherent $\mathcal{O}_X$-modules. Indeed, note first that if we have a short exact sequence of coherent $\mathcal{D}_X$-modules

$$0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0, \tag{2.5}$$

after choosing a good filtration on $\mathcal{M}$ and taking the induced filtrations on $\mathcal{M}'$ and $\mathcal{M}''$, we obtain the short exact sequence

$$0 \to \mathrm{Gr}^F_\bullet(\mathcal{M}') \to \mathrm{Gr}^F_\bullet(\mathcal{M}) \to \mathrm{Gr}^F_\bullet(\mathcal{M}'') \to 0, \tag{2.6}$$

so $\dim(\mathcal{M}) = \max\big\{\dim(\mathcal{M}'), \dim(\mathcal{M}'')\big\}$. In particular, Theorem 2.13 implies that $\mathcal{M}$ is holonomic if and only if both $\mathcal{M}'$ and $\mathcal{M}''$ are holonomic. It is then easy to see that the holonomic $\mathcal{D}_X$-modules form an abelian category.

EXERCISE 2.15. Suppose that we have algebraic coordinates $x_1, \ldots, x_n$ on X, centered at a point $P \in X$.

i) Show that we have an isomorphism of left $\mathcal{D}_X$-modules
$$\mathcal{H}^n_{\{P\}}(\mathcal{O}_X) \simeq \mathcal{D}_X/\mathcal{D}_X \cdot (x_1, \ldots, x_n).$$
ii) Show that the characteristic variety of $\mathcal{H}^n_{\{P\}}(\mathcal{O}_X)$ is $T^*_P X$; in particular, $\mathcal{H}^n_{\{P\}}(\mathcal{O}_X)$ is a holonomic $\mathcal{D}_X$-module.

2.3.2. The characteristic cycle. A more refined invariant than the characteristic variety is the characteristic cycle of a coherent $\mathcal{D}_X$-module. Given a good filtration on the coherent $\mathcal{D}_X$-module $\mathcal{M}$, if $\mathcal{F}$ is the coherent sheaf on T^*X corresponding to $\mathrm{Gr}^F_\bullet(\mathcal{M})$, the *characteristic cycle* $\mathrm{CC}(\mathcal{M})$ is

$$\mathrm{CC}(\mathcal{M}) := \sum_Z \ell(\mathcal{F}_{\eta_Z})[Z],$$

where the sum is over the irreducible components Z of $\mathrm{SS}(\mathcal{M})$ and η_Z is the generic point of Z. It is a bit more subtle than in the case of the characteristic variety, but one can show that the characteristic cycle, too, is independent of the choice of good

filtration. Given a short exact sequence (2.5) of holonomic $\mathcal{D}_X$ modules, we deduce from the corresponding exact sequence (2.6) that $\mathrm{CC}(\mathcal{M}) = \mathrm{CC}(\mathcal{M}') + \mathrm{CC}(\mathcal{M}'')$. Since we clearly have $\mathrm{CC}(\mathcal{M}) = 0$ if and only if $\mathcal{M} = 0$, it is easy to deduce that in the category of holonomic $\mathcal{D}_X$-modules, every object has finite length.

2.3.3. The graded de Rham complex. If $\mathcal{M}$ is a coherent $\mathcal{D}_X$-module and $F_\bullet\mathcal{M}$ is a good filtration on $\mathcal{M}$, then the de Rham complex $\mathrm{DR}_X(\mathcal{M})$ becomes a graded complex, whose p^{th} graded piece is

$$F_p\mathrm{DR}_X(\mathcal{M}): \ 0 \to F_p\mathcal{M} \to \Omega^1_X \otimes_{\mathcal{O}_X} F_{p+1}\mathcal{M} \to \ldots \to \Omega^n_X \otimes_{\mathcal{O}_X} F_{p+n}\mathcal{M} \to 0.$$

In order to see that this is a subcomplex of $\mathrm{DR}_X(\mathcal{M})$, note that if $x_1, \ldots, x_n$ are local algebraic coordinates on $U \subseteq X$, then

$$\nabla(\eta \otimes u) = d\eta \otimes u + \sum_{i=1}^{n} (dx_i \wedge \eta) \otimes \partial_i u.$$

Moreover, this formula shows that each graded piece $\mathrm{Gr}^F_p\mathrm{DR}_X(\mathcal{M})$

$$0 \to \mathrm{Gr}^F_p(\mathcal{M}) \to \Omega^1_X \otimes_{\mathcal{O}_X} \mathrm{Gr}^F_{p+1}(\mathcal{M}) \to \ldots \to \Omega^n_X \otimes_{\mathcal{O}_X} \mathrm{Gr}^F_{p+n}(\mathcal{M}) \to 0$$

is a complex of $\mathcal{O}_X$-modules: indeed, the differential is given on U by

$$\eta \otimes \overline{u} \mapsto \sum_{j=1}^{n} (dx_i \wedge \eta) \otimes \overline{\partial_i u}$$

and this is clearly $\mathcal{O}_X$-linear.

Example 2.16. An important example of de Rham complex is

$$\mathrm{DR}_X(\mathcal{D}_X): \ 0 \to \mathcal{D}_X \to \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{D}_X \to \ldots \to \Omega^n_X \otimes_{\mathcal{O}_X} \mathcal{D}_X \to 0.$$

Note that this is a complex of right $\mathcal{D}_X$-modules. Moreover, the canonical right $\mathcal{D}_X$-module structure on $\omega_X = \Omega^n_X$ gives a canonical map of right $\mathcal{D}_X$-modules $\Omega^n_X \otimes_{\mathcal{O}_X} \mathcal{D}_X \to \omega_X$. We claim that in this way, $\mathrm{DR}_X(\mathcal{D}_X)$ gives a resolution of ω_X by locally free right $\mathcal{D}_X$-modules. In order to see this, we consider the order filtration on $\mathcal{D}_X$, as well as the standard filtration on ω_X (corresponding to the filtration on $\mathcal{O}_X$ given by $F_0\mathcal{O}_X = \mathcal{O}_X$ and $F_1\mathcal{O}_X = 0$), and it is easy to see that it is enough to show that the induced morphism

$$C^\bullet = \bigoplus_p \mathrm{Gr}^F_p\mathrm{DR}_X(\mathcal{D}_X) \to \omega_X$$

is a quasi-isomorphism. The assertion follows by considering local coordinates $x_1, \ldots, x_n$ on $U \subseteq X$ and by identifying $C^\bullet$ on U with the twist by ω_U of the Koszul complex of $y_1, \ldots, y_n$ on $R[y_1, \ldots, y_n]$.

2.4. Holonomic $\mathcal{D}$-modules over the Weyl algebra

In this section, we discuss modules over the Weyl algebra. In this setting, it is easier to give direct proofs for some of the basic results involving coherent $\mathcal{D}$-modules. Our presentation follows the one in [**Cou95**]. As before, let k be a field of characteristic 0 (in this section, the hypothesis that k is algebraically closed does not play any role).

Let us fix a positive integer n. Recall that the Weyl algebra A_n is the ring of differential operators of $R = k[x_1, \ldots, x_n]$. Note that A_n has a basis over k given by $x^\alpha \partial^\beta$, for $\alpha, \beta \in \mathbf{Z}^n_{\geq 0}$.

2.4.1. The Bernstein filtration. What distinguishes the case of the affine space is that we have another natural filtration on A_n, whose terms are finite-dimensional vector spaces over k. More precisely, we have the following

DEFINITION 2.17. The *Bernstein filtration* on A_n is given by

$$B_pA_n = \bigoplus_{|\alpha|+|\beta|\leq p} kx^\alpha\partial^\beta.$$

Note that in this case B_pA_n is not a module over $k[x_1,\dots,x_n]$ anymore, but just a k-vector space. The new feature is that $\dim_k B_pA_n < \infty$ for all p. Again, it is easy to show, by induction on $p+q$, that the following hold for every $p,q\geq 0$:

i) $B_pA_n \cdot B_qA_n \subseteq B_{p+q}A_n$.
ii) $[B_pA_n, B_qA_n] \subseteq B_{p+q-2}A_n$.

Assertion i) above implies that $\mathrm{Gr}^B_\bullet(A_n)$ has an induced ring structure and assertion ii) implies that $\mathrm{Gr}^B_\bullet(A_n)$ is commutative. Since $\mathrm{Gr}^B_m(A_n)$ is free, with a basis given by the monomials of degree m in $x_1,\dots,x_n,y_1,\dots,y_n$, where $y_i = \overline{\partial_i}$ for $1\leq i\leq n$, it follows that the unique morphism of graded k-algebras

$$k[x_1,\dots,x_n,y_1,\dots,y_n] \to \mathrm{Gr}^B_\bullet(A_n)$$

that maps the x_i to x_i and the y_i to y_i is an isomorphism. Using this, we easily deduce that A_n is both left and right Noetherian.

2.4.2. Filtrations on A_n-modules. In order to fix ideas, we only consider left A_n-modules. For every such finitely generated A_n-module M, we may consider *good filtrations* $B_\bullet M$ with respect to $B_\bullet A_n$. These are increasing filtrations that satisfy

i) $B_p\mathcal{M} = 0$ for $p \ll 0$ and $\mathcal{M} = \bigcup_{p\in\mathbf{Z}} B_p\mathcal{M}$.
ii) $B_pA_n \cdot B_q\mathcal{M} \subseteq B_{p+q}\mathcal{M}$ for all $p,q\in\mathbf{Z}$.
iii) $\mathrm{Gr}^B_\bullet(\mathcal{M}) := \bigoplus_{p\in\mathbf{Z}} B_p\mathcal{M}/B_{p-1}\mathcal{M}$ is locally finitely generated over $\mathrm{Gr}^B_\bullet(A_n)$.

For example, if M is generated as an A_n-module by $u_1,\dots,u_r$, then we may take

$$B_pM = \sum_{i=1}^r B_pA_n\cdot u_i.$$

We note that each B_pM is a k-vector subspace of M, but it is not true anymore that B_pM is a module over $k[x_1,\dots,x_n]$. In fact, if $B_\bullet M$ is a good filtration, then $\dim_k B_pM < \infty$ for all p.

Given a finitely generated A_n-module M and a good filtration $B_\bullet M$ on M, we again consider the support of the finitely generated $\mathrm{Gr}^B_\bullet(A_n)$-module $\mathrm{Gr}^B_\bullet(M)$ (with the reduced structure). This is the *characteristic variety* $\mathrm{Char}_B(M)$. Again, the characteristic variety is independent of the choice of good filtration and $\mathrm{Char}_B(M)\neq\emptyset$ if and only if $M\neq 0$. We define the *dimension* $\dim(M)$ as the dimension of $\mathrm{Char}_B(M)$ (we make the convention that $\dim(M) = -1$ if $M = 0$).

REMARK 2.18. If M is a finitely generated A_n-module, we have two notions of characteristic variety of M, depending on whether we work with the order filtration or the Bernstein filtration. The resulting varieties are really different: consider for example the module $M = A_1(k)/A_1(k)\cdot(x^2+\partial^2)$. In this case, $\mathrm{SS}(M)$ is defined by the ideal $(y)\subseteq k[x,y]$, while $\mathrm{Char}_B(M)$ is defined by (x^2+y^2) (in particular, one variety is irreducible, while the other one is not). However, one can show that the

two notions of dimension coincide, since they both admit the same cohomological interpretation.

The advantage in the setting of the Bernstein filtration is that $\mathrm{Gr}^B_\bullet(A_n)$ is a polynomial ring in $2n$ variables with the *standard grading* (note that, with respect to the order filtration, half of the variables have degree 0). In this case, it is a basic result of commutative algebra that if $B_\bullet M$ is a good filtration on the nonzero A_n-module M and if P is the Hilbert polynomial of the graded $\mathrm{Gr}^B_\bullet(A_n)$-module $\mathrm{Gr}^B_\bullet(M)$, we have

$$\dim\big(\mathrm{Gr}^B_\bullet(M)\big) = \deg(P) + 1$$

(with the convention that the degree of the zero polynomial is -1). Recall that by definition $P \in \mathbf{Q}[y]$ is such that

$$\dim_k \mathrm{Gr}^B_i(M) = P(i) \quad \text{for all} \quad i \gg 0.$$

For us, it will be more convenient to consider the function $f\colon \mathbf{Z} \to \mathbf{Z}$ given by $f(i) = \dim_k B_i M$. Since $f(i) - f(i-1) = P(i)$ for $i \gg 0$, it follows that there is a polynomial $Q \in \mathbf{Q}[y]$ of degree $d = \dim(M)$ such that $f(i) = Q(i)$ for $i \gg 0$. It is well-known (and easy to see) that since $Q(i) \in \mathbf{Z}_{>0}$ for $i \gg 0$, the top degree term of Q is of the form $\frac{e(M)}{d!}y^d$, for some $e(M) \in \mathbf{Z}_{>0}$. By definition, $e(M)$ is the *multiplicity* of M.

Remark 2.19. Given a short exact sequence

$$0 \to M' \to M \to M'' \to 0$$

of finitely generated A_n-modules, if $B_\bullet M$ is a good filtration on M and if we consider the induced filtrations $B_\bullet M'$ and $B_\bullet M''$ on M' and M'', respectively, for every i we have a short exact sequence of k-vector spaces

$$0 \to B_i M' \to B_i M \to B_i M'' \to 0,$$

hence $\dim_k B_i M = \dim_k B_i M' + \dim_k B_i M''$. This implies that

$$\dim(M) = \max\big\{\dim(M'), \dim(M'')\big\}.$$

Moreover, if all three modules have the same dimension, then $e(M) = e(M') + e(M'')$.

2.4.3. The Bernstein inequality for modules over the Weyl algebra. The following is the special case of Theorem 2.13 in the setting of modules over the Weyl algebra.

Theorem 2.20. *If M is a nonzero finitely generated A_n-module, then* $\dim(M) \geq n$.

Remark 2.21. Note that, unlike in Theorem 2.13, we here do not bound the dimension of all irreducible components of $\mathrm{Char}_B(M)$.

Proof of Theorem 2.20. Let $B_\bullet M$ be a good filtration on M with respect to $B_\bullet A_n$. After possibly replacing all $B_p M$ by $B_{p+p_0} M$ for some p_0, we may and will assume that $B_0 M \neq 0$. The key point is the following
Claim. For every $p \geq 0$, the map

$$B_p A_n \to \mathrm{Hom}_k\big(B_p M, B_{2p} M\big), \quad Q \mapsto (u \mapsto Qu)$$

is injective.

We prove the claim by induction on p, the case $p = 0$ being clear, since $B_0A_n = k$ and $B_0M \neq 0$. Suppose now that $p \geq 1$ and we know the assertion for $p-1$. If $0 \neq Q \in B_pA_n$, we need to show that there is $u \in B_pM$ such that $Qu \neq 0$. This is clear if $Q \in k$. Otherwise, if we write $Q = \sum_{\alpha,\beta} c_{\alpha,\beta}x^\alpha\partial^\beta$, there is $(\alpha',\beta') \neq (0,0)$ such that $c_{\alpha',\beta'} \neq 0$.

If $\alpha' \neq 0$, let i be such that $\alpha'_i \neq 0$. In this case, using (2.1), we see that

$$[Q,\partial_i] = -\sum_{\alpha,\beta} \alpha_i c_{\alpha,\beta} x^{\alpha-e_i}\partial^\beta$$

is a nonzero element of $B_{p-1}A_n$. By induction, it follows that we have $v \in B_{p-1}A_n$ such that $[Q,\partial_i]v \neq 0$. Since $Q\partial_i v \neq \partial_i Qv$, it follows that either $Q\partial_i v \neq 0$ (in which case we may take $u = \partial_i v \in B_pM$) or $\partial_i Qv \neq 0$, hence $Qv \neq 0$, so we may take $u = v \in B_{p-1}M \subseteq B_pM$.

The case when $\beta' \neq 0$ is similar. This completes the proof of the claim. Note now that $\dim_k B_pA_n$ grows like p^{2n}, while if $d = \dim(M)$, then it follows that $\dim_k \mathrm{Hom}_k(B_pM, B_{2p}M)$ grows like $p^d \cdot (2p)^d = 2^d p^{2d}$, and therefore $2d \geq 2n$. This gives the inequality in the theorem. □

As in Section 2.3.1, we say that a finitely generated A_n-module is *holonomic* if either $M = 0$ or $\dim(M) = n$.

Proposition 2.22. The following hold:

i) Given an exact sequence

$$0 \to M' \to M \to M'' \to 0$$

of finitely generated A_n-modules, M is holonomic if and only if both M' and M'' are holonomic. Moreover, in this case we have $e(M) = e(M') + e(M'')$.

ii) The category of all holonomic A_n-modules is an Abelian subcategory of the category of all A_n-modules. Moreover, all objects in this category have finite length; in fact, for every M holonomic, we have $\ell(M) \leq e(M)$.

Proof. The assertion in i) follows from Remark 2.19. This implies that the category of holonomic A_n-modules has kernels and cokernels. Then the fact that it is an Abelian category is immediate.

We also see that if $M_1 \subsetneq M_2 \subseteq M$ are A_n-modules, with M holonomic, then $e(M_1) < e(M_2) \leq e(M)$ and these are positive integers. This implies that $\ell(M) \leq e(M)$. □

Example 2.23. The A_n-module $R = k[x_1,\ldots,x_n] \simeq A_n/A_n(\partial_1,\ldots,\partial_n)$ is holonomic. Indeed, the Bernstein filtration on A_n induces a filtration on R given by $B_pR = \{f \in R \mid \deg(f) \leq p\}$, hence $\dim_k B_pR = \binom{n+p}{p}$. Therefore R is holonomic and $e(R) = 1$.

Example 2.24. The A_1-module $M = k[x,x^{-1}]/k[x]$ is isomorphic to A_1/A_1x. It is easy to see that if we take on M the filtration induced by the Bernstein filtration on A_1, then $\mathrm{Gr}^B_\bullet(M) \simeq k[x,y]/(x) \simeq k[y]$, hence M is holonomic and $e(M) = 1$. We then conclude using Proposition 2.22 that $k[x,x^{-1}]$ is a holonomic A_1-module, of multiplicity 2.

EXAMPLE 2.25. Let $P \in A_n$ be nonzero, where $n \geq 1$, and let $M = A_n/A_n \cdot P$. Let $d \geq 0$ be such that $P \in B_d A_n \smallsetminus B_{d-1}A_n$. If $d = 0$, then $M = 0$, hence from now on we assume $d \geq 1$. If we write $P = P_0 + Q$, where $P_0 \in B_{d-1}A_n$ and $Q \in \bigoplus_{|\alpha|+|\beta|=d} k \cdot x^\alpha \partial^\beta$, and if we consider on M the filtration induced from A_n, then $\mathrm{Gr}^B_\bullet(M) \simeq k[x_1, \ldots, x_n, y_1, \ldots, y_n]/(Q(x,y))$, hence $\dim(M) = 2n-1$ and $e(M) = d$. In particular, we see that M is holonomic if and only if $n = 1$.

The first surprising example is provided by the localization of R at a nonzero element (note that, in this case, it's not *a priori* clear that this is a finitely generated A_n-module).

THEOREM 2.26. *If $f \in R$ is nonzero, then R_f is a holonomic R-module.*

PROOF. Recall that R_f has a natural structure of A_n-module (see Example 2.5). Let $d = \deg(f)$ and let us consider for every $q \geq 0$

$$B_q R_f = \left\{ \frac{g}{f^q} \mid \deg(g) \leq (d+1)q \right\}.$$

It is clear that this is a k-linear subspace of R_f. Note that $\bigcup_{q\geq 0} B_q R_f = R_f$. Indeed, given $\frac{g}{f^p} \in R_f$, we can rewrite this as $\frac{gf^m}{f^{p+m}}$ and we have

$$\deg(gf^m) = md + \deg(g) \leq (d+1)(p+m) \quad \text{for} \quad m \gg 0.$$

Moreover, we have $B_p A_n \cdot B_q R_f \subseteq B_{p+q} R_f$ for all $p, q \geq 0$. In order to see this, it is enough to show that $x_i \cdot B_q R_f \subseteq B_{q+1} R_f$ and $\partial_i \cdot B_q R_f \subseteq B_{q+1} R_f$ for all i. Indeed, if $\frac{g}{f^q} \in B_q R_f$, then

$$x_i \tfrac{g}{f^q} = \tfrac{x_i g f}{f^{q+1}} \quad \text{and} \quad \deg(x_i g f) = d + 1 + \deg(g) \leq (d+1)(q+1)$$

and

$$\partial_i \cdot \tfrac{g}{f^q} = \frac{f\frac{\partial g}{\partial x_i} - qg\frac{\partial f}{\partial x_i}}{f^{q+1}} \quad \text{and} \quad \deg\left(f\frac{\partial g}{\partial x_i} - qg\frac{\partial f}{\partial x_i}\right) \leq d + \deg(g) - 1 \leq (d+1)(q+1).$$

Therefore $B_\bullet R_f$ is a filtration on R_f with respect to $B_\bullet A_n$.

It follows from the definition that $\dim_k F_q R_f = \binom{(d+1)q+n}{n}$ for every $q \geq 0$. Note that this is a polynomial of degree n in q, with the top degree term $\frac{(d+1)^n q^n}{n!}$. If M is a finitely generated A_n-submodule of R_f and $B_\bullet M$ is a good filtration on M, then it is easy to see that there is $\ell \geq 0$ such that $B_q M \subseteq B_{q+\ell} R_f \cap M$, hence there is $C > 0$ such that $\dim_k B_q M \leq \frac{(d+1)^n q^n}{n!} + Cq^{n-1}$ for all $q \gg 0$. This implies that M is holonomic and $e(M) \leq (d+1)^n$.

We deduce from this that R_f is a finitely generated A_n-module: otherwise there is an infinite sequence

$$M_1 \subsetneq M_2 \subsetneq \ldots \subsetneq R_f$$

of finitely generated A_n-submodules. Since they are all holonomic, of multiplicity $\leq (d+1)^n$, it follows from Proposition 2.22 that

$$e(M_1) < e(M_2) \ldots \leq (d+1)^n$$

is a bounded strictly increasing sequence of positive integers, a contradiction. Once we know that R_f is finitely generated, it follows from what we have already proved that R_f is holonomic. $\square$

2.5. Functors on $\mathcal{D}$-modules

We now give a brief description of the main functors between categories of $\mathcal{D}$-modules: the pull-back, the push-forward, and duality.

2.5.1. The pull-back. Let $f\colon X \to Y$ be a morphism of smooth, irreducible algebraic varieties over k. If $\mathcal{M}$ is a left $\mathcal{D}_Y$-module, then the $\mathcal{O}_X$-module $f^*(\mathcal{M}) = \mathcal{O}_X \otimes_{f^{-1}(\mathcal{O}_Y)} f^{-1}(\mathcal{M})$ has a natural structure of left $\mathcal{D}_X$-module: the connection $\nabla_{\mathcal{M}}$ on $\mathcal{M}$ induces the connection $\nabla_{f^*(\mathcal{M})}$ on $f^*(\mathcal{M})$ such that if we have $\nabla_{\mathcal{M}}(u) = \sum_i dg_i \otimes v_i$, then

$$\nabla_{f^*(\mathcal{M})}\big(h \otimes f^{-1}(u)\big) = dh \otimes f^{-1}(u) + \sum_i hd(g_i \circ f) \otimes f^{-1}(v_i) \quad \text{for} \quad h \in \mathcal{O}_X.$$

We thus get a right-exact functor from $\mathcal{D}_Y$-modules to $\mathcal{D}_X$-modules that extends to an exact functor between triangulated categories $\mathbf{L}f^*\colon \mathcal{D}^b(\mathcal{D}_Y) \to \mathcal{D}^b(\mathcal{D}_X)$. Note that we denote by $\mathcal{D}^b(\mathcal{D}_X)$, respectively $\mathcal{D}^b(\mathcal{D}_X^r)$, the bounded derived category of left, respectively right, $\mathcal{D}_X$-modules.

2.5.2. The push-forward. Let $f\colon X \to Y$ be as above. In particular, it follows from the previous discussion that $f^*(\mathcal{D}_Y) = \mathcal{O}_X \otimes_{f^{-1}(\mathcal{O}_Y)} f^{-1}(\mathcal{D}_Y)$ is a left $\mathcal{D}_X$-module; in fact, it is a $\mathcal{D}_X - f^{-1}(\mathcal{D}_Y)$-bimodule, using also the right $\mathcal{D}_Y$-module structure of $\mathcal{D}_Y$. As such, it is denoted by $\mathcal{D}_{X\to Y}$. It follows that if $\mathcal{M}$ is a *right* $\mathcal{D}_X$-module, then $\mathcal{M} \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y}$ is a right $f^{-1}(\mathcal{D}_Y)$-module, and thus $f_*(\mathcal{M} \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y})$ is a right $\mathcal{D}_Y$-module. Since this construction combines a left exact functor with a right-exact functor, the correct functor is defined at the level of derived categories. More precisely, we have the exact functor between triangulated categories

$$f_+ = \mathbf{R}f_*(- \otimes^L_{\mathcal{D}_X} \mathcal{D}_{X\to Y})\colon \mathcal{D}^b(\mathcal{D}_X^r) \to \mathcal{D}^b(\mathcal{D}_Y^r).$$

Finally, the equivalence of categories between left and right $\mathcal{D}$-modules (see Section 2.2.1) induces equivalences $\mathcal{D}^b(\mathcal{D}_X) \simeq \mathcal{D}^b(\mathcal{D}_X^r)$ and $\mathcal{D}^b(\mathcal{D}_Y) \simeq \mathcal{D}^b(\mathcal{D}_Y^r)$. Using these, we obtain the push-forward functor for left $\mathcal{D}$-modules $f_+\colon \mathcal{D}^b(\mathcal{D}_X) \to \mathcal{D}^b(\mathcal{D}_Y)$.

If $g\colon Y \to Z$ is another such morphism, then we have a canonical isomorphism $(g \circ f)_+ \simeq g_+ \circ f_+$. Since every morphism $f\colon X \to Y$ admits the decomposition $X \overset{i}{\hookrightarrow} X \times Y \xrightarrow{p} Y$, where p is the projection and i is the closed immersion given by $i(x) = \big(x, f(x)\big)$, one can reduce understanding the push-forward by arbitrary morphisms to the case of projections and closed immersions. We next discuss each of these two cases in more detail.

We consider first the case of a closed immersion $i\colon Z \hookrightarrow X$ between smooth varieties, defined by $\mathcal{I}_Z$. Working locally, we may assume that we have algebraic coordinates $x_1, \dots, x_r, z_1, \dots, z_m$ on X such that $\mathcal{I}_Z = (x_1, \dots, x_m)$. In this case, we have

$$\mathcal{D}_{Z\to X} = \mathcal{D}_X/(x_1, \dots, x_r)\mathcal{D}_X = \bigoplus_{\beta \in \mathbf{Z}^m_{\geq 0}} \mathcal{D}_Z \partial_x^\beta,$$

where the right multiplication by x_i and ∂_{x_i} are given by

$$P\partial_x^\beta \cdot \partial_{x_i} = P\partial_x^{\beta+e_i} \quad \text{and} \quad P\partial_x^\beta \cdot x_i = \beta_i P\partial_x^{\beta-e_i}.$$

In particular, we see that $\mathcal{D}_{Z\to X}$ is a free left $\mathcal{D}_Z$-module. Since both functors i_* and $- \otimes_{\mathcal{D}_Z} \mathcal{D}_{Z\to X}$ are exact, we see that i_+ is induced at the level of derived

categories by an exact functor at the level of abelian categories. It is now easy to see that if $\mathcal{N}$ is a left $\mathcal{D}_Z$-module, then

$$i_+(\mathcal{N}) \simeq \mathcal{N} \otimes_k [\partial_{x_1}, \dots, \partial_{x_r}]$$

and the operations are defined by

$$\partial_{x_i} \cdot (u \otimes \partial_x^\beta) = u \otimes \partial_x^{\beta+e_i} \quad \text{and} \quad x_i \cdot (u \otimes \partial_x^\beta) = -\beta_i (u \otimes \partial_x^{\beta-e_i}).$$

An important result in this setting is Kashiwara's Equivalence Theorem (see [**HTT08**, Section 1.6]): this says that for every closed immersion $i\colon Z \hookrightarrow X$ of smooth varieties, i_+ gives an equivalence between the category of coherent (holonomic) $\mathcal{D}_Z$-modules and the category of coherent (respectively, holonomic) $\mathcal{D}_X$-modules with support contained in Z; the inverse functor is given by

$$\mathcal{N} \mapsto \{u \in \mathcal{N} \mid \mathcal{I}_Z \cdot u = 0\}$$

(this equivalence of categories is a major difference when working with $\mathcal{D}$-modules as opposed to $\mathcal{O}$-modules).

Exercise 2.27. Show that if $Z = \{P\}$, for some point $P \in X$, then $i_+(\mathcal{O}_Z) \simeq \mathcal{H}^n_{\{P\}}(\mathcal{O}_X)$, where $n = \dim(X)$.

More generally, there is a natural isomorphism $i_+ \circ \mathbf{L}i^*[-r] \simeq \mathbf{R}\Gamma_Z$ of functors on $\mathcal{D}^b(\mathcal{D}_X)$, where $\mathbf{R}\Gamma_Z$ is the right derived functor of the functor that maps $\mathcal{M}$ to the subsheaf $\Gamma_Z\mathcal{M}$ of sections supported on Z. In particular, we have

$$i_+(\mathcal{O}_Z) \simeq \mathcal{H}^r_Z(\mathcal{O}_X), \tag{2.7}$$

where the right-hand side is the r^{th} local cohomology sheaf of $\mathcal{O}_X$ with respect to Z (cf. Example 2.6).

Remark 2.28. If Z is a smooth hypersurface in X defined by (t), with $i\colon Z \hookrightarrow X$ the inclusion, and $\mathcal{M}$ is a coherent $\mathcal{D}_X$-module, then it is easy to see that we have isomorphisms

$$\mathcal{H}^0_Z(\mathcal{M}) \simeq i_+\{u \in \mathcal{M} \mid tu = 0\} \quad \text{and} \quad \mathcal{H}^1_Z(\mathcal{M}) \simeq i_+(\mathcal{M}/t\mathcal{M}).$$

Indeed, the first isomorphism follows directly from Kashiwara's theorem. For the second isomorphism, using again Kashiwara's theorem, we see that it is enough to show that the map

$$\mathcal{M}/t\mathcal{M} \to \big\{w \in \operatorname{Coker}(\mathcal{M} \to \mathcal{M}_t) \mid tw = 0\big\}, \quad u \mapsto [u/t]$$

is an isomorphism. This is easy to check, modulo the fact that we use again Kashiwara's theorem to deduce that if $\mathcal{N} = \Gamma_Z(\mathcal{M})$, then $\mathcal{N} \simeq i_+(\mathcal{N}_0)$ for some $\mathcal{D}_Z$-module, and thus $\mathcal{N} = t\mathcal{N}$.

Suppose now that Y and Z are smooth varieties and $p\colon X = Y \times Z \to Y$ is the projection onto the first factor. In this case we have $\mathcal{D}_{X\to Y} \simeq \mathcal{D}_Y \otimes_k \mathcal{O}_Z$. Note that the de Rham complex $\mathrm{DR}_Z(\mathcal{D}_Z)$ is a complex of right $\mathcal{D}_Z$-modules and the corresponding complex of left $\mathcal{D}_Z$-modules gives a free resolution of $\mathcal{O}_Z$ (see Example 2.16). This implies that if Y is a point and if $\mathcal{M}$ is a left $\mathcal{D}_Z$-module, then

$$p_+(\mathcal{M}) = \mathbf{R}\Gamma\big(\mathrm{DR}_Z(\mathcal{M})\big),$$

so the cohomology of $p_+(\mathcal{M})$ gives, up to a shift, the de Rham cohomology of $\mathcal{M}$.

2.5.3. Preservation of holonomicity. Let $\mathcal{D}^b_h(\mathcal{D}_X)$ denote the bounded derived category of complexes of $\mathcal{D}$-modules on X with holonomic cohomology. It is an important result of the theory that pull-back and push-forward of $\mathcal{D}$-modules preserves holonomic $\mathcal{D}$-modules: if $f\colon X \to Y$ is a morphism of smooth, irreducible varieties, then we have induced functors

$$\mathbf{L}f^*\colon \mathcal{D}^b_h(\mathcal{D}_Y) \to \mathcal{D}^b_h(\mathcal{D}_X) \quad \text{and} \quad f_+\colon \mathcal{D}^b_h(\mathcal{D}_X) \to \mathcal{D}^b_h(\mathcal{D}_Y).$$

An important example is the following: suppose that X is the open subset of Y where $g \neq 0$, for some nonzero $g \in \mathcal{O}_Y(Y)$. Note that in this case $\mathcal{D}_{X\to Y} = \mathcal{D}_X$ and f_+ coincides with the usual push-forward functor at the level of $\mathcal{O}$-modules. We thus see that if $\mathcal{M}$ is a holonomic $\mathcal{D}_X$-module, then $f_+(\mathcal{M}) = f_*(\mathcal{M})$ is a holonomic $\mathcal{D}_Y$-module. Applying this for $\mathcal{M} = \mathcal{O}_X$, we see that $\mathcal{O}_Y[1/g]$ is a holonomic $\mathcal{D}_Y$-module (note that, a priori, it is not even clear that this is a coherent $\mathcal{D}_Y$-module). We have seen this in the case $Y = \mathbf{A}^n$ in Theorem 2.26. The proof in the general case makes use of the notion of b-function, which we discuss in Section 3.1.

2.5.4. Duality. Another important functor on holonomic $\mathcal{D}$-modules is provided by duality. More generally, we have a duality functor on the bounded derived category of left $\mathcal{D}_X$-modules with coherent cohomology $\mathcal{D}^b_c(\mathcal{D}_X)$. Note that for every left $\mathcal{D}_X$-module $\mathcal{M}$, we have a right $\mathcal{D}_X$-module structure on $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X)$ induced by the canonical right $\mathcal{D}_X$-module structure on $\mathcal{D}_X$, and thus a left $\mathcal{D}_X$-module structure on $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X) \otimes_{\mathcal{O}_X} \omega_X^{-1}$. We thus obtain a *duality functor*

$$\mathcal{D}^b_c(\mathcal{D}_X) \to \mathcal{D}^b_c(\mathcal{D}_X), \quad \mathcal{M} \mapsto \mathbf{D}(\mathcal{M}) := \mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(-, \mathcal{D}_X) \otimes_{\mathcal{O}_X} \omega_X^{-1}[n]. \tag{2.8}$$

One can show that there is a natural isomorphism $\mathbf{D} \circ \mathbf{D} \simeq \mathrm{Id}$; in particular, the functor $\mathbf{D}$ is an anti-equivalence of categories. The reason for the shift by n in the definition of duality is due to the following two properties: if $\mathcal{M}$ is a coherent $\mathcal{D}_X$-module, then

i) $\mathcal{E}xt^i_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X) = 0$ for all $i > n$.
ii) If $\mathcal{M} \neq 0$, then $\min\{i \geq 0 \mid \mathcal{E}xt^i_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X) \neq 0\} = 2n - \dim(\mathcal{M})$.

The key point for ii) is that, by a spectral sequence argument, if one takes a good filtration $F_\bullet\mathcal{M}$ on $\mathcal{M}$, then the invariant on the left-hand side does not change when replacing $\mathcal{M}$ and $\mathcal{D}_X$ by $\mathrm{Gr}^F_\bullet(\mathcal{M})$ and $\mathrm{Gr}^F_\bullet(\mathcal{D}_X)$, respectively. The assertion then follows from the fact that if S is a regular ring of pure dimension d and M is a nonzero finitely generated S-module, then

$$\mathrm{depth}\big(\mathrm{Ann}_S(M), S\big) + \dim(M) = d.$$

As a side remark: when working on the affine space, the analogous equality holds with respect to the Bernstein filtration. Since the invariant on the left-hand side of ii) clearly does not depend on the filtration, we see that the two notions of dimension, defined using the order filtration or the Bernstein filtration, coincide.

In particular, it follows from i) and ii) above that if $\mathcal{M}$ is holonomic, then $\mathcal{E}xt^i_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X) = 0$ for $i \neq n$, hence $\mathbf{D}(\mathcal{M})$ is a $\mathcal{D}_X$-module; in fact, one can show that it is a holonomic $\mathcal{D}_X$-module. We thus get an induced anti-equivalence on the abelian category of holonomic $\mathcal{D}_X$-modules and also an anti-equivalence on $\mathcal{D}^b_h(\mathcal{D}_X)$.

2.6. The Riemann-Hilbert correspondence

We now assume that $k = \mathbf{C}$ and write X^{an} for the complex manifold corresponding to the smooth complex algebraic variety X. Recall that a sheaf $\mathcal{F}$ of $\mathbf{C}$-vector spaces on X^{an} is *constructible* if there is a finite decomposition $X = \bigsqcup_i X_i$, with each X_i being a locally closed algebraic subvariety, such that the restriction of $\mathcal{F}$ to each X_i is a local system (that is, it is a locally constant sheaf of finite rank). Let us write $\mathcal{D}^b_c(X^{\mathrm{an}})$ for the bounded derived category of complexes of sheaves of $\mathbf{C}$-vector spaces on X^{an} whose cohomology sheaves are constructible. These categories satisfy a 6-functor formalism. In particular, we have the *Verdier duality* functor $\mathbf{D}_{X^{\mathrm{an}}} = \mathbf{R}\mathcal{H}om_{\mathbf{C}}\big(-, \underline{\mathbf{C}}_X[n]\big)$ on $\mathcal{D}^b_c(X^{\mathrm{an}})$ and if $f\colon X \to Y$ is a morphism of smooth, irreducible complex algebraic varieties, we have exact functors

$$\mathbf{R}f_*, \mathbf{R}f_!\colon \mathcal{D}^b_c(X^{\mathrm{an}}) \to \mathcal{D}^b_c(Y^{\mathrm{an}}) \quad \text{and} \quad f^{-1}, f^!\colon \mathcal{D}^b_c(Y^{\mathrm{an}}) \to \mathcal{D}^b_c(X^{\mathrm{an}})$$

that satisfy certain adjointness properties and compatibilities with duality.

Beilinson, Bernstein, Deligne, and Gabber defined in [**BBD82**] an abelian subcategory $\mathrm{Perv}(X^{\mathrm{an}})$ of $\mathcal{D}^b_c(X^{\mathrm{an}})$ whose objects are known as *perverse sheaves*. They are complexes with constructible cohomology that satisfy certain bounds on the dimensions of the supports of the cohomology sheaves and whose Verdier duals satisfy the same conditions (in particular, the subcategory is preserved by Verdier duality).

The *classical Riemann-Hilbert correspondence* is an equivalence of categories between local systems on X^{an} and analytic vector bundles on X^{an}, with integrable connection. In one direction, to a local system $\mathcal{L}$ on X^{an} one associates the analytic vector bundle $\mathcal{L} \otimes_{\mathbf{C}} \mathcal{O}_{X^{\mathrm{an}}}$, with the integrable connection $\mathrm{Id}_{\mathcal{L}} \otimes \nabla_{\mathrm{dR}}$, where ∇_{dR} is the standard connection on $\mathcal{O}_{X^{\mathrm{an}}}$. The functor in the opposite direction associates to $(\mathcal{E}, \nabla)$ the subsheaf $\mathcal{E}^{\nabla} = \{s \in \mathcal{E} \mid \nabla s = 0\}$ of $\mathcal{E}$. The fact that this is a local system of rank equal to $\mathrm{rk}(\mathcal{E})$ and

$$(\mathcal{E}, \nabla) \simeq (\mathcal{E}^{\nabla} \otimes_{\mathbf{C}} \mathcal{O}_{X^{\mathrm{an}}}, \mathrm{Id}_{\mathcal{E}^{\nabla}} \otimes \nabla_{\mathrm{dR}})$$

is a consequence of the Frobenius Theorem on the existence and uniqueness of integral submanifolds. Note that once we know that $\mathcal{E} \simeq \mathcal{E}^{\nabla} \otimes_{\mathbf{C}} \mathcal{O}_{X^{\mathrm{an}}}$ as vector bundles with integrable connection, it follows from Poincaré's Lemma that the canonical map $\mathcal{E}^{\nabla} \to \mathrm{DR}_{X^{\mathrm{an}}}(\mathcal{E})[-n]$ is a quasi-isomorphism. If X is a proper variety, then one can deduce using GAGA that we have an equivalence of categories between analytic vector bundles with integrable connection on X^{an} and algebraic vector bundles with integrable connection on X.

We next formulate the general version of the Riemann-Hilbert correspondence for algebraic varieties. If $\mathcal{M}$ is a left $\mathcal{D}_X$-module, then we also consider the analytification of the de Rham complex $\mathrm{DR}_X(\mathcal{M})$, that we denote by $\mathrm{DR}^{\mathrm{an}}_X(\mathcal{M})$. In his doctoral thesis, Kashiwara proved that if $\mathcal{M}$ is holonomic, then $\mathrm{DR}^{\mathrm{an}}_X(\mathcal{M}) \in \mathcal{D}^b_c(X^{\mathrm{an}})$. In fact, he showed that $\mathrm{DR}^{\mathrm{an}}_X(\mathcal{M})$ is a perverse sheaf, though this notion was formalized only later.

In order to obtain an equivalence of categories one needs to impose one extra condition: *regular singularities*. We do not give the definition since it is somewhat involved and we will not need it. For a thorough discussion, see [**HTT08**, Chapter 6]. If we denote by $\mathcal{D}^b_{rh}(\mathcal{D}_X)$ the subcategory of $\mathcal{D}^b_h(\mathcal{D}_X)$ whose cohomology sheaves are regular holonomic $\mathcal{D}_X$-modules, then $\mathcal{D}^b_{rh}(\mathcal{D}_X)$ is preserved by $\mathbf{D}_X$ and for a morphism $f\colon X \to Y$ between smooth, irreducible complex algebraic varieties,

we have induced functors

(2.9)
$$f_+\colon \mathcal{D}^b_{rh}(\mathcal{D}_X) \to \mathcal{D}^b_{rh}(\mathcal{D}_Y) \quad \text{and} \quad f^\dagger = \mathbf{L}f^*[\dim(X)-\dim(Y)]\colon \mathcal{D}^b_{rh}(\mathcal{D}_Y) \to \mathcal{D}^b_{rh}(\mathcal{D}_X).$$

The Riemann-Hilbert correspondence, proved independently by Kashiwara and Mebkhout, says that the analytic de Rham functor induces an equivalence of triangulated categories

$$\mathrm{DR}^{\mathrm{an}}_X\colon \mathcal{D}^b_{rh}(\mathcal{D}_X) \simeq \mathcal{D}^b_c(X^{\mathrm{an}}).$$

Moreover, via this equivalence, the functor $(-)^*$ defined in (2.8) corresponds to $\mathbf{D}_{X^{\mathrm{an}}}$ and if $f\colon X \to Y$ is a morphism of smooth, irreducible algebraic varieties, then f_+ and $f^\dagger$ correspond, respectively, to $\mathbf{R}f_*$ and $f^!$. Finally, $\mathrm{DR}^{\mathrm{an}}_X$ induces an equivalence between the abelian category of regular holonomic $\mathcal{D}_X$-modules and $\mathrm{Perv}(X^{\mathrm{an}})$. For a detailed discussion and for proofs, see [**HTT08**, Chapter 7].

CHAPTER 3

The Bernstein-Sato polynomial and the V-filtration

Our goal in this chapter is to discuss the V-filtration of Malgrange and Kashiwara. As we will see in the next chapter, this notion plays an important role in Saito's theory of Hodge modules. Moreover, it has independent interest, as it provides interesting invariants of hypersurface singularities. Its existence relies on the existence of the Bernstein-Sato polynomial and the rationality of its roots. We begin with an elementary discussion of the Bernstein-Sato polynomial. As in the previous chapter, we assume that X is a smooth, irreducible, n-dimensional algebraic variety, over an algebraically closed field k of characteristic 0.

3.1. The Bernstein-Sato polynomial

3.1.1. Definition and main existence result. Let $f \in \mathcal{O}_X(X)$ be nonzero. We consider the quasicoherent $\mathcal{O}_X$-module $\mathcal{O}_X[1/f,s]f^s$, that is free over $\mathcal{O}_X[1/f,s] = \mathcal{O}_X[1/f] \otimes_k k[s]$, with basis the symbol f^s. For every $m \in \mathbf{Z}$, it is common to write f^{s+m} for the element $f^m \cdot f^s \in \mathcal{O}_X[1/f]f^s$.

Exercise 3.1. Show that the $\mathcal{O}_X[s]$-module structure on $\mathcal{O}_X[1/f,s]f^s$ can be extended to a $\mathcal{D}_X[s]$-module structure such that every $D \in \mathcal{T}_X$ acts by the "expected" rule

$$D \cdot P(s)hf^s = P(s)D(h)f^s + sP(s)\frac{D(f)h}{f}f^s$$

for every $P \in k[s]$ and $h \in \mathcal{O}_X$.

Remark 3.2. Note that for every $m \in \mathbf{Z}$, we have a map $\mathcal{O}_X[1/f,s]f^s \to \mathcal{O}_X[1/f]$ given by $s \mapsto m$: this maps $P(s)hf^s$ to $P(m)hf^m$ for every $P \in k[s]$ and every $h \in \mathcal{O}_X$. It follows from the definition of the action of $\mathcal{D}_X$ on $\mathcal{O}_X[1/f,s]f^s$ that this map is $\mathcal{D}_X$-linear.

Note that on $\mathcal{O}_X[1/f,s]f^s$ we have an $\mathcal{O}_X$-linear automorphism T induced by $s \mapsto s+1$, that is

$$T\big(P(s)f^s\big) = P(s+1)f^{s+1},$$

and its inverse is similarly induced by $s \mapsto s-1$. It is easy to see that $T(D \cdot u) = D \cdot T(u)$ for every $D \in \mathrm{Der}_k(\mathcal{O}_X)$ and $u \in \mathcal{O}_X[1/f,s]f^s$, hence T is $\mathcal{D}_X$-linear.

The following theorem was proved by Sato for certain polynomials f, by Bernstein for all polynomials, and by Kashiwara in general.

Theorem 3.3. *If $f \in \mathcal{O}_X(X)$ is nonzero, then there is a nonzero $b(s) \in k[s]$ such that*

$$b(s)f^s \in \mathcal{D}_X[s]f^{s+1}. \tag{3.1}$$

It is clear that the set of polynomials $b(s)$ that satisfy the condition in the theorem is an ideal in $k[s]$. The theorem says that this ideal is nonzero: its monic generator is called the *b-function* of f (or the *Bernstein-Sato polynomial* of f) and is denoted by $b_f(s)$.

EXERCISE 3.4 (Specializing the functional equation). Let X be affine, with $R = \mathcal{O}_X(X)$. Show that a polynomial $b(s) \in k[s]$ is divisible by $b_f(s)$ if and only if there is $P \in D_R[s]$ such that

$$b(m)f^m = P(m) \cdot f^{m+1} \text{ in } R_f \tag{3.2}$$

for infinitely many $m \in \mathbf{Z}$ (equivalently, for all $m \in \mathbf{Z}$).

We do not give the proof of Theorem 3.3 in general, but only treat the case of affine space, by building on the results proved in Section 2.4.3.

PROOF OF THEOREM 3.3 FOR $X = \mathbf{A}^n$. Let $K = k(s)$ and consider the localization

$$k[s, x_1, \ldots, x_n]_f f^s \subseteq M := K[x_1, \ldots, x_n]_f f^s.$$

Note that M is a left $A_n(K)$-module. An argument similar to that in the proof of Theorem 2.26 implies that M is a holonomic $A_n(K)$-module. By Proposition 2.22, the decreasing sequence of $A_n(K)$-submodules of M:

$$A_n(K) \cdot f^s \supseteq A_n(K) \cdot f^{s+1} \supseteq A_n(K) f^{s+2} \supseteq \ldots$$

is eventually stationary. Therefore there is $m \in \mathbf{Z}_{\geq 0}$ and $P \in A_n(K)$ such that

$$f^{s+m} = P \cdot f^{s+m+1}.$$

Clearly, there is a nonzero polynomial $q(s) \in k[s]$ such that $q(s)P = Q \in A_n[s]$. Applying T^{-m} to the identity $q(s)f^{s+m} = Q \cdot f^{s+m+1}$, we conclude that if $b(s) = q(s-m)$, then $b(s)f^s \in A_n[s] \cdot f^{s+1}$. □

3.1.2. First properties and examples. We first show that b_f always has a "trivial" root:

PROPOSITION 3.5. If f is not invertible, then $b_f(-1) = 0$.

PROOF. Specializing s to -1 in (3.1) (see Remark 3.2), we conclude that

$$b_f(-1)\tfrac{1}{f} \in \mathcal{D}_X \cdot 1 \subseteq \mathcal{O}_X.$$

Since f is not invertible, this implies $b_f(-1) = 0$. □

DEFINITION 3.6. If $f \in \mathcal{O}_X(X)$ is nonzero and noninvertible, then it follows from the above proposition that $b_f(s)$ is divisible by $(s+1)$. The *reduced Bernstein-Sato polynomial* of f is $\widetilde{b}_f(s) = b_f(s)/(s+1)$.

REMARK 3.7 (Effective generation of the localization via the roots of the b-function). As we have mentioned in Section 2.5.3, the $\mathcal{D}_X$-module $\mathcal{O}_X[1/f]$ is holonomic; in particular, it is coherent, and thus there is $m \in \mathbf{Z}_{>0}$ such that $\frac{1}{f^m}$ generates $\mathcal{O}_X[1/f]$ as a $\mathcal{D}_X$-module. In fact, this m can be determined if we know the b-function b_f: it is enough to take m such that b_f has no integer roots $< -m$. Indeed, we know that locally there is $P \in \mathcal{D}_X[s]$ such that $b_f(s)f^s = P \cdot f^{s+1}$. By making $s = -i$, it follows from Remark 3.2 that we have

$$b_f(-i)\tfrac{1}{f^i} = P(-i) \cdot \tfrac{1}{f^{i-1}} \in \mathcal{D}_X \cdot \tfrac{1}{f^{i-1}}.$$

By assumption, $b_f(-i) \neq 0$ for $i > m$, and thus $\mathcal{O}_X[1/f] \subseteq \mathcal{D}_X \cdot \frac{1}{f^m}$.

REMARK 3.8 (Independence of equation). The Bernstein-Sato polynomial $b_f(s)$ only depends on the hypersurface defined by f. Indeed, suppose that $g = pf$, with p invertible. It is easy to check that we have an isomorphism of sheaves of rings $\tau_0\colon \mathcal{D}_X[s] \to \mathcal{D}_X[s]$ that is the identity on $\mathcal{O}_X[s]$ and such that $\tau_0(Q) = Q - Q(p)p^{-1}s$ for all $Q \in \mathrm{Der}_k(\mathcal{O}_X)$. We then have a bijective additive map

$$\tau\colon \mathcal{O}_X[1/f,s]f^s \to \mathcal{O}_X[1/g,s]g^s, \quad \tau(uf^s) = ug^s,$$

which is linear with respect to τ_0, that is, $\tau(P\cdot w) = \tau_0(P)\cdot\tau(w)$ for all $P \in \mathcal{D}_X[s]$ and $w \in \mathcal{O}_X[1/f,s]f^s$. The fact that $b_g(s) = b_f(s)$ then follows easily from the definition.

REMARK 3.9 (Behavior of b-function under open covers). If $X = U_1 \cup \ldots \cup U_r$ is an open cover and $f_i = f|_{U_i}$, then

$$b_f = \mathrm{lcm}\big\{b_{f_i} \mid 1 \leq i \leq r\big\}.$$

This is clear from the definition.

We can now define the Bernstein-Sato polynomial of an arbitrary hypersurface.

DEFINITION 3.10. If Z is a hypersurface in X, consider a finite open cover $X = U_1 \cup \ldots \cup U_m$ such that we have $f_i \in \mathcal{O}_X(U_i)$ with $Z \cap U_i$ defined by (f_i). In this case, we define the *b-function* (or *Bernstein-Sato polynomial*) of Z by

$$b_Z(s) = \mathrm{lcm}\big\{b_{f_i} \mid 1 \leq i \leq m\big\}.$$

Note that $b_{f_i}(s)$ is independent of the choice of f_i by Remark 3.8 and then it is easy to deduce from Remark 3.9 that the definition of $b_Z(s)$ is independent of any choices. Note that if Z is nonempty, then it follows from Proposition 3.5 that $b_Z(-1) = 0$ and we put $\widetilde{b}_Z(s) = b_Z(s)/(s+1)$.

EXAMPLE 3.11 (The Bernstein-Sato polynomial of invertible functions). It is straightforward to see that if f is an invertible function, then $b_f(s) = 1$: we have $f^s = \frac{1}{f}\cdot f^{s+1}$.

EXAMPLE 3.12 (Smooth hypersurface). Suppose that Z is a smooth (nonempty) hypersurface in X. In this case, we have $b_Z(s) = s+1$. In order to show this, we may assume that we have a system of algebraic coordinates $x_1,\ldots,x_n$ on X such that $f = x_1$. Since

$$\partial_1 \cdot x_1^{s+1} = (s+1)x_1^s,$$

it follows that $b_Z(s)|(s+1)$. On the other hand, Proposition 3.5 implies that $(s+1)|b_Z(s)$, hence $b_Z(s) = s+1$.

EXAMPLE 3.13 (The b-function of a monomial). More generally, suppose that we have algebraic coordinates $x_1,\ldots,x_n$ on X and $f = x_1^{a_1}\cdots x_n^{a_n}$. It is then easy to check that if

$$b(s) = \prod_{i=1}^{n}\prod_{j=1}^{a_i}(a_i s + j) \quad \text{and} \quad P = \partial_1^{a_1}\cdots\partial_n^{a_n},$$

then $b(s)f^s = P\cdot f^{s+1}$, hence $b_f(s)$ divides $\prod_{i=1}^{n}\prod_{j=1}^{a_i}\big(s+\frac{j}{a_i}\big)$. In fact, this is an equality, but this does not follow directly from the definition.

EXAMPLE 3.14 (The multiplicative Thom-Sebastiani property). It is a recent result (see [Lee24]) that if X and Y are smooth varieties, $Z = X \times Y$, and $f \in \mathcal{O}_X(X)$ and $g \in \mathcal{O}_Y(Y)$ are nonzero, and $h \in \mathcal{O}_Z(Z)$ is given by $h(x,y) = f(x)g(y)$, then

$$b_h(s) = b_f(s)b_g(s).$$

EXAMPLE 3.15 (The b-function of a quadric). Let $f = \sum_{i=1}^n x_i^2 \in k[x_1, \ldots, x_n]$. Note that

$$\partial_i \cdot f^{s+1} = 2x_i(s+1)f^s \quad \text{and} \quad \partial_i^2 \cdot f^{s+1} = 4x_i^2 s(s+1)f^{s-1} + 2(s+1)f^s.$$

We thus conclude that if $\Delta = \sum_{i=1}^n \partial_i^2$, then

$$\Delta \cdot f^{s+1} = (s+1)(4s+n)f^s,$$

hence $b_f(s)$ divides $(s+1)\big(s+\frac{n}{2}\big)$. We know that $(s+1)$ divides $b_f(s)$. Furthermore, it is a general result that we will discuss later, saying that $b_f(s) = s+1$ if and only if the hypersurface defined by f is smooth. This implies that in this case we have $b_f(s) = (s+1)\big(s+\frac{n}{2}\big)$.

EXAMPLE 3.16 (Weighted homogeneous polynomials with isolated singularities). More generally, let's consider the case when $f \in R = k[x_1, \ldots, x_n]$ is a weighted-homogeneous polynomial, with an isolated singularity. The former condition says that there are $w_1, \ldots, w_n \in \mathbf{Q}_{>0}$ and d such that every monomial x^a that appears in f with nonzero coefficient satisfies $\langle a, w\rangle := \sum_i a_i w_i = d$. After rescaling the w_i, we may and will assume that $d = 1$. We also assume that f has an isolated singularity at 0. Since f is weighted homogeneous, this is equivalent to the fact that the hypersurface defined by f in $\mathbf{A}^n \smallsetminus \{0\}$ is smooth; furthermore, the Euler relation $f = \sum_{i=1}^n w_i x_i \frac{\partial f}{\partial x_i}$ implies that in this case the quotient R/J_f, where $J_f = \big(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\big)$, is a finite-dimensional k-vector space. Let Σ be the set of all rational numbers of the form $\langle v, w\rangle + \sum_{i=1}^n w_i$, where x^v is a monomial not in J_f (to get Σ, it is enough to let x^v run over any monomial basis of R/J_f). It is a result of [BGM86] (see also [CDM24, Theorem 6.2] for the proof) that

$$b_f(s) = (s+1) \cdot \prod_{\lambda \in \Sigma} (s+\lambda).$$

As a special case, consider $f = \sum_{i=1}^n x_i^{a_i}$, which satisfies the required condition with $w_i = \frac{1}{a_i}$ for all i. A basis of R/J_f is given by

$$\{x^v \mid 0 \leq v_p \leq a_p - 2 \text{ for } 1 \leq p \leq n\}$$

and we see that

$$\Sigma = \Big\{ \tfrac{i_1}{a_1} + \ldots + \tfrac{i_n}{a_n} \mid 0 \leq i_p \leq a_p - 2 \text{ for } 1 \leq p \leq n \Big\}.$$

EXAMPLE 3.17 (The generic determinantal hypersurface). Let $X = \mathbf{A}^{n^2}$ be the affine space of $n \times n$ matrices, with coordinates x_{ij}, for $1 \leq i \leq m$, $1 \leq j \leq n$. If $f = \det(x_{ij})$, then a formula that goes back to Cayley gives

$$\prod_{i=1}^n (s+i) f^s = \det(\partial_{ij}) \cdot f^{s+1},$$

and thus $b_f(s)$ divides $\prod_{i=1}^n (s+i)$. It is known that, in fact, we have $b_f(s) = \prod_{i=1}^n (s+i)$ (see [Kim03, Appendix]).

It is sometimes convenient to also have a local version of the Bernstein-Sato polynomial. Note that if $f \in \mathcal{O}_X(X)$ and $U \subseteq V$ are open subsets of X, then we clearly have $b_{f|_U}(s) | b_{f|_V}(s)$. It is then clear that for every point $P \in X$, there is an open neighborhood U_P of P such that $b_{f|_U}(s) = b_{f|_{U_P}}(s)$ for all neighborhoods U of P, with $U \subseteq U_P$. We put $b_{f,P}(s) := b_{f|_{U_P}}(s)$; this is the *b-function* (or *Bernstein-Sato polynomial*) of f at P. Equivalently, it follows from the definition of the Bernstein-Sato polynomial that $b_{f,P}(s)$ is the monic polynomial of smallest degree such that $b_{f,P}(s)f^s \in \mathcal{D}_X[s] \cdot f^{s+1}$ in some neighborhood of P. It follows from Proposition 3.5 that if $f(P) = 0$, then $b_{f,P}(-1) = 0$ and we put $\widetilde{b}_{f,P}(s) = b_{f,P}(s)/(s+1)$. Note also that it follows easily from Remark 3.9 that

$$b_f(s) = \operatorname{lcm}\{b_{f,P}(s) \mid P \in X\}.$$

If Z is a hypersurface in X, then we define $b_{Z,P}(s)$ and $\widetilde{b}_{Z,P}(s)$ by choosing a local equation f defining Z is a neighborhood of P.

3.1.3. The roots of the Bernstein-Sato polynomial. An important result of Kashiwara [**Kas77**] says that all roots of the Bernstein-Sato polynomial are negative rational numbers. In fact, Kashiwara gave a more precise result involving the data of a log resolution. We state an improved version of this result, due to Lichtin [**Lic89**]

Let Z be a hypersurface in the smooth, irreducible variety X. Let $\pi\colon Y \to X$ be a log resolution of (X, Z) that is an isomorphism over the complement of Z: this means that π is a proper morphism that is an isomorphism over $X \smallsetminus Z$, Y is smooth, and the divisor $\pi^*(Z)$ has simple normal crossings. Note that the relative divisor $K_{Y/X}$ is supported on the exceptional locus hence, by assumption, is supported on $\pi^*(Z)$. Note that such a resolution always exists by Hironaka's Theorem [**Hir64**]. We write

$$\pi^*(Z) = \sum_{i=1}^{d} a_i E_i \quad \text{and} \quad K_{Y/X} = \sum_{i=1}^{d} k_i E_i.$$

THEOREM 3.18. *With the above notation, every root of $b_Z(s)$ is of the form $-\frac{k_i+\ell}{a_i}$ for some i, with $1 \leq i \leq d$, and some $\ell \in \mathbf{Z}_{\geq 0}$.*

In these notes, our interest in the Bernstein-Sato polynomial comes from the fact that it provides an interesting and subtle invariant for hypersurface singularities. As we will see, it is related to several other invariants of singularities. However, it is worth pointing out that it tends to be somewhat worse behaved and more difficult to study than other more classical invariants.

3.1.4. Historical motivation: meromorphic extension of complex powers. The original motivation for the Bernstein-Sato polynomial came from the problem of meromorphic continuation of complex powers. We consider the following setting: suppose that X is a smooth affine complex algebraic variety. We assume that we have coordinates $z_1, \ldots, z_n$ on X and write $dz = dz_1 \wedge \ldots \wedge dz_n$. The classical case is that when $X = \mathbf{C}^n$.

Suppose that $f \in \mathcal{O}_X(X)$ is a regular function. For every real function φ on X, which is smooth (that is, $\mathcal{C}^\infty$) and with compact support, and every $s \in \mathbf{C}$ with $\operatorname{Re}(s) > 0$, we consider

$$Z_{f,\varphi}(s) := \int_{X^{\mathrm{an}}} |f|^{2s} \varphi \, dz d\overline{z}.$$

Recall that for $a \in \mathbf{R}_{>0}$ and $\lambda \in \mathbf{C}$, we have $a^\lambda := \exp\big(\lambda \cdot \log(a)\big)$. Of course, in the above integral we can ignore those $P \in X^{\rm an}$ with $f(P) = 0$, which form a set of measure 0.

PROPOSITION 3.19. The function $Z_{f,\varphi}$ is well-defined and holomorphic in the half-plane $H_0 = \{s \mid {\rm Re}(s) > 0\}$.

PROOF. Note that if $s \in H_0$, then $||f(x)|^{2s}| = |f(x)|^{2{\rm Re}(s)}$, so if $|f(x)| \leq M$ for $x \in {\rm Supp}(\varphi)$, we have

$$||f(x)|^{2s}\varphi(x)| \leq M^{2{\rm Re}(s)}{\rm sup}|\varphi|.$$

Since we integrate on the support of φ, which is compact, it follows that $Z_{f,\varphi}(s)$ is well-defined for $s \in H_0$.

By Morera's theorem, in order to show that $Z_{f,\varphi}$ is holomorphic in H_0, it is enough to show that $Z_{f,\varphi}$ is continuous on H_0 and for every closed smooth curve γ in H_0, we have $\int_\gamma Z_{f,\varphi}(s)ds = 0$. Both these assertions follow easily using Lebesgue's dominated convergence theorem. □

It was a question of I. Gel'fand (ICM, Amsterdam, 1954) whether $Z_{f,\varphi}$ admits a meromorphic extension to $\mathbf{C}$. In fact, one would like to do this uniformly in φ (more precisely, given any $s_0 \in \mathbf{C}$, one would like to find $N = N(s_0)$ such that $(s - s_0)^N Z_{f,\varphi}$ is holomorphic in a neighborhood of s_0 for all φ). This is easy to do, using an argument via integration by parts, when f is a monomial $z_1^{a_1} \cdots z_n^{a_n}$. A few years after Hironaka's proof of resolution of singularities in [**Hir64**], an affirmative answer to I. Gel'fand's question was given independently by Bernstein-S. Gel'fand [**BG69**] and Atiyah [**Ati70**], based on Hironaka's result. The idea is to use a log resolution and the Change of Variable formula to reduce the assertion to the monomial case.

A second solution to I. Gel'fand's question was given shortly afterward by Bernstein [**Ber72**], directly extending the integration by parts argument mentioned above, using the existence of the b-function. We now sketch his argument.

By definition of b_f, we have a relation of the form

$$b_f(s)f^s = P \cdot f^{s+1}.$$

Note that if we are in an open subset U of X where a branch of $\log(f)$ is defined, so $f^s = \exp\big(s \cdot \log(f)\big)$ is defined, then we can write

$$b_f(s)\overline{b_f}(s)|f|^{2s} = b_f(s)f^s \cdot \overline{b_f}(s)\overline{f}^s = (P \cdot f^{s+1}) \cdot (\overline{P} \cdot \overline{f}^{s+1}) = Q \cdot |f|^{2(s+1)},$$

where $Q = P \cdot \overline{P}$. The last equality follows from the fact that if $P \in \mathcal{O}_X[\partial_{z_1}, \ldots, \partial_{z_n}]$ and g and h are holomorphic functions, then $P \cdot (g\overline{h}) = \overline{h}(P \cdot g)$ and $\overline{P} \cdot (g\overline{h}) = g(\overline{P} \cdot \overline{h}) = g \cdot \overline{P \cdot h}$. Using the Stokes theorem, we thus see that if $s \in H_0$, then we can write

$$b_f(s)\overline{b_f}(s) \cdot Z_{f,\varphi}(s) = \int_{X^{\rm an}} \big(Q \cdot |f|^{2(s+1)}\big) \cdot \varphi dz d\overline{z} = \int_{X^{\rm an}} |f|^{2(s+1)}\psi \, dz d\overline{z}, \tag{3.3}$$

with $\psi = \widetilde{Q} \cdot \varphi$, where $\widetilde{Q}$ is the adjoint of Q (we use here the fact that φ has compact support). Note that the right-hand side of (3.3) is in fact holomorphic in the half-space $\big\{s \mid {\rm Re}(s) > -1\big\}$ by Proposition 3.19, since ψ is a smooth function, with compact support. We can now multiply by $b_f(s+1)\overline{b_f}(s+1)$ and repeat. The conclusion is that indeed, $Z_{f,\varphi}$ admits a meromorphic extension such that

for every positive integer m, in the half-space $\{s \mid \mathrm{Re}(s) > -m\}$, the function $\prod_{i=0}^{m-1} b_f(s+i)\overline{b_f}(s+i) \cdot Z_{f,\varphi}(s)$ is holomorphic. In particular, we see that every pole of $Z_{f,\varphi}$ is of the form $\lambda - j$, for some root λ of b_f and some nonnegative integer j (recall that such λ is a negative rational number by Theorem 3.18).

3.1.5. General b-functions. As we have already mentioned, the main role of b-functions in $\mathcal{D}$-module theory is to show that holonomicity is preserved under open immersions. This requires a more general notion of b-function that we now briefly introduce.

Suppose that $f \in \mathcal{O}_X(X)$ is nonzero and $\mathcal{M}$ is a $\mathcal{D}_X$-module that is also an $\mathcal{O}_X[1/f]$-module (that is, the canonical localization map $\mathcal{M} \to \mathcal{M}_f$ is an isomorphism). The case we discussed previously is that when $\mathcal{M} = \mathcal{O}_X[1/f]$. In the general setting, we put

$$\mathcal{M}[s]f^s := \mathcal{M} \otimes_k k[s]f^s,$$

where $k[s]f^s$ is a free $k[s]$-module with basis the symbol f^s. As in Exercise 3.1, one can check that we have a $\mathcal{D}_X[s]$-module structure on $\mathcal{M}[s]f^s$ such that for every $D \in \mathcal{D}er_k(\mathcal{O}_X)$, every $u \in \mathcal{M}$, and every $P \in k[s]$, we have

$$D \cdot P(s)uf^s = P(s)(Du)f^s + sP(s)\frac{D(f)}{f}uf^s.$$

Let $U = (f \neq 0) \subseteq X$. The following result is due to Kashiwara:

THEOREM 3.20. *With the above notation, if $\mathcal{M}|_U$ is a holonomic $\mathcal{D}_U$-module, then for every $u \in \Gamma(X, \mathcal{M})$, there is a nonzero $b(s) \in k[s]$ such that*

$$b(s)uf^s \in \mathcal{D}_X[s] \cdot (fu)f^s.$$

Note that if we take $\mathcal{M} = \mathcal{O}_X[1/f]$ and $u = 1$, then we recover Theorem 3.3. We will discuss an even more general version of this result, when we consider arbitrary elements of $\mathcal{M}[s]f^s$, in Section 3.2.2 below.

3.2. V-filtrations

We next discuss the V-filtration of Malgrange and Kashiwara. Originally, it was introduced in order to describe two important functors on perverse sheaves, the nearby and vanishing cycles, at the level of $\mathcal{D}$-modules, see [**Mal83**] and [**Kas83**]. This allowed Malgramge to prove that λ is a root of $b_f(s)$ if and only if $\exp(-2\pi i\lambda)$ is an eigenvalue for the monodromy action on the Milnor fiber of f at some point P with $f(P) = 0$. Subsequently, the V-filtration, with the $\mathbf{Q}$-indexing introduced by Saito that we discuss below, turned out to be crucial for the theory of Hodge modules (see [**Sai88**] and [**Sai90**]). For an introduction to V-filtrations, we refer to [**CDM24**, Chapters 3-5]: we follow this closely, though we omit many details.

3.2.1. The V-filtration with respect to a smooth hypersurface. Let X be a smooth, irreducible, algebraic variety over k and let $t \in \mathcal{O}_X(X)$ be nonzero, defining a smooth, irreducible hypersurface H. We put $\dim(X) = n + 1$, so $\dim(H) = n$.

For every $m \in \mathbf{Z}$, we put

$$V^m\mathcal{D}_X = \{P \in \mathcal{D}_X \mid P \cdot (t)^q \subseteq (t)^{q+m} \text{ for all } q \in \mathbf{Z}\}$$

(with the convention that $(t)^i = \mathcal{O}_X$ if $i \leq 0$). It follows from the definition that we have $V^{m_1}\mathcal{D}_X \cdot V^{m_2}\mathcal{D}_X \subseteq V^{m_1+m_2}\mathcal{D}_X$ for all $m_1, m_2 \in \mathbf{Z}$.

REMARK 3.21. Note that the decreasing filtration $V^\bullet\mathcal{D}_X$ only depends on H, but not on the function t defining H.

It is easy to describe $V^m\mathcal{D}_X$ locally. Note first that if $U \subseteq X$ is an open subset such that $H \cap U = \emptyset$, then $V^m\mathcal{D}_X|_U = \mathcal{D}_U$ for all $m \in \mathbf{Z}$. On the other hand, if U is an affine open subset with algebraic coordinates $x_1, \ldots, x_n, t$, then given $P \in \Gamma(U, \mathcal{D}_X)$, we write $P = \sum_{\alpha,j} P_{\alpha,j}\partial_x^\alpha \partial_t^j$, where the sum is over $\alpha \in \mathbf{Z}^n_{\geq 0}$ and $j \in \mathbf{Z}_{\geq 0}$, and all $P_{\alpha,j}$ lie in $\mathcal{O}_X(U)$. In this case, it is straightforward to see that $P \in \Gamma(U, V^m\mathcal{D}_X)$ if and only if $P_{\alpha,j} \in (t^{m+j})$ for all α and j.

Given coordinates, as above, it is convenient to also describe $V^m\mathcal{D}_X$ in terms of the Euler operator $t\partial_t$. In fact, for reasons that will become clearer later, we will use instead $s := -\partial_t t = -t\partial_t - 1$. For future reference, we collect in the next exercise some easy identities involving s, t, and ∂_t.

EXERCISE 3.22. Use induction to show that the operator $s = -\partial_t t$ satisfies the following properties:

i) For every $P \in k[s]$ and every $m \in \mathbf{Z}_{\geq 0}$, we have $P(s)t^m = t^m P(s-m)$ and $P(s)\partial_t^m = \partial_t^m P(s+m)$.
ii) For every $m \in \mathbf{Z}_{>0}$, we have

$$t^m\partial_t^m = (-1)^m \prod_{j=1}^{m}(s+j) \quad \text{and} \quad \partial_t^m t^m = (-1)^m \prod_{j=0}^{m-1}(s-j).$$

Using the above exercise, we see that $t^i\partial_t^j$ lies in $k[s]\cdot t^{i-j}$ if $i \geq j$ and it lies in $k[s]\cdot\partial_t^{j-i}$ if $i \leq j$. We thus conclude that, given coordinates $x_1, \ldots, x_n, t$ as above,

$$V^0\mathcal{D}_U = \mathcal{O}_X\langle\partial_{x_1}, \ldots, \partial_{x_n}, s\rangle. \tag{3.4}$$

We also have

$$V^m\mathcal{D}_U = V^0\mathcal{D}_U \cdot t^m = t^m \cdot V^0\mathcal{D}_U \quad \text{for all} \quad m \geq 0 \quad \text{and} \tag{3.5}$$

$$V^{-m}\mathcal{D}_U = \sum_{i=0}^{m} V^0\mathcal{D}_U \cdot \partial_t^i = \sum_{i=0}^{m} \partial_t^i \cdot V^0\mathcal{D}_U \quad \text{for all} \quad m \geq 0. \tag{3.6}$$

REMARK 3.23. Note that the operators ∂_t and $s = -\partial_t t$ depend on the choice of coordinates. More precisely, if we choose a different set of local coordinates $x_1', \ldots, x_n', t$ and write ∂_t' for the corresponding derivation with respect to this system of coordinates, we have $\partial_t' = \partial_t + \sum_{i=1}^n \partial_t'(x_i)\partial_{x_i}$, hence

$$\partial_t' - \partial_t \in V^0\mathcal{D}_X \quad \text{and} - \partial_t' t + \partial_t t \in V^1\mathcal{D}_X.$$

We also note that if we replace t by $t' = ut$, for some invertible function u, then a similar computation shows that, with respect to the same coordinates $x_1, \ldots, x_n$, we have $\partial_{t'} - \partial_t u^{-1} \in V^0\mathcal{D}_X$ and thus $\partial_{t'}t' - \partial_t t \in V^1\mathcal{D}_X$.

We now come to the definition of the V-filtration with respect to t. The role of the V-filtration is, roughly speaking, to put the operator $\partial_t t$ on a given $\mathcal{D}$-module in upper triangular form.

DEFINITION 3.24. A *V-filtration* on a coherent $\mathcal{D}_X$-module $\mathcal{M}$ (with respect to $t \in \mathcal{O}_X(X)$) is a decreasing filtration $V^\bullet\mathcal{M} = (V^\alpha\mathcal{M})_{\alpha\in\mathbf{Q}}$ by quasi-coherent

$\mathcal{O}_X$-modules, indexed by rational numbers, which is exhaustive[1], discrete and left-continuous[2], and satisfies the following properties:

i) We have $V^i\mathcal{D}_X \cdot V^\alpha\mathcal{M} \subseteq V^{\alpha+i}\mathcal{M}$ for every $\alpha \in \mathbf{Q}$ and $i \in \mathbf{Z}$.
ii) For every $\alpha \in \mathbf{Q}$, the operator $(s+\alpha)$ is nilpotent on $\mathrm{Gr}_V^\alpha(\mathcal{M})$.
iii) Each $V^\alpha\mathcal{M}$ is locally finitely generated over $V^0\mathcal{D}_X$.
iv) We have $t \cdot V^\alpha\mathcal{M} = V^{\alpha+1}\mathcal{M}$ for all $\alpha > 0$.

Note that in the above definition, we put $\mathrm{Gr}_V^\alpha(\mathcal{M}) := V^\alpha\mathcal{M}/V^{>\alpha}\mathcal{M}$, where

$$V^{>\alpha}\mathcal{M} = \bigcup_{\beta>\alpha} V^\beta\mathcal{M} = V^{\alpha+\epsilon}\mathcal{M} \quad \text{for} \quad 0 < \epsilon \ll 1.$$

By assumption, there is a positive integer ℓ such that $\mathrm{Gr}_V^\alpha(\mathcal{M}) = 0$ unless $\ell\alpha \in \mathbf{Z}$.

REMARK 3.25. Since $\mathrm{Gr}_V^\alpha(\mathcal{M})$ is supported on H, we only put the condition in ii) in Definition 3.24 at the points in H; note that the operator s is defined around these points. While this operator is only defined locally and depends on the choice of local coordinates, its action on $\mathrm{Gr}_V^\alpha(\mathcal{M})$ is well-defined: indeed, as we have noticed in Remark 3.23, if s' is the same operator corresponding to a different choice of coordinates, then $s - s' \in V^1\mathcal{D}_X$, hence the actions of s and s' on $\mathrm{Gr}_V^\alpha(\mathcal{M})$ agree by condition i).

Furthermore, it follows from the same remark that if we replace t by ut, for some invertible function u, and if we denote by s' the new operator, then the actions of s and s' on $\mathrm{Gr}_V^\alpha(\mathcal{M})$ again coincide. Therefore the notion of V-filtration on $\mathcal{M}$ only depends on H and it does not depend on the choice of function t.

PROPOSITION 3.26. If $V^\bullet\mathcal{M}$ is a V-filtration on the coherent $\mathcal{D}_X$-module $\mathcal{M}$, then for every $\alpha \neq 0$, the maps

$$\mathrm{Gr}_V^\alpha(\mathcal{M}) \xrightarrow{t\cdot} \mathrm{Gr}_V^{\alpha+1}(\mathcal{M}) \xrightarrow{\partial_t\cdot} \mathrm{Gr}_V^\alpha(\mathcal{M}) \tag{3.7}$$

are isomorphisms of $\mathcal{D}_H$-modules.

PROOF. Let ν_t and ν_{∂_t} be the first, respectively the second, map in (3.7). It is clear that they are morphisms of $\mathcal{D}_H$-modules, hence we only need to show that they are bijective. It follows from condition ii) in Definition 3.24 that since $\alpha \neq 0$, the composition $\nu_{\partial_t} \circ \nu_t$ is invertible, hence ν_t is injective and ν_{∂_t} is surjective. On the other hand, since $t\partial_t = \partial_t t - 1$ and $\alpha + 1 \neq 1$, it follows from condition ii) in Definition 3.24 that $\nu_t \circ \nu_{\partial_t}$ is invertible, hence ν_t is surjective and ν_{∂_t} is injective. Therefore both ν_t and ν_{∂_t} are bijective. □

It follows from the above proposition that the interesting maps are the following morphisms of $\mathcal{D}_H$-modules:

$$\mathrm{can}\colon \mathrm{Gr}_V^1(\mathcal{M}) \xrightarrow{\partial_t\cdot} \mathrm{Gr}_V^0(\mathcal{M}) \quad \text{and}$$

$$\mathrm{Var}\colon \mathrm{Gr}_V^0(\mathcal{M}) \xrightarrow{t\cdot} \mathrm{Gr}_V^1(\mathcal{M}).$$

PROPOSITION 3.27. A coherent $\mathcal{D}_X$-module $\mathcal{M}$ admits at most one V-filtration with respect to t.

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[1] This means that $\cup_{\alpha\in\mathbf{Q}} V^\alpha\mathcal{M} = \mathcal{M}$.

[2] These conditions mean that there is a positive integer ℓ such that $V^\alpha\mathcal{M}$ takes a constant value for α in an interval of the form $\big(i/\ell, (i+1)/\ell\big]$, where i is any integer.

PROOF. Suppose that $V^\bullet\mathcal{M}$ and $\widetilde{V}^\bullet\mathcal{M}$ are two V-filtrations on $\mathcal{M}$. By symmetry, it is enough to show that for every $\alpha \in \mathbf{Q}$, we have

$$V^\alpha\mathcal{M} \subseteq \widetilde{V}^\alpha\mathcal{M} \quad \text{for all} \quad \alpha \in \mathbf{Q}. \tag{3.8}$$

This is clear on $X \smallsetminus H$ (since the conditions in the definition easily imply that $\widetilde{V}^\alpha\mathcal{M} = \mathcal{M}$ on $X \smallsetminus H$ for all α), hence we only need to check it around the points in H. Note first that

$$\frac{V^\beta\mathcal{M} \cap \widetilde{V}^\gamma\mathcal{M}}{(V^{>\beta}\mathcal{M} \cap \widetilde{V}^\gamma\mathcal{M}) + (V^\beta\mathcal{M} \cap \widetilde{V}^{>\gamma}\mathcal{M})} \tag{3.9}$$

is a subquotient of both $\mathrm{Gr}_V^\beta(\mathcal{M})$ and $\mathrm{Gr}_{\widetilde{V}}^\gamma(\mathcal{M})$. Property ii) in Definition 3.24 thus implies that both $s+\beta$ and $s+\gamma$ are nilpotent on (3.9), and thus (3.9) is 0 for all $\beta, \gamma \in \mathbf{Q}$, with $\beta \neq \gamma$.

Let us fix now $\alpha \in \mathbf{Q}$. Note first that since the filtration $\widetilde{V}^\bullet\mathcal{M}$ is an exhaustive filtration by $V^0\mathcal{D}_X$-submodules and $V^\alpha\mathcal{M}$ is locally finitely generated over $V^0\mathcal{D}_X$, there is $\gamma \in \mathbf{Q}$ such that

$$V^\alpha\mathcal{M} \subseteq \widetilde{V}^\gamma\mathcal{M}. \tag{3.10}$$

If $\gamma \geq \alpha$, then (3.8) holds and we are done, hence we may and will assume that $\gamma < \alpha$ and that γ is maximal such that (3.10) holds, aiming for a contradiction (we may assume this since the filtration $\widetilde{V}^\bullet\mathcal{M}$ is discrete and left continuous). On the other hand, since $V^\bullet\mathcal{M}$ satisfies property iv), it follows that there is $m_0 \geq 0$ such that for every $m \in \mathbf{Z}_{>0}$, we have

$$V^{\alpha+m_0+m}\mathcal{M} \subseteq t^m \cdot V^{\alpha+m_0}\mathcal{M} \subseteq t^m \cdot \widetilde{V}^\gamma\mathcal{M} \subseteq \widetilde{V}^{>\gamma}\mathcal{M}.$$

Since $V^\alpha\mathcal{M} \not\subseteq \widetilde{V}^{>\gamma}\mathcal{M}$ and the filtration $V^\bullet\mathcal{M}$ is discrete and left continuous, it follows that there is $\beta \geq \alpha$ such that

$$V^\beta\mathcal{M} \not\subseteq \widetilde{V}^{>\gamma}\mathcal{M} \quad \text{and} \tag{3.11}$$

$$V^{>\beta}\mathcal{M} \subseteq \widetilde{V}^{>\gamma}\mathcal{M}. \tag{3.12}$$

Note that $\gamma < \beta$, hence the quotient (3.9) for β and γ is 0, and therefore

$$V^\beta\mathcal{M} \subseteq V^\beta\mathcal{M} \cap \widetilde{V}^\gamma\mathcal{M} \subseteq (V^{>\beta}\mathcal{M} \cap \widetilde{V}^\gamma\mathcal{M}) + (V^\beta\mathcal{M} \cap V^{>\gamma}\mathcal{M}) \subseteq V^{>\gamma}\mathcal{M},$$

where the first inclusion follows from (3.10), since $\beta \geq \alpha$, and the last inclusion follows from (3.12). However, this contradicts (3.11), and thus completes the proof of the proposition. □

EXAMPLE 3.28 (The V-filtration on coherent $\mathcal{O}_X$-modules). If $\mathcal{M}$ is a $\mathcal{D}_X$-module that is coherent as an $\mathcal{O}_X$-module, then $\mathcal{M}$ has a V-filtration given by $V^\alpha\mathcal{M} = (t)^{\lceil\alpha\rceil-1} \cdot \mathcal{M}$ for all $\alpha \in \mathbf{Q}$, with the convention that $(t)^m = \mathcal{O}_X$ if $m \leq 0$. It is straightforward to check that the conditions in Definition 3.24 are satisfied. We only note that each $V^\alpha\mathcal{M}$ is clearly locally finitely generated over $V^0\mathcal{D}_X$ since it is coherent as an $\mathcal{O}_X$-module. Also, condition ii) in the definition holds since if $m \in \mathbf{Z}_{>0}$ and $u \in \mathcal{M}$, then $(\partial_t t - m)t^{m-1}u = t^m\partial_t u \in V^{m+1}\mathcal{M}$, hence $(s+m) \cdot \mathrm{Gr}_V^m(\mathcal{M}) = 0$.

EXAMPLE 3.29. A variation on the previous example, which will be needed later, is the following: suppose that $f \in \mathcal{O}_X(X)$ is such that the hypersurface defined by f is smooth and intersects H transversely. If $\mathcal{M} = \mathcal{O}_X[1/f]$, then $\mathcal{M}$

has a V-filtration given by $V^\alpha\mathcal{M} = (t)^{\lceil\alpha\rceil-1}\cdot\mathcal{M}$ for all $\alpha\in\mathbf{Q}$, with the convention that $(t)^m = \mathcal{O}_X$ if $m\leq 0$. The argument is similar to the one in the previous example. For example, $V^\alpha\mathcal{M}$ is finitely generated over $V^0\mathcal{D}_X$ since it is clearly generated by $t^{\lceil\alpha\rceil-1}\frac{1}{f}$.

EXAMPLE 3.30 (The V-filtration for $\mathcal{D}_X$-modules supported on H). If $\mathcal{N}$ is a coherent $\mathcal{D}_X$-module supported on H, then $\mathcal{N}$ has a V-filtration with respect to t given by

$$V^\alpha\mathcal{N} = \bigoplus_{i=0}^{-\lceil\alpha\rceil}\mathcal{N}_{-i} \quad \text{for} \quad \alpha\leq 0,$$

where $\mathcal{N}_i = \{u\in\mathcal{N} \mid (s+i)u = 0\}$, and $V^\alpha\mathcal{N} = 0$ for $\alpha > 0$. This is a consequence of Kashiwara's Equivalence Theorem, which says that $\mathcal{N}\simeq i_+(\mathcal{N}_0)$ for some $\mathcal{D}_H$-modules $\mathcal{N}_0$, where $i\colon H\hookrightarrow X$ is the inclusion. For details, see for example [**CDM24**, Example 3.18].

REMARK 3.31 (The category of modules that have a V-filtration). One can deduce using Example 3.30 that if $\mathcal{M}$ has a V-filtration with respect to t, $\mathcal{M}'$ is a $\mathcal{D}_X$-submodule of $\mathcal{M}$, and $p\colon\mathcal{M}\to\mathcal{M}'' = \mathcal{M}/\mathcal{M}'$ is the canonical projection, then $\mathcal{M}'$ and $\mathcal{M}''$ have V-filtrations with respect to t given by

$$V^\alpha\mathcal{M}' = V^\alpha\mathcal{M}\cap\mathcal{M}' \quad \text{and} \quad V^\alpha\mathcal{M}'' = p(V^\alpha\mathcal{M}) \quad \text{for all} \quad \alpha\in\mathbf{Q}. \tag{3.13}$$

For a proof, see for example [**CDM24**, Proposition 3.19]. In particular, every morphism of $\mathcal{D}_X$-modules that admit V-filtrations with respect to t is strict[3] with respect to these filtrations. This implies that the category of coherent $\mathcal{D}_X$-modules that carry a V-filtration with respect to t is an abelian category.

REMARK 3.32 (The V-filtration and the restriction to the smooth hypersurface). The V-filtration allows computing the restriction to H; if $\mathcal{M}$ has a V-filtration with respect to t and $i\colon H\hookrightarrow X$ is the inclusion, then $\mathbf{L}i^*(\mathcal{M})$ is computed by the complex

$$\mathrm{Gr}_V^0(\mathcal{M})\xrightarrow{t\cdot}\mathrm{Gr}_V^1(\mathcal{M}),$$

placed in cohomological degrees -1 and 0. This follows by showing that the following vertical maps given by the canonical surjections, respectively inclusions

$$\begin{array}{ccc}
\mathrm{Gr}_V^0(\mathcal{M}) & \xrightarrow{t} & \mathrm{Gr}_V^1(\mathcal{M}) \\
\uparrow & & \uparrow \\
V^0\mathcal{M} & \xrightarrow{t} & V^1\mathcal{M} \\
\downarrow & & \downarrow \\
\mathcal{M} & \xrightarrow{t} & \mathcal{M}
\end{array}$$

are quasi-isomorphisms. For a proof, see for example [**CDM24**, Proposition 3.22].

[3] This means that if $\varphi\colon\mathcal{M}\to\mathcal{N}$ is such a morphism, then $\varphi(V^\alpha\mathcal{M}) = \varphi(\mathcal{M})\cap V^\alpha\mathcal{N}$ for all $\alpha\in\mathbf{Q}$.

3.2.2. The V-filtration with respect to an arbitrary hypersurface. In these lectures, we are mainly interested in the V-filtration on the structure sheaf $\mathcal{O}_X$, but with respect to a singular hypersurface. To handle singular hypersurfaces, the trick is to use the graph embedding.

Let X be a smooth, irreducible, n-dimensional variety, and $f \in \mathcal{O}_X(X)$ be nonzero, defining the (possibly singular) hypersurface Z in X. Consider the closed immersion $\iota = \iota_f \colon X \hookrightarrow X \times \mathbf{A}^1$ given by $\iota(x) = \big(x, f(x)\big)$. We denote by t the coordinate on $\mathbf{A}^1$, defining the smooth hypersurface $H = X \times \{0\}$, that we identify with X via $i_0 \colon X \hookrightarrow X \times \mathbf{A}^1$, given by $i_0(x) = (x, 0)$. Given a coherent $\mathcal{D}_X$-module $\mathcal{M}$, we consider the coherent $\mathcal{D}_{X\times\mathbf{A}^1}$-module $\iota_+\mathcal{M}$. With a slight abuse of terminology, we will say that $\mathcal{M}$ *has a V-filtration with respect to f* if $\iota_+(\mathcal{M})$ has a V-filtration with respect to t.

Note that the formulas (3.4), (3.5), and (3.6) become in our setting

$$V^0\mathcal{D}_{X\times\mathbf{A}^1} = \mathcal{D}_X\langle s, t\rangle \subseteq \mathcal{D}_X\langle t, \partial_t\rangle, \tag{3.14}$$

$$V^m\mathcal{D}_{X\times\mathbf{A}^1} = V^0\mathcal{D}_{X\times\mathbf{A}^1} \cdot t^m = t^m \cdot V^0\mathcal{D}_{X\times\mathbf{A}^1} \quad \text{for all} \quad m \geq 0 \quad \text{and} \tag{3.15}$$

$$V^{-m}\mathcal{D}_{X\times\mathbf{A}^1} = \sum_{i=0}^{m} V^0\mathcal{D}_{X\times\mathbf{A}^1} \cdot \partial_t^i \quad \text{for all} \quad m \geq 0. \tag{3.16}$$

Remark 3.33. It follows from formula (3.14) that if $\mathcal{M}$ has a V-filtration with respect to f, then all $V^\alpha\iota_+(\mathcal{M})$ are $\mathcal{D}_X\langle s, t\rangle$-submodules of $\iota_+(\mathcal{M})$.

Let us describe explicitly $\iota_+(\mathcal{M})$. Let's consider first the case when $\mathcal{M} = \mathcal{O}_X$. It follows from (2.7) that we have an isomorphism $\iota_+(\mathcal{O}_X) \simeq \mathcal{H}^1_{\iota(X)}(\mathcal{O}_{X\times\mathbf{A}^1})$, hence

$$B_f := \iota_+(\mathcal{O}_X) \simeq \mathcal{O}_X[t]_{f-t}/\mathcal{O}_X[t] \tag{3.17}$$

(recall that we view this as a sheaf of $\mathcal{D}_X\langle t, \partial_t\rangle$-modules on X).

We will denote by δ the class of $\frac{1}{f-t}$ in B_f. Since $\mathcal{O}_X[t]_{f-t}/\mathcal{O}_X[t] = \bigoplus_{j\geq 1} \mathcal{O}_X \frac{1}{(f-t)^j}$ and $\frac{(j-1)!}{(f-t)^j} = \partial_t^{j-1} \cdot \frac{1}{(f-t)}$, we conclude that

$$B_f = \bigoplus_{j\geq 0} \mathcal{O}_X \partial_t^j \delta.$$

The action of $\mathcal{O}_X$ and ∂_t with respect to this decomposition is clear, while the actions of $\mathcal{D}er_k(\mathcal{O}_X)$ and of t are given by

$$D \cdot h\partial_t^j\delta = D(h)\partial_t^j\delta - hD(f)\partial_t^{j+1}\delta \quad \text{and} \quad t \cdot h\partial_t^j\delta = hf\partial_t^j\delta - jh\partial_t^{j-1}\delta \tag{3.18}$$

for every $D \in \mathcal{D}er_k(\mathcal{O}_X)$ and $h \in \mathcal{O}_X$.

If $\mathcal{M}$ is an arbitrary coherent $\mathcal{D}_X$-module, then we have an isomorphism

$$\iota_+(\mathcal{M}) \simeq \mathcal{M} \otimes_{\mathcal{O}_X} B_f = \bigoplus_{j\geq 0} \mathcal{M} \otimes \partial_t^j\delta,$$

with the actions of $\mathcal{D}er_k(\mathcal{O}_X)$ and of t being given by the analogues of the formulas in (3.18). Besides B_f, one important example is that of

$$B'_f := \iota_+\big(\mathcal{O}_X[1/f]\big) = \bigoplus_{j\geq 0} \mathcal{O}_X[1/f]\partial_t^j\delta.$$

Remark 3.34. Note that if $\mathcal{M}$ has a V-filtration with respect to f, then each $\mathrm{Gr}_V^\alpha\big(\iota_+(\mathcal{M})\big)$ is a $\mathcal{D}_X$-module supported on Z. This follows from the fact that it is a $\mathcal{D}_H$-module with support on $\iota(X) \cap H$.

EXAMPLE 3.35. Note that $\mathrm{Supp}(\mathcal{M}) \subseteq Z$ if and only if $\mathrm{Supp}\big(\iota_+(\mathcal{M})\big) \subseteq H$ and in this case it follows from Example 3.30 that $\mathcal{M}$ has a V-filtration with respect to f. Moreover, we have $V^\alpha\iota_+(\mathcal{M}) = 0$ for all $\alpha > 0$ and

$$\mathrm{Gr}_V^0\big(\iota_+(\mathcal{M})\big) \simeq \big\{u \in \iota_+(\mathcal{M}) \mid t \cdot u = 0\big\}$$
$$= \left\{\sum_{j\geq 0} \tfrac{f^j}{j!} u_0 \otimes \partial_t^j \delta \mid u_0 \in \mathcal{M}\right\}$$

(note that $f^j u = 0$ for $j \gg 0$ by the assumption on $\mathcal{M}$). We thus have an isomorphism of $\mathcal{D}_X$-modules

$$\mathrm{Gr}_V^0\big(\iota_+(\mathcal{M})\big) \simeq \mathcal{M}.$$

REMARK 3.36 (Compatibility of the two definitions). If f defines a smooth hypersurface in X, then one can show that we have a V-filtration on $\mathcal{M}$ with respect to f if and only if we have a V-filtration on $\iota_+(\mathcal{M})$ with respect to t and the two V-filtrations determine each other as follows: if we choose local coordinates $x_1, \ldots, x_{n-1}, y$ on X, with $y = f$, then

$$V^\alpha\iota_+(\mathcal{M}) = \sum_{j\geq 0} \partial_y^j \cdot (V^\alpha\mathcal{M} \otimes \delta).$$

For example, if we take $\mathcal{M} = \mathcal{O}_X$, then it follows from Example 3.28 that if the hypersurface defined by f is smooth, then the V-filtration on B_f is given by

$$V^\alpha\iota_+(\mathcal{O}_X) = \mathcal{D}_X \cdot y^{\lceil\alpha\rceil-1}\delta = \sum_{j\geq 0} \mathcal{O}_X \cdot \partial_y^j y^{\lceil\alpha\rceil-1}\delta \quad \text{for all} \quad \alpha \in \mathbf{Q},$$

with the convention that $y^m = 1$ if $m \leq 0$.

EXAMPLE 3.37 (The V-filtration with respect to a monomial hypersurface). For a more interesting example, let us consider the case of a simple normal crossing divisor. Suppose that $x_1, \ldots, x_n$ are algebraic coordinates on X, and $f = \prod_{i=1}^r x_i^{a_i}$, where $a_1, \ldots, a_r$ are positive integers. For every $\lambda \in \mathbf{Q}$, we put

$$I(f^\lambda) = \big(x_1^{\lceil\lambda a_1\rceil-1} \cdots x_r^{\lceil\lambda a_r\rceil-1}\big),$$

with the convention that this is $\mathcal{O}_X$ for $\lambda < 0$ (the reader familiar with multiplier ideals might note that $I(f^\lambda) = \mathcal{J}(f^{\lambda-\epsilon})$ for $0 < \epsilon \ll 1$, where the right-hand side is the multiplier ideal that we will discuss in Section 3.3.1). It is clear from the formula that $I(f^{\lambda_1}) \subseteq I(f^{\lambda_2})$ if $\lambda_1 \geq \lambda_2$. Note also that we have $f \cdot I(f^\lambda) \subseteq I(f^{\lambda+1})$, with equality if $\lambda > 0$.

Let us show that if we put

$$V^\lambda\iota_+(\mathcal{O}_X) = \sum_{j\geq 0} \mathcal{D}_X \cdot I(f^{\lambda+j})\partial_t^j\delta \quad \text{for all} \quad \lambda \in \mathbf{Q}, \tag{3.19}$$

then $V^\bullet\iota_+(\mathcal{O}_X)$ is a V-filtration with respect to t. A more general result is proved in [**Sai90**, Theorem 3.4], with a rather involved proof. We here give a direct proof, by checking that the formula in (3.19) satisfies the properties of the V-filtration. It is clear that this is a decreasing, exhaustive filtration. Note also that if $N = \mathrm{lcm}(a_1, \ldots, a_r)$, then $I(f^\lambda)$ is constant for $\lambda \in (i/N, (i+1)/N]$ for every $i \in \mathbf{Z}$, hence the filtration we defined on $\iota_+(\mathcal{O}_X)$ has the same property, and thus it is discrete and left continuous.

Note first that the sum in the formula (3.19) can be replaced by a finite sum. Indeed, for every i, with $1 \leq i \leq r$, and every $b \in \mathbf{Z}^r_{\geq 0}$, we have

$$(x_i\partial_{x_i} - b_i) \cdot x^b\partial_t^j\delta = -a_i x^{a+b}\partial_t^{j+1},$$

which implies

$$I(f^{\lambda+1})\partial_t^{j+1}\delta \subseteq \mathcal{D}_X \cdot I(f^\lambda)\partial_t^j\delta \quad \text{for} \quad \lambda > 0.$$

This implies that for $\lambda > 0$ we have $V^\lambda\iota_+(\mathcal{O}_X) = \mathcal{D}_X \cdot I(f^\lambda)\delta$ and, more generally, we have

$$V^\lambda\iota_+(\mathcal{O}_X) = \sum_{j \leq \max\{1-\lambda,0\}} \mathcal{D}_X \cdot I(f^{\lambda+j})\partial_t^j \quad \text{for all} \quad \lambda \in \mathbf{Q}.$$

This implies that every $V^\lambda\iota_+(\mathcal{O}_X)$ is finitely generated over $\mathcal{D}_X$ (hence also over $V^0\mathcal{D}_{X\times\mathbf{A}^1}$).

Note next that for every $j \geq 0$, we have

$$t \cdot I(f^{\lambda+j})\partial_t^j\delta \subseteq I(f^{\lambda+j+1})\partial_t^j\delta + I(f^\lambda)\partial_t^{j-1}\delta \subseteq V^{\lambda+1}\iota_+(\mathcal{O}_X),$$

so $t \cdot V^\lambda\iota_+(\mathcal{O}_X) \subseteq V^{\lambda+1}\iota_+(\mathcal{O}_X)$ for every $\lambda \in \mathbf{Q}$. Moreover, this is an equality for $\lambda > 0$. Indeed, we have seen that in this case we have

$$V^{\lambda+1}\iota_+(\mathcal{O}_X) = \mathcal{D}_X \cdot I(f^{\lambda+1})\delta = f \cdot I(f^\lambda)\delta = t \cdot I(f^\lambda)\delta = t \cdot V^\lambda\iota_+(\mathcal{O}_X).$$

Note also that we have

$$\partial_t \cdot I(f^{\lambda+j})\partial_t^j = I(f^{\lambda+j})\partial_t^{j+1}\delta \subseteq V^{\lambda-1}\iota_+(\mathcal{O}_X),$$

hence $\partial_t \cdot V^\lambda\iota_+(\mathcal{O}_X) \subseteq V^{\lambda-1}\iota_+(\mathcal{O}_X)$ for all $\lambda \in \mathbf{Q}$.

In order to conclude, it is enough to show that if $b \in \mathbf{Z}^r_{\geq 0}$ is such that $x^b \in I(f^{\lambda+j})$, then

$$(\partial_t t - \lambda)^r x^b\partial_t^j\delta \in V^{>\alpha}\iota_+(\mathcal{O}_X).$$

Note that $x^b\partial_t^j\delta \in V^{>\lambda}\iota_+(\mathcal{O}_X)$, unless there is i such that $b_i = (\lambda+j)a_i - 1$. In this case, if $e_1, \ldots, e_r$ is the standard basis of $\mathbf{Z}^r$, a simple computation gives

$$\partial_{x_i} \cdot x^{b+e_i}\partial_t^j\delta = (b_i+1)x^b\partial_t^j\delta - a_i x^{a+b}\partial_t^{j+1}\delta = -a_i(\partial_t t - \lambda)x^b\partial_t^j\delta.$$

We thus see that if $J = \big\{i \mid b_i = (\lambda+j)a_i - 1\big\}$, then

$$(\partial_t t - \lambda)^{|J|} \cdot x^b\partial_t^j\delta \in \mathcal{D}_X \cdot x^{b'}\partial_t^j\delta,$$

for some $b' \in \mathbf{Z}^r_{\geq 0}$ such that $x^{b'} \in I(f^{\lambda'+j})$ for some $\lambda' > \lambda$. We thus conclude that $(\partial_t t - \lambda)^{|J|} \cdot x^b\partial_t^j\delta \in V^{>\lambda}\iota_+(\mathcal{O}_X)$.

3.2.3. The V-filtration and b-functions. The existence of V-filtrations is equivalent to the existence of b-functions and the rationality of their roots. Moreover, the V-filtration can be described via the roots of the b-functions. We now explain this.

Let X be a smooth, irreducible variety over k and let $f \in \mathcal{O}_X(X)$ be nonzero. Suppose that $\mathcal{M}$ is a coherent $\mathcal{D}_X$-module on which multiplication by f is bijective. Recall that in this case we have a $\mathcal{D}_X[s]$-module $\mathcal{M}[s]f^s$ on X. In fact, we have on $\mathcal{M}[s]f^s$ an action of $\mathcal{D}_X\langle t, s\rangle$, where t acts on $\mathcal{M}[s]f^s$ via the invertible map T given by $s \mapsto s+1$: more precisely, we have

$$T\big(P(s)uf^s\big) = P(s+1)fuf^s \quad \text{for all} \quad P \in k[s] \text{ and } \quad u \in \mathcal{M}.$$

We note that since $ts = (s+1)t$, it follows from Lemma 3.22 that we have an injective homomorphism $\mathcal{D}_X\langle t, s\rangle \hookrightarrow \mathcal{D}_X\langle t, \partial_t\rangle$, with s mapping to $-\partial_t t$.

PROPOSITION 3.38. If $\mathcal{M}$ is a coherent $\mathcal{D}_X$-module on which the multiplication by f is bijective (for example, we could take $\mathcal{M} = \mathcal{O}_X[1/f]$), then we have an isomorphism of $\mathcal{D}_X\langle t, s\rangle$-modules

$$\tau\colon \mathcal{M}[s]f^s \xrightarrow{\sim} \iota_+(\mathcal{M}), \quad P(s)uf^s \mapsto P(-\partial_t t)(u\otimes\delta). \tag{3.20}$$

Moreover, the action of ∂_t on the right-hand side corresponds to the action of $(-s)T^{-1}$ on the left-hand side.

PROOF. Let's check the compatibility of τ with the $\mathcal{D}_X\langle t, s\rangle$-action. It is clear that τ is $\mathcal{O}_X[s]$-linear, hence we only need to check the compatibility with the action of $\mathcal{D}er_k(\mathcal{O}_X)$ and that of t. Note that

$$\tau\big(t\cdot P(s)uf^s\big) = \tau\big(P(s+1)fuf^s\big) = P(-\partial_t t+1)fu\otimes\delta$$
$$= P(-\partial_t t+1)tu\otimes\delta = tP(-\partial_t t)u\otimes\delta = t\cdot\tau\big(P(s)uf^s\big),$$

where the second to last equality follows from Lemma 3.22i).

Suppose now that $D\in\mathcal{D}er_k(\mathcal{O}_X)$. We have

$$\tau\big(D\cdot P(s)uf^s\big) = \tau\big(P(s)Duf^s + sP(s)\tfrac{D(f)}{f}uf^s\big) = P(-\partial_t t)Du\otimes\delta - P(-\partial_t t)\partial_t t\tfrac{D(f)}{f}u\otimes\delta$$
$$= P(-\partial_t t)Du\otimes\delta - P(-\partial_t t)D(f)u\otimes\partial_t\delta, \quad \text{while}$$
$$D\cdot\tau\big(P(s)uf^s\big) = D\cdot P(-\partial_t t)u\otimes\delta = P(-\partial_t t)\big(Du\otimes\delta - D(f)u\otimes\partial_t\delta\big),$$

hence $\tau\big(D\cdot P(s)uf^s\big) = D\cdot\tau\big(P(s)uf^s\big)$.

Note next that it follows from the second formula in Lemma 3.22ii) that

$$(-1)^m\tau\big(s(s-1)\cdots(s-m+1)uf^s\big) = f^m u\otimes\partial_t^m\delta \quad \text{for all} \quad m.$$

Since $k[s]$ has a basis given by $\prod_{j=0}^{m-1}(s-j)$, for $m\geq 0$, and since multiplication by f is invertible on M, it follows that τ is a bijective map. This completes the proof of the proposition. □

DEFINITION 3.39. Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module. We say that a section $w\in\iota_+(\mathcal{M})$ *has a b-function* if there is a nonzero $b(s)\in k[s]$ such that $b(s)w\in V^1\mathcal{D}_{X\times\mathbf{A}^1}\cdot w$. In this case it follows that the set of such polynomials $b(s)$ is a nonzero ideal of $k[s]$. Its monic generator is the *b-function* $b_w(s)$.

REMARK 3.40. With the notation in the above definition, we note that in fact $b_w(s)$ satisfies $b_w(s)\cdot\frac{V^0\mathcal{D}_{X\times\mathbf{A}^1}\cdot w}{V^1\mathcal{D}_{X\times\mathbf{A}^1}\cdot w} = 0$. This is due to the fact that $V^0\mathcal{D}_{X\times\mathbf{A}^1} = \mathcal{D}_X[s] + V^1\mathcal{D}_{X\times\mathbf{A}^1}$ and $b_w(s)$ commutes with the elements of $\mathcal{D}_X[s]$.

Moreover, if $\mathcal{N} = V^0\mathcal{D}_{X\times\mathbf{A}^1}\cdot w$, then b_w is the minimal polynomial of the action of s on $\mathcal{N}/t\mathcal{N}$. This follows from the fact that $V^1\mathcal{D}_{X\times\mathbf{A}^1}\cdot w = \big(t\cdot\mathcal{D}_X\langle s,t\rangle\big)\cdot w = t\mathcal{N}$.

EXERCISE 3.41. With the notation in Definition 3.39, show that if multiplication by f is invertible on $\mathcal{M}$ and $w = u\otimes\delta\in\iota_+\mathcal{M}$ corresponds to uf^s via the isomorphism τ in Proposition 3.38, then $b_w(s)$ is the monic polynomial of minimal degree such that $b_w(s)uf^s\in\mathcal{D}_X[s]\cdot fuf^s$ (hence we recover the definition in Section 3.1.5).

The following result relates the existence of V-filtrations and b-functions.

THEOREM 3.42. *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module.*

i) *If $\mathcal{M}$ has a V-filtration with respect to f, then every section $w\in\iota_+(\mathcal{M})$ has a b-function. Moreover, if $w\in V^\alpha\iota_+(\mathcal{M})$, then all roots of $b_w(s)$ are rational numbers $\gamma\leq -\alpha$ such that $\mathrm{Gr}_V^{-\gamma}(\iota_+\mathcal{M})\neq 0$.*

ii) *Conversely, if $w_1, \dots, w_r \in \Gamma(X, \mathcal{M})$ generate $\mathcal{M}$ as a $\mathcal{D}_X$-module and if each $w_i \otimes \delta \in \iota_+(\mathcal{M})$ has a b-function whose roots are all rational, then $\mathcal{M}$ has a V-filtration with respect to f.*

PROOF. The assertion in i) is easy to see. Indeed, note first that there is β such that

$$V^\beta \iota_+(\mathcal{M}) \cap \mathcal{D}_{X\times \mathbf{A}^1} \cdot w \subseteq V^1 \mathcal{D}_{X\times \mathbf{A}^1} \cdot w. \tag{3.21}$$

Indeed, by Remark 3.31, the V-filtration with respect to t on $\mathcal{M}_w := \mathcal{D}_{X\times \mathbf{A}^1} \cdot w$ is the restriction of the V-filtration on $\iota_+(\mathcal{M})$. Since $V^1\mathcal{M}_w$ is finitely generated over $V^0\mathcal{D}_{X\times \mathbf{A}^1}$, it follows that there is γ such that $V^1\mathcal{M}_w \subseteq V^\gamma \mathcal{D}_{X\times \mathbf{A}^1} \cdot w$. We conclude that if $\beta \in \mathbf{Z}_{>1}$ is such that $\beta + \gamma \geq 2$, then

$$V^\beta \iota_+(\mathcal{M}) \cap \mathcal{M}_w = V^\beta \mathcal{M}_w = t^{\beta-1} \cdot V^1\mathcal{M}_w \subseteq (t^{\beta-1} \cdot V^\gamma \mathcal{D}_{X\times \mathbf{A}^1}) \cdot w \subseteq V^1 \mathcal{D}_{X\times \mathbf{A}^1} \cdot w,$$

hence (3.21) holds.

Since $(s+\gamma)$ is nilpotent on $\mathrm{Gr}_V^\gamma(\iota_+\mathcal{M})$ for all $\gamma \in \mathbf{Q}$ and since the V-filtration is discrete, it follows that if $\gamma_1, \dots, \gamma_r$ are the rational numbers γ with $\alpha \leq \gamma < \beta$ and with $\mathrm{Gr}_V^\gamma(\iota_+\mathcal{M}) \neq 0$, then there is $N \geq 1$ such that

$$(s+\gamma_1)^N \cdots (s+\gamma_r)^N w \in V^\beta \iota_+\mathcal{M} \cap \mathcal{D}_{X\times \mathbf{A}^1} \cdot w \subseteq V^1 \mathcal{D}_{X\times \mathbf{A}^1} \cdot w.$$

This implies that $b_w(s)$ divides $\prod_{i=1}^r (s+\gamma_i)^N$, proving the assertion in i).

For the proof of the assertion in ii), the first step is to find a decreasing, exhaustive filtration $(W^i)_{i\in\mathbf{Z}}$ on $\iota_+(\mathcal{M})$, with a polynomial $p \in \mathbf{Q}[x]$ having all roots in $[0,1)$, such that $p(\partial_t t - i) \cdot W^i \subseteq W^{i+1}$ for all $i \in \mathbf{Z}$. The V-filtration is then constructed by refining $W^\bullet$ to a rational filtration using the decomposition of each W^i/W^{i+1} into generalized eigenspaces with respect to the action of $\partial_t t$. For details, we refer to [**CDM24**, Theorem 5.11]. □

COROLLARY 3.43. The $\mathcal{D}_X$-module $\mathcal{O}_X$ has a V-filtration with respect to any nonzero $f \in \mathcal{O}_X(X)$.

PROOF. This follows by combining the assertion in Theorem 3.42ii) with the fact that the roots of $b_f(s)$ are rational by Theorem 3.18. □

An interesting consequence of the above theorem is the following description of the V-filtration due to Sabbah:

COROLLARY 3.44. If a coherent $\mathcal{D}_X$-module $\mathcal{M}$ has a V-filtration with respect to f, then for every $\alpha \in \mathbf{Q}$, $V^\alpha\iota_+(\mathcal{M})$ consists of those sections $w \in \iota_+(\mathcal{M})$ with the property that all roots of b_w are $\leq -\alpha$.

PROOF. We have already seen in Theorem 3.42i) that if $w \in V^\alpha\iota_+(\mathcal{M})$, then all roots of b_w are rational numbers $\leq -\alpha$. Conversely, suppose that all roots of b_w are $\leq -\alpha$. Let $\beta \in \mathbf{Q}$ be such that $w \in V^\beta\iota_+(\mathcal{M})$. If $\beta \geq \alpha$, then we are done. If $\beta < \alpha$, since the V-filtration is discrete, we may assume that $w \notin V^{>\beta}\iota_+(\mathcal{M})$ and aim for a contradiction. Since $b_w(s)w \in V^1\mathcal{D}_{X\times \mathbf{A}^1} \cdot w \subseteq V^{\beta+1}\iota_+(\mathcal{M})$, we have $b_w(s)\overline{w} = 0$ in $\mathrm{Gr}_V^\beta\big(\iota_+(\mathcal{M})\big)$. Since $s+\beta$ is nilpotent on $\mathrm{Gr}_V^\beta(\iota_+\mathcal{M})$, while $b_w(s)\overline{w} = 0$ and $b_w(s) = \prod_{i=1}^r (s+\alpha_i)$ with $\alpha_i \geq \alpha > \beta$ for all i, we conclude that $\overline{w} = 0$ in $\mathrm{Gr}_V^\beta\big(\iota_+(\mathcal{M})\big)$, a contradiction. □

REMARK 3.45. Another easy consequence of the criterion in Theorem 3.42 is the fact that if

$$0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$$

is an exact sequence of coherent $\mathcal{D}_X$-modules such that $\mathcal{M}'$ and $\mathcal{M}''$ have V-filtrations with respect to f, then so does $\mathcal{M}$ (see [**CDM24**, Corollary 5.20] for a proof).

This implies that if $\mathcal{M}$ is a holonomic $\mathcal{D}_X$-module (so $\mathcal{M}_f$ is again coherent), then $\mathcal{M}$ has a V-filtration with respect f if and only if so does $\mathcal{M}_f$: indeed, both the kernel and the cokernel of the canonical map $\mathcal{M} \to \mathcal{M}_f$ have support on the hypersurface defined by f, and thus have a V-filtration with respect to f by Example 3.35. Moreover, we see that if $u_\alpha \colon V^\alpha\iota_+(\mathcal{M}) \to V^\alpha\iota_+(\mathcal{M}_f)$ is induced by the canonical map $\mathcal{M} \to \mathcal{M}_f$, then u_α is an isomorphism for all $\alpha > 0$, while

$$\operatorname{Ker}(u_0) \simeq \mathcal{H}^0_Z(\mathcal{M}) \quad \text{and} \quad \operatorname{Coker}(u_0) \simeq \mathcal{H}^1_Z(\mathcal{M}).$$

We end this section by stating a generalization of Lichtin's description of the roots of $b_f(s)$ in Theorem 3.18 to b-functions of elements of the form $g\partial_t^m\delta \in B_f$. This was proved in [**DM22**], following closely Kashiwara's argument in [**Kas77**], as improved by Lichtin [**Lic89**].

THEOREM 3.46. *Let X be a smooth, irreducible algebraic variety and $f \in \mathcal{O}_X(X)$ nonzero, defining the hypersurface Z. With the notation in Theorem 3.18, for every $m \in \mathbf{Z}_{\geq 0}$ and every $g \in \mathcal{O}_X(X)$, if b_i is the coefficient of E_i in $\pi^*\big(\operatorname{div}(g)\big)$, then the following hold:*

i) *Every root of $b_{g\partial_t^m\delta}$ is $\leq -\min\big\{1, \frac{k_i+b_i+1}{a_i} - m \mid 1 \leq i \leq N\big\}$.*
ii) *Every root of $b_{g\delta}$ is $\leq -\min\big\{\frac{k_i+b_i+1}{a_i} \mid 1 \leq i \leq N\big\}$.*
iii) *Every root of $b_{\partial_t^m\delta}$ is either a negative integer or it is of the form $m - \frac{k_i+\ell}{a_i}$, for some i with $1 \leq i \leq N$ and some positive integer ℓ. Furthermore, if Z is reduced and the strict transforms of the components of H on X are disjoint[4], then we may take i such that the divisor E_i is exceptional.*

REMARK 3.47. The reason why in the above theorem, in the cases i) and ii) where g appears, we only get an inequality (instead of a more precise expression for the roots, as in Theorem 3.18) is that for a more precise expression, we would need π to be a log resolution of fg. On the other hand, in order to run the argument of Kashiwara and Lichtin we need π to be an isomorphism over the complement of Z, and we can find such π that is a log resolution of fg only when the hypersurface defined by g has simple normal crossings on $X \smallsetminus Z$.

We will make use of Theorem 3.46 in the next section, to relate invariants coming from the V-filtration to invariants coming from log resolutions.

3.3. The V-filtration and invariants of singularities

We now turn to the connection between the V-filtration and invariants of singularities.

3.3.1. The V-filtration and multiplier ideals. We first discuss the connection between the V-filtration and the Bernstein-Sato polynomial, on one side, and multiplier ideals, on the other side. We begin with a quick introduction to multiplier ideals. For a more detailed discussion and for the proofs of some of the results we state, we refer to [**Laz04**, Chapter 9].

[4]This condition can always be achieved after performing finitely many further blow-ups.

Let X be a smooth, irreducible algebraic variety over an algebraically closed field k of characteristic 0. Suppose that Z is a (nonempty) hypersurface on X. We consider a log resolution $\pi\colon Y \to X$ of the pair (X, Z) and write

$$\pi^*(Z) = \sum_{i=1}^{d} a_i E_i \quad \text{and} \quad K_{Y/X} = \sum_{i=1}^{d} k_i E_i. \tag{3.22}$$

DEFINITION 3.48. For every $\lambda \in \mathbf{Q}_{\geq 0}$, the *multiplier ideal* $\mathcal{J}(X, \lambda Z)$ is given by

$$\mathcal{J}(X, \lambda Z) = \pi_* \mathcal{O}_Y\big(K_{Y/X} - \lfloor \lambda \pi^*(Z) \rfloor\big).$$

If Z is defined by $f \in \mathcal{O}_X(X)$, we also write this as $\mathcal{J}(X, f^\lambda)$.

We recall that, by definition

$$\lfloor \lambda \pi^*(Z) \rfloor = \sum_{i=1}^{N} \lfloor \lambda a_i \rfloor E_i.$$

We also note that since $\lfloor \lambda \pi^*(Z) \rfloor$ is an effective divisor and since $K_{Y/X}$ is an effective exceptional divisor, we have

$$\mathcal{J}(X, \lambda Z) \subseteq \pi_* \mathcal{O}_Y(K_{Y/X}) = \mathcal{O}_X,$$

hence $\mathcal{J}(X, \lambda Z)$ is indeed a coherent ideal of $\mathcal{O}_X$. It is a basic fact that the definition of multiplier ideals is independent of the choice of log resolution (see [**Laz04**, Theorem 9.2.18]).

In what follows we list a few properties of multiplier ideals. Most of these follow in a straightforward way from the definition:

1) If $\lambda \geq \mu$, then
$$\mathcal{J}(X, \lambda Z) \subseteq \mathcal{J}(X, \mu Z).$$
This is a consequence of the fact that the divisor $\lfloor \lambda \pi^*(Z) \rfloor - \lfloor \mu \pi^*(Z) \rfloor$ is effective.
2) It is an immediate consequence of the properties of the round-down function that for every $\lambda \in \mathbf{Q}_{\geq 0}$, there is $\epsilon > 0$ such that
$$\mathcal{J}(X, \lambda Z) = \mathcal{J}(X, \mu Z) \quad \text{for} \quad \lambda \leq \mu \leq \lambda + \epsilon.$$
3) In particular, we have $\mathcal{J}(X, \mu Z) = \mathcal{O}_X$ for $0 \leq \mu \ll 1$.
4) We say that $\lambda \in \mathbf{Q}_{>0}$ is a *jumping number* of (X, Z) if
$$\mathcal{J}(X, \lambda Z) \subsetneq \mathcal{J}\big(X, (\lambda - \epsilon)Z\big) \quad \text{for all} \quad \epsilon > 0.$$
Note that in this case we have $a_i \lambda \in \mathbf{Z}$ for some i with $1 \leq i \leq N$. Therefore the set of jumping numbers of (X, Z) is contained in $\frac{1}{\ell}\mathbf{Z}_{>0}$ for some positive integer ℓ.
5) The smallest jumping number of (X, Z) is the *log canonical threshold* $\operatorname{lct}(X, Z)$ (also written $\operatorname{lct}(Z)$ when X is understood):
$$\operatorname{lct}(X, Z) = \min\big\{\lambda > 0 \mid \mathcal{J}(X, \lambda Z) \neq \mathcal{O}_X\big\}.$$
Note that we have $1 \in \mathcal{J}(X, \lambda Z)$ if and only if $k_i \geq \lfloor \lambda a_i \rfloor$ for all i; equivalently, $k_i > \lambda a_i - 1$ for all i. We thus deduce that
$$\operatorname{lct}(X, Z) = \min_i \tfrac{k_i + 1}{a_i}. \tag{3.23}$$

6) 1 is always a jumping number of (X, Z). In order to see this, we may replace X by any open subset U such that $U \cap Z \neq \emptyset$. We may thus assume that $Z = mH$ for some smooth hypersurface H and some positive integer m. In this case, we may take π to be the identity and we see that

$$\mathcal{J}(X, Z) = \mathcal{O}_X(-mH) \quad \text{and} \quad \mathcal{J}\big(X, (1-\epsilon)Z\big) = \mathcal{O}_X\big(-(m-1)H\big) \text{ for } 0 < \epsilon \ll 1.$$

7) For every $\lambda \geq 1$, it follows from the definition and the projection formula that
$$\mathcal{J}(X, \lambda Z) = \mathcal{O}_X(-Z) \cdot \mathcal{J}\big(X, (\lambda - 1)Z\big).$$
In particular, we see that $\lambda > 1$ is a jumping number of (X, Z) if and only if $\lambda - 1$ has this property. This means that as invariants of singularities, it is enough to consider the multiplier ideals $\mathcal{J}(X, \lambda Z)$ for $\lambda < 1$.

8) If $k = \mathbf{C}$, then there is an analytic description of $\mathcal{J}(X, \lambda Z)$ that is more intuitive than the algebraic one that we gave. Suppose, for simplicity, that Z is defined by $f \in \mathcal{O}_X(X)$. In this case, we have
$$\mathcal{J}(X, \lambda Z) = \big\{g \in \mathcal{O}_X(X) \mid \tfrac{|g|^2}{|f|^{2\lambda}} \text{ is locally integrable}\big\}.$$
The local integrability condition means that for every $P \in X$, if $z_1, \ldots, z_n$ are local coordinates around P, then there is an open neighborhood U of P such that $\int_U \frac{|g|^2}{|f|^{2\lambda}} dz d\overline{z} < \infty$. The equivalence with the formula in the algebraic definition is shown using the Change of Variable formula and the fact that, in one variable, $\frac{1}{|z|^{2\lambda}}$ is locally integrable if and only if $\lambda < 1$ (see [**Laz04**, Chapter 9.3.D] for details).

The first result relating the Bernstein-Sato polynomial to the invariants in birational geometry is the following theorem due to Kollár [**Kol97**]:

THEOREM 3.49. *The largest root of $b_Z(s)$ is $-\operatorname{lct}(X, Z)$.*

The fact that the largest root of $b_Z(s)$ is $\leq -\operatorname{lct}(Z)$ follows from Theorem 3.18 and the formula for the log canonical threshold in (3.23). The opposite inequality follows by a variant of Bernstein's integration by parts argument that we described in Section 3.1.4.

The following result of Budur and Saito [**BS05**] gives a description of multiplier ideals in terms of the V-filtration:

THEOREM 3.50. *If Z is defined by $f \in \mathcal{O}_X(X)$ and $\iota\colon X \hookrightarrow X \times \mathbf{A}^1$ is the graph of f, then for every $\lambda \in \mathbf{Q}_{\geq 0}$, we have*
$$\mathcal{J}(X, \lambda Z) = \big\{g \in \mathcal{O}_X \mid g\delta \in V^{>\lambda}\iota_+(\mathcal{O}_X)\big\}.$$

The proof of this result in [**BS05**] makes use of results in Saito's theory of mixed Hodge modules. The following more elementary proof is from [**DM22**], extending Kollár's proof of Theorem 3.49.

SKETCH OF PROOF OF THEOREM 3.50. We may and will assume that X is affine, with $R = \mathcal{O}_X(X)$ and let $\pi\colon Y \to X$ be a log resolution of (X, Z) that is an isomorphism over $X \smallsetminus Z$. Let $g \in R$ be nonzero. We use the notation in (3.22) and denote by b_i the coefficient of E_i in $\pi^*\big(\operatorname{div}(g)\big)$. By definition, we have $g \in \mathcal{J}(X, \lambda Z)$ if and only if $b_i + k_i \geq \lfloor \lambda a_i \rfloor$ for all i, which is the case if and only if $\lambda < \min_i \frac{b_i + k_i + 1}{a_i} =: \operatorname{lct}_g(X, Z)$. On the other hand, it follows from Corollary 3.44 that $g\delta \in V^{>\lambda}\iota_+(\mathcal{O}_X)$ if and only if all roots of $b_{g\delta}$ are $< -\lambda$. We thus conclude

that the assertion in the theorem is equivalent to the fact that for every nonzero $g \in \mathcal{O}_X(X)$, the largest root of $b_{g\delta}$ is $-\operatorname{lct}_g(X,Z)$.

The fact that every root of $b_{g\delta}$ is $\leq -\operatorname{lct}_g(X,Z)$ follows from Theorem 3.46iii). In order to complete the proof, it is enough to show that if $g \in \mathcal{J}\big(X,(\lambda-\epsilon)Z\big)$ for every $\epsilon > 0$, but $g \notin \mathcal{J}(X,\lambda Z)$, then $b_{g\delta}(-\lambda) = 0$. For this, we use the Lefschetz Principle to reduce to the case when the ground field is $\mathbf{C}$, when the assertion follows via a variant of the integration by parts argument that we described in Section 3.1.4. □

REMARK 3.51. Suppose that X and Z are as in Theorem 3.50. We have seen in the proof of the theorem that its statement is equivalent to the fact that for every $g \in \mathcal{O}_X(X)$ nonzero, the largest root of $b_{g\delta}$ is $-\operatorname{lct}_g(X,Z)$. The special case $g = 1$ corresponds to the assertion in Theorem 3.49

The assertion in Theorem 3.50 implies the following result of Ein, Lazarsfeld, Smith, and Varolin, extending the fact that the negative of the log canonical threshold is a root of the b-function:

THEOREM 3.52. *If $\lambda \leq 1$ is a jumping number of (X,Z), then $b_Z(-\lambda) = 0$.*

PROOF. We may and will assume that X is an affine variety, with $R = \mathcal{O}_X(X)$, and Z is defined by $f \in R$. Since λ is a jumping number, it follows that there is $g \in R$ such that $g \in \mathcal{J}\big(X,(\lambda-\epsilon)Z\big)$ for $0 < \epsilon \ll 1$, but $g \notin \mathcal{J}(X,\lambda Z)$. For simplicity, we write V^α and $\operatorname{Gr}_V^\alpha$ for $V^\alpha\iota_+(\mathcal{O}_X)$ and $\operatorname{Gr}_V^\alpha\big(\iota_+(\mathcal{O}_X)\big)$, respectively. By Theorem 3.50, we have $g\delta \in V^\lambda \smallsetminus V^{>\lambda}$. On the other hand, by definition of b_f, there is $P \in D_R[s]$ such that

$$b_f(s)\delta = P \cdot t\delta.$$

Note that $\delta \in V^{>0}$: this follows from Theorem 3.50 since $\mathcal{J}(X,\lambda H) = \mathcal{O}_X$ for $\lambda = 0$; alternatively, it follows from the description of the V-filtration in Proposition 3.44 and the fact that the roots of b_f are negative by Theorem 3.49. Therefore $t\delta \in V^{>1} \subseteq V^{>\lambda}$. Since $b_f(s)g\delta = gP \cdot t\delta \in V^{>\lambda}$, it follows that $b_f(s)$ annihilates a nonzero element in $\operatorname{Gr}_V^\lambda$. Since $(s+\lambda)^N$ annihilates $\operatorname{Gr}_V^\lambda$ for some N, it follows that $\gcd\big(b_f(s),(s+\lambda)^N\big)$ annihilates a nonzero element, hence $(s+\lambda)$ divides $b_f(s)$. □

3.3.2. The minimal exponent. We next define the invariant that will be one of our main focus points in these lectures. It was introduced by Saito in [**Sai93**]. As usual, we consider a smooth, irreducible algebraic variety X. Let Z be a nonempty hypersurface in X.

DEFINITION 3.53. The *minimal exponent* $\widetilde{\alpha}(Z)$ is the negative of the largest root of $\widetilde{b}_Z(s)$ (with the convention that this is ∞ if $\widetilde{b}_Z(s) = 1$). Similarly, if $P \in Z$, then we define $\widetilde{\alpha}_P(Z)$ as the negative of the largest root of $\widetilde{b}_{Z,P}(s)$. If Z is defined by $f \in \mathcal{O}_X(X)$, we also write $\widetilde{\alpha}(f)$ and $\widetilde{\alpha}_P(f)$ for $\widetilde{\alpha}(Z)$ and $\widetilde{\alpha}_P(Z)$, respectively.

REMARK 3.54. It follows from the definition that

$$\widetilde{\alpha}(Z) = \min_{P \in Z} \widetilde{\alpha}_P(Z) \quad \text{and} \quad \widetilde{\alpha}_P(Z) = \max_{U \ni P} \widetilde{\alpha}(Z|_U),$$

where the maximum is over the open neighborhoods of P.

REMARK 3.55. It is a consequence of Theorem 3.18 that $\widetilde{\alpha}(Z)$ is a positive rational number or infinity. Moreover, it follows from Theorem 3.49 that

$$\operatorname{lct}(X,Z) = \min\{\widetilde{\alpha}(Z),1\}. \tag{3.24}$$

Note that $\widetilde{\alpha}(Z) = \infty$ if Z is smooth (see Example 3.12) and we will show later that the converse holds too.

REMARK 3.56. It follows from (3.24) that the minimal exponent only provides interesting new information when $\mathrm{lct}(X, Z) = 1$. Because of this, when proving properties of the minimal exponent that are already known for the log canonical threshold, we may assume that Z is reduced (note that if Z has a component of multiplicity $m > 1$, then $\mathrm{lct}(X, Z) \leq \frac{1}{m}$).

The first result that illustrates the fact that $\widetilde{\alpha}(Z)$ is an interesting invariant is the following theorem due to Saito [**Sai93**]. We will discuss extensions of this result in a later chapter.

THEOREM 3.57. *We have $\widetilde{\alpha}(Z) > 1$ if and only if the hypersurface Z has rational singularities.*

The following result relates the minimal exponent and the V-filtration.

THEOREM 3.58. *If Z is defined by $f \in \mathcal{O}_X(X)$ and $\iota\colon X \hookrightarrow X \times \mathbf{A}^1$ is the graph of f, then for every $p \in \mathbf{Z}_{\geq 0}$ and every $\lambda \in (0, 1] \cap \mathbf{Q}$, we have*

$$\widetilde{\alpha}(Z) \geq p + \lambda \quad \text{if and only if} \quad \partial_t^p \delta \in V^\lambda(\iota_+ \mathcal{O}_X).$$

This description was proved by Saito in [**Sai16**] using a version of the V-filtration, the *microlocal V-filtration*. The assertion also follows via the description of the V-filtration in terms of b-functions in Corollary 3.44 and the following more precise result from [**CDM24**] relating the b-functions of δ and $\partial_t^p \delta$.

THEOREM 3.59. *If $f \in \mathcal{O}_X(X)$ is nonzero and noninvertible, then for every $p \geq 0$, we have*

$$b_{\partial_t^p \delta}(s) = (s + 1) \cdot \widetilde{b}_f(s - p).$$

SKETCH OF PROOF. We only prove the easier assertion, which first appeared in [**MP20**], that $b_{\partial_t^q \delta}(s)$ divides the polynomial $(s + 1)\widetilde{b}(s - q)$. For the opposite divisibility, we refer to the proof of [**CDM24**, Theorem 10.17].

By definition of $\widetilde{b}_f(s)$, we may assume that there is a global section P of $\mathcal{D}_X[s]$ such that

$$(s + 1)\widetilde{b}(s)\delta = P(s) f \delta.$$

Since $s + 1 = -t\partial_t$ and $P(s) f \delta = P(s) t \delta = t P(s - 1)\delta$ by Lemma 3.22, we have

$$-t\partial_t \widetilde{b}(s)\delta = t P(s - 1)\delta.$$

Since the action of t on $\iota_+(\mathcal{O}_X)$ is injective, we deduce that

$$\partial_t \widetilde{b}(s)\delta = -P(s - 1)\delta.$$

Using again Lemma 3.22, we have

$$\begin{aligned}(s + 1)\widetilde{b}(s - q)\partial_t^q \delta &= (s + 1)\partial_t^q \widetilde{b}(s)\delta = -(s + 1)\partial_t^{q-1} P(s - 1)\delta \\ &= t\partial_t^q P(s - 1)\delta = P(s - q) t \partial_t^q \delta.\end{aligned}$$

Since $P(s - q)t \in V^1 \mathcal{D}_{X \times \mathbf{A}^1}$, we conclude that $b_{\partial_t^q \delta}(s)$ divides $(s + 1)\widetilde{b}(s - q)$. □

We will return to the general properties of the minimal exponent in later lectures. We end this section with a lower bound in terms of log resolutions. This was proved in [**MP20**] using results on Hodge ideals. The more elementary proof that we give is from [**DM22**]

THEOREM 3.60. *Let X be a smooth, irreducible algebraic variety and Z a nonempty reduced hypersurface in X. Let $\pi\colon Y \to X$ be a log resolution of (X,Z) that is an isomorphism over $X \smallsetminus Z$ and such that the strict transform $\widetilde{Z}$ of Z on Y is smooth*[5]*. If we write $\pi^*(Z) = \widetilde{Z} + \sum_{i=1}^d a_i E_i$ and $K_{Y/X} = \sum_{i=1}^d k_i E_i$, then*

$$\widetilde{\alpha}(Z) \geq \min_i \tfrac{k_i+1}{a_i}. \tag{3.25}$$

PROOF. After taking a suitable open cover of X, we may and will assume that Z is defined by $f \in \mathcal{O}_X(X)$. Let α be the right-hand side of (3.25) and let us write $\alpha = p + \lambda$, with $p \in \mathbf{Z}_{\geq 0}$ and $\lambda \in (0,1]$. By combining Theorem 3.58 and Corollary 3.44, we see that $\widetilde{\alpha}(Z) \geq \alpha$ if and only if all roots of $b_{\partial_t^p \delta}(s)$ are $\leq -\lambda$. However, by Theorem 3.46, such a root is either $-m$, for some positive integer m (in which case it is clearly $\leq -\lambda$) or of the form $p - \frac{k_i+\ell}{a_i}$, for some positive integer ℓ and some i with E_i exceptional (in which case, it is $\leq p - \alpha = -\lambda$). This completes the proof. □

REMARK 3.61 (Computation of the minimal exponent as the minimum on a log resolution). Unlike in the case of formula (3.23) that computes $\mathrm{lct}(X,Z)$ in terms of any log resolution, we can't hope that the inequality in (3.25) is an equality for an arbitrary resolution. Indeed, if the minimum on the right-hand side of (3.25) is > 1 (so $\widetilde{\alpha}(X,Z) > 1$), then one can construct a sequence of blow-ups with smooth centers of codimension 2, such that the corresponding minima converge to 1. Indeed, given a log resolution Y, consider the blow-up along a component of the intersection of an exceptional divisor E_j with a connected component $\widetilde{Z}_0$ of $\widetilde{Z}$, with exceptional divisor G_1. Consider next the blow-up along a component of the intersection of G with the strict transform of E_0, with exceptional divisor G_2, etc. An easy computation shows that the coefficient of G_ℓ in the inverse image of H on Y_ℓ is $a_i + \ell$ and its coefficient in $K_{Y_\ell/X}$ is $k_i + \ell$. Note that $\lim_{\ell\to\infty} \frac{k_i+1+\ell}{a_i+\ell} = 1$. However, it is an interesting question whether given a reduced hypersurface Z on X, there is a log resolution $\pi\colon Y \to X$ of (X,Z) that is an isomorphism over $X \smallsetminus Z$, such that we have equality in (3.25).

3.3.3. Refined multiplier ideals. The characterization of the minimal exponent in Theorem 3.58 suggests the definition of an extension of multiplier ideals whose triviality detects the minimal exponent in the same way that the triviality of multiplier ideals detects the log canonical threshold.

Suppose first that Z is a hypersurface in X defined by $f \in \mathcal{O}_X(X)$. For simplicity, we write V^α for $V^\alpha \iota_+(\mathcal{O}_X)$, where $\iota\colon X \hookrightarrow X \times \mathbf{A}^1$ is the graph embedding corresponding to f.

DEFINITION 3.62. For every $\alpha \in \mathbf{Q}_{\geq 0}$, we write $\alpha = p + \lambda$, where $p \in \mathbf{Z}_{\geq 0}$ and $\lambda \in [0,1)$, and we denote by $\widetilde{\mathcal{J}}(X, \alpha Z)$ the coherent ideal of $\mathcal{O}_X$ consisting of those $h \in \mathcal{O}_X$ with the property that there are $h_0, \ldots, h_{q-1} \in \mathcal{O}_X$ such that $h_0\delta + \ldots + h_{q-1}\partial_t^{q-1}\delta + h\partial_t^q\delta \in V^{>\alpha}$.

These ideals have been introduced by Saito in [**Sai16**] as *microlocal multiplier ideals*, due to the fact that the definition was expressed in terms of the microlocal V-filtration (however, we will not use this terminology). Under the name of

[5]Note that this condition can always be achieved after some further blow-ups with smooth centers.

higher multiplier ideals (and with a different indexing), the ideals have been studied systematically by Schnell and Yang in their recent preprint [**SY23**] using twisted Hodge modules.

REMARK 3.63 (The connection with the classical version). Note that Theorem 3.50 implies that for $\lambda < 1$, we have

$$\widetilde{\mathcal{J}}(X, \lambda Z) = \mathcal{J}(X, \lambda Z).$$

REMARK 3.64 (Independence of the equation). The ideals $\widetilde{\mathcal{J}}(X, \alpha Z)$ do not depend on the equation f defining Z. Indeed, suppose that $g \in \mathcal{O}_X(X)$ defines the same hypersurface, hence we can write $g = pf$, for some invertible $p \in \mathcal{O}_X(X)$. Arguing as in Remark 3.8, we can compare the V-filtrations corresponding to f and g: we have an isomorphism of $\mathcal{O}_X$-modules

$$\tau \colon \mathcal{O}_X[1/f, s]f^s \to \mathcal{O}_X[1/g, s]g^s, \quad Pf^s \mapsto Pg^s,$$

which is linear over the isomorphism of sheaves of rings

$$\tau_0 \colon \mathcal{D}_X\langle t, \partial_t\rangle \to \mathcal{D}_X\langle t, \partial_t\rangle,$$

where τ_0 is the identity on $\mathcal{O}_X$, $\tau_0(Q) = Q - p^{-1}Q(p)s$ for $Q \in \mathcal{D}er_k(\mathcal{O}_X)$, and $\tau_0(t) = p^{-1}t$ and $\tau_0(\partial_t) = p\partial_t$. It is then easy to check that if ι_f and ι_g are the graph embeddings corresponding to f and g, respectively, then τ induces an isomorphism $(\iota_f)_+(\mathcal{O}_X) \simeq (\iota_g)_+(\mathcal{O}_X)$ linear over τ_0 that maps each $V^\alpha(\iota_f)_+(\mathcal{O}_X)$ onto $V^\alpha(\iota_g)_+(\mathcal{O}_X)$. Since $\tau(\partial_t^m\delta_f) = p^m\partial_t^m\delta_g$, it is now clear that f and g lead to the same ideals $\widetilde{\mathcal{J}}(X, \alpha Z)$.

It follows from the above remark that we can define $\widetilde{\mathcal{J}}(X, \lambda Z)$ for every hypersurface Z, by gluing the corresponding ideals on a suitable open cover such that Z is a principal divisor in each of these open subsets.

REMARK 3.65. For every hypersurface Z, there is a positive integer ℓ such that $\widetilde{\mathcal{J}}(X, \alpha Z)$ is constant for $\alpha \in \left[\frac{i}{\ell}, \frac{i+1}{\ell}\right)$ for every $i \in \mathbf{Z}_{\geq 0}$. Indeed, it is enough to check this when Z is defined by $f \in \mathcal{O}_X(X)$, in which case the assertion follows from the fact that $V^\bullet\iota_+(\mathcal{O}_X)$ is discrete and left continuous. In particular, we see that for every $\alpha \in \mathbf{Q}_{\geq 0}$, there is $\alpha' > \alpha$ such that $\widetilde{\mathcal{J}}(X, \alpha Z) = \widetilde{\mathcal{J}}(X, \beta Z)$ for every $\alpha \leq \beta \leq \alpha'$.

REMARK 3.66. It is not hard to see that for every hypersurface Z and every $\alpha \geq \beta$, we have

$$\widetilde{\mathcal{J}}(X, \alpha Z) \subseteq \widetilde{\mathcal{J}}(X, \beta Z).$$

We refer to [**CDM24**, Proposition 10.11] for the proof.

PROPOSITION 3.67. For every hypersurface Z in X, we have

$$\widetilde{\alpha}(Z) = \min\left\{\alpha > 0 \mid \widetilde{\mathcal{J}}(X, \alpha Z) \neq \mathcal{O}_X\right\}.$$

PROOF. We may and will assume that X is affine. The assertion follows from Theorem 3.58 if we show that if there are $h_0, \ldots, h_{p-1} \in \mathcal{O}_X(X)$ such that

$$u = h_0\delta + \ldots + h_{p-1}\partial_t^{p-1}\delta + \partial_t^p\delta \in V^{>\alpha}, \tag{3.26}$$

then $\partial_t^i\delta \in V^{>\alpha}$ for all $i \leq p$ (in fact, we only need this for $i = p$). We argue by induction on p, the case $p = 0$ being trivial. For the induction step, note that $u \in V^{>\alpha}$ implies $(f - t)u \in V^{>\alpha}$ and

$$(f - t)u = \partial_t^{p-1}\delta + \text{lower order terms}.$$

By induction, it follows that $\partial_t^i \delta \in V^{>\alpha}$ for all $i \leq p-1$. The fact that $u \in V^{>\alpha}$ now implies that also $\partial_t^p \delta \in V^{>\alpha}$. $\square$

CHAPTER 4

A quick overview of Hodge module theory

The theory of Hodge modules was developed by Morihiko Saito in [Sai88] and [Sai90]. Roughly speaking, the theory provides a 6-functor formalism for doing Hodge theory. The objects can be viewed as "enriched" perverse sheaves: they are $\mathcal{D}$-modules with several other pieces of data, the most important for us being a canonical good filtration, the *Hodge filtration*. This will allow us to obtain and exploit coherent information in the realm of algebraic geometry.

4.1. Generalities

We do not aim in this chapter for a detailed or motivated introduction to Hodge modules. For such an introduction, we recommend Schnell's notes [Sch19]. Our goal is to simply explain what the objects of this theory are and how to work with them.

To begin with, let X be a smooth, irreducible, n-dimensional complex algebraic variety. Hodge modules come in two flavors: pure (of a certain weight) and mixed. If X is a point, then a pure Hodge module on X is a pure Hodge structure (of a certain weight), while a mixed Hodge module on X is a mixed Hodge structure. Several of the main results of the theory require the Hodge modules to be polarizable: when X is a point, this means that the pure or mixed Hodge structures are required to be polarizable.

A *mixed Hodge module* (or simply *Hodge module*) M consists of several pieces of information: $M = (\mathcal{M}, F_\bullet\mathcal{M}, \mathcal{P}, \alpha, W_\bullet\mathcal{M})$. Here $\mathcal{M}$ is a left[1] regular holonomic $\mathcal{D}_X$-module, $F_\bullet\mathcal{M}$ is a good filtration on $\mathcal{M}$ (the *Hodge filtration*), $\mathcal{P}$ is a perverse sheaf with $\mathbf{Q}$-coefficients, and α is an isomorphism

$$\alpha\colon \mathrm{DR}_X^{\mathrm{an}}(\mathcal{M}) \simeq \mathcal{P} \otimes_{\mathbf{Q}} \mathbf{C}.$$

The filtration $W_\bullet\mathcal{M}$ (the *weight filtration*) is a finite increasing filtration by $\mathcal{D}_X$-submodules. We will refer to $\mathcal{M}$ and $(\mathcal{M}, F_\bullet\mathcal{M})$ as the underlying $\mathcal{D}$-module, respectively filtered $\mathcal{D}$-module. The Hodge module M is *pure, of weight* ℓ, if $W_{\ell-1}\mathcal{M} = 0$ and $W_\ell\mathcal{M} = \mathcal{M}$. For every Hodge module M as above and every q, the quotient $\mathrm{Gr}_q^W(\mathcal{M})$, with the induced Hodge filtration, underlies a pure Hodge module of weight q, denoted $\mathrm{Gr}_q^W(M)$. We will come back to explaining what it means for a Hodge module M to be polarizable.

A morphism of Hodge modules is simply a morphism of $\mathcal{D}_X$-modules compatible with the $\mathbf{Q}$-perverse sheaf and with the Hodge and weight filtrations. One shows that every such morphism is automatically strict with respect to the Hodge

[1] We make the choice to work with left $\mathcal{D}$-modules, but many other sources, including [Sai88], [Sai90], and [Sch19], consider right $\mathcal{D}$-modules instead. Of course, we have an equivalence between the two definitions via the equivalence of categories in Section 2.2.1; however, one should keep in mind the convention for shifting the filtrations as in (2.4).

and weight filtrations. This is the key fact behind the fact that the categories of pure/mixed Hodge modules are abelian. We write $\mathrm{MHM}(X)$ for the abelian category of polarizable Hodge modules on X and $D^b\big(\mathrm{MHM}(X)\big)$ for the corresponding bounded derived category.

The role of the perverse sheaf $\mathcal{P}$ and of the isomorphism α is to provide a "$\mathbf{Q}$-structure". It is an important fact that the functor rat from the abelian category of Hodge modules to the category of $\mathbf{Q}$-perverse sheaves is exact and faithful.

For an integer d, the *Tate twist* $M(d)$ of a Hodge module M given by the tuple $(\mathcal{M}, F_\bullet, \mathcal{P}, \alpha, W_\bullet)$ is $M(d) = (\mathcal{M}, F_{\bullet-d}, \mathcal{P} \otimes_{\mathbf{Q}} \mathbf{Q}(d), \alpha, W_{\bullet+2d})$, where $\mathbf{Q}(d) = (2\pi i)^d\mathbf{Q} \subseteq \mathbf{C}$. In particular, note that if M is pure of weight w, then $M(d)$ is pure of weight $w - 2d$.

Naturally, not all objects as described above are allowed and which ones are allowed is determined by induction on dimension (more precisely, by induction on the dimension of the support of the underlying $\mathcal{D}$-module). Let us explain in more detail how this works in the case of pure Hodge modules. In dimension 0 and in the pure setting, our data is equivalent to that of a finite-dimensional $\mathbf{Q}$-vector space V and of a separated, exhaustive, increasing filtration on $V_{\mathbf{C}}$. This is a pure Hodge module of weight w if and only if it is a pure Hodge structure of weight w (note that the increasing filtration has to be turned into a decreasing filtration in the standard way, by putting $F^pV = F_{-p}V$). The key ingredient for the inductive condition is the assumption that the vanishing and nearby cycles exist for every local function $f \in \mathcal{O}_X(U)$; in this case, they automatically carry a natural Hodge filtration. Then M is pure, of suitable weight, if for every such f that doesn't vanish identically on any component of the support of M, the graded pieces of the vanishing/nearby cycles with respect to the monodromy filtration (which have support of smaller dimension) are pure Hodge modules, of the corresponding weight. We do not go into details but will review the construction of vanishing/nearby cycles in this setting later. We only mention that due to this inductive definition, one always assumes the following:

i) If $\mathcal{M}$ is the underlying $\mathcal{D}$-module of a mixed Hodge module, then for every $0 \neq f \in \mathcal{O}_X(U)$, the restriction $\mathcal{M}|_U$ has a V-filtration with respect to f.
ii) In the setting of i), one makes two compatibility assumptions between the Hodge filtration $F_\bullet\mathcal{M}$ and the V-filtration with respect to f. We do not state these conditions, since we will not need them, but refer to [**Sch19**, Section 11] for a nice discussion of these conditions and of their motivation.

Remark 4.1. While we do not make precise the compatibility conditions between the Hodge and V-filtrations, one consequence is worth mentioning: if $(\mathcal{M}, F_\bullet\mathcal{M})$ is the filtered $\mathcal{D}_X$-module underlying a Hodge module and if $\mathrm{Supp}(\mathcal{M}) \subseteq V(f)$, for some $f \in \mathcal{O}_X(X)$, then $f \cdot F_p\mathcal{M} \subseteq F_{p-1}\mathcal{M}$ for all $p \in \mathbf{Z}$.

Example 4.2. The most basic example of a pure Hodge module is $\mathbf{Q}_X^H[n]$. The underlying filtered $\mathcal{D}_X$-module is $\mathcal{O}_X$, with the filtration given by $F_p\mathcal{O}_X = \mathcal{O}_X$ for $p \geq 0$ and $F_p\mathcal{O}_X = 0$ for $p < 0$. The corresponding perverse sheaf is $\mathbf{Q}_X[n]$, with the isomorphism α being induced by the inclusion $\mathbf{C}_X \hookrightarrow \mathcal{O}_X$. This is a pure Hodge module of weight n.

Example 4.3. More generally, every (polarizable) variation of $\mathbf{Q}$-Hodge structures of weight w gives a (polarizable) pure Hodge module of weight $w + \dim(X)$:

the $\mathcal{D}_X$-module in this case is a vector bundle with integrable connection, the Hodge filtration is the one given by the structure of a variation of Hodge structures (turned into an increasing filtration), and the perverse sheaf is $\mathcal{L}[n]$, where $\mathcal{L}$ is the local system of $\mathbf{Q}$-vector spaces associated to the variation of Hodge structures.

We will explain below what is the meaning of a polarization for a pure Hodge module. A Hodge module M is *polarizable* (the standard name is *graded polarizable*) if all $\mathrm{Gr}_i^W(M)$ are polarizable.

REMARK 4.4. We note that if we have a tuple $M = (\mathcal{M}, F_\bullet\mathcal{M}, \mathcal{P}, \alpha, W_\bullet\mathcal{M})$ as in the definition of a Hodge module such that each $\mathrm{Gr}_i^W(M)$ is a pure Hodge module of weight i, it does not automatically mean that M is a Hodge module. There is an extra condition that has to be put again with respect to all functions $f \in \mathcal{O}_X(U)$ and which has to do with the *admissibility* condition in the theory of variations of mixed Hodge structures, but we do not discuss this.

4.2. Functors on Hodge modules

We now discuss some of the main functors for Hodge modules. Let X be a smooth, irreducible, n-dimensional complex algebraic variety (as above, for now, we consider functors for Hodge modules on smooth varieties, and discuss a more general framework in Section 4.4 below).

4.2.1. The graded de Rham complex. If $(\mathcal{M}, F_\bullet\mathcal{M})$ is the underlying filtered $\mathcal{D}_X$-module of a Hodge module M, for every $p \in \mathbf{Z}$, we may consider the complex $\mathrm{Gr}_p^F\mathrm{DR}_X(\mathcal{M})$. Since morphisms of Hodge modules are strict with respect to the Hodge filtration, it follows that short exact sequences of Hodge modules induce short exact sequences of complexes. We thus get an exact functor of triangulated categories

$$\mathrm{Gr}_p^F\mathrm{DR}_X \colon D^b\big(\mathrm{MHM}(X)\big) \to D^b_{\mathrm{coh}}(X),$$

where on the right-hand side we have the bounded derived category of complexes of $\mathcal{O}_X$-modules with coherent cohomology. With a slight abuse of notation, we sometimes write $\mathrm{Gr}_p^F\mathrm{DR}_X(\mathcal{M})$ instead of $\mathrm{Gr}_p^F\mathrm{DR}_X(M)$. For a Hodge module M, the various graded de Rham complexes provide interesting invariants and, as we will see, they interact well with some of the important functors.

EXAMPLE 4.5. It is immediate to see, using the definition of the filtration on $\mathrm{DR}_X(\mathcal{O}_X)$, that

$$\mathrm{Gr}_{-p}\mathrm{DR}_X\big(\mathbf{Q}_X^H[n]\big) \simeq \Omega_X^p[n-p] \quad \text{for all} \quad p \in \mathbf{Z}. \tag{4.1}$$

EXAMPLE 4.6. It follows from the definition of the Tate twist that for every $d \in \mathbf{Z}$ and every Hodge module M, we have

$$\mathrm{Gr}_p^F\mathrm{DR}_X\big(M(d)\big) = \mathrm{Gr}_{p-d}^F\mathrm{DR}_X(M).$$

4.2.2. Duality. In general, there is no good definition of a duality functor for filtered $\mathcal{D}$-modules. What allows the definition of such a functor for Hodge modules is the fact that if $(\mathcal{M}, F_\bullet)$ underlies a Hodge module, then $\mathrm{Gr}_\bullet^F(\mathcal{M})$ is Cohen-Macaulay (when viewed as a sheaf on the cotangent bundle).

It follows that there is a good duality functor for Hodge modules which, at the level of $\mathcal{D}_X$-modules, corresponds to the duality functor we discussed in Section 2.5.4 and, at the level of perverse sheaves, corresponds to the Verdier duality

$\mathbf{R}\mathcal{H}om_{\mathbf{Q}}\big(-,\mathbf{Q}_{X^{\mathrm{an}}}[n]\big)$. This is an exact functor $\mathrm{MHM}(X)\to\mathrm{MHM}(X)$ and thus induces an exact functor

$$\mathbf{D}=\mathbf{D}_X\colon D^b\big(\mathrm{MHM}(X)\big)\to D^b\big(\mathrm{MHM}(X)\big).$$

The behavior of $\mathbf{D}$ with respect to the weight filtration is easy to describe:

$$W_p\mathbf{D}(M)=\mathrm{Ker}\big(\mathbf{D}(M)\to\mathbf{D}(W_{-p-1}M)\big)\simeq\mathbf{D}(M/W_{-p-1}M).$$

We thus have

$$\mathrm{Gr}^W_p\mathbf{D}(M)\simeq\mathbf{D}\big(\mathrm{Gr}^W_{-p}(M)\big).$$

In particular, if M is pure, of weight w, then $\mathbf{D}(M)$ is pure, of weight $-w$.

We do not describe what duality does on the Hodge filtration, but only mention that it is compatible with Tate twists:

$$\mathbf{D}\big(M(q)\big)\simeq\mathbf{D}(M)(-q)\quad\text{for all}\quad q\in\mathbf{Z}$$

and state a useful result concerning the compatibility with taking the graded de Rham complex: if M is a Hodge module (or, more generally, an object in $D^b\big(\mathrm{MHM}(X)\big)$, then we have a canonical isomorphism

$$\mathrm{Gr}^F_p\mathrm{DR}_X\big(\mathbf{D}(M)\big)\simeq\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}\big(\mathrm{Gr}^F_{-p}\mathrm{DR}_X(M),\omega_X[n]\big).\tag{4.2}$$

Example 4.7. If M is a pure Hodge module of weight w, then a polarization of M is given by an isomorphism

$$\mathbf{D}(M)\simeq M(w)\tag{4.3}$$

that has to satisfy extra properties. Recall that for Hodge structures, such an isomorphism is a polarization if it satisfies some positivity condition encoded in the Hodge-Riemann bilinear relations. In higher dimension, the condition that one puts is that after getting to dimension 0 via successive applications of vanishing/nearby cycles, the resulting isomorphism of Hodge structures is a polarization.

4.2.3. Push-forward by a proper morphism. Given a proper morphism $f\colon X\to Y$ of smooth, irreducible algebraic varieties, there is a functor

$$f_*\colon D^b\big(\mathrm{MHM}(X)\big)\to D^b\big(\mathrm{MHM}(Y)\big)$$

that agrees with f_+ at the level of underlying $\mathcal{D}$-modules and such that the Hodge filtration on its cohomology can be computed as follows. Suppose that we have a complex $C^\bullet$ of filtered *right* $\mathcal{D}_X$-modules that corresponds to a complex of left such $\mathcal{D}_X$-modules underlying a bounded complex of Hodge modules representing $M\in D^b\big(\mathrm{MHM}(X)\big)$. This is always filtered quasi-isomorphic to a bounded above complex $\widetilde{C}^\bullet$ of *induced filtered $\mathcal{D}_X$-modules*: these are $\mathcal{D}_X$-modules of the form $\mathcal{E}\otimes_{\mathcal{O}_X}\mathcal{D}_X$, where $\mathcal{E}$ is a filtered coherent $\mathcal{O}_X$-module and the filtration on the tensor product is the tensor product filtration (note that $\mathcal{D}_X$ is filtered by the order filtration). In this case

$$\mathcal{H}^i\big(f_*(M)\big)\simeq\mathbf{R}^if_*(\widetilde{C}^\bullet\otimes_{\mathcal{O}_X}D_{X\to Y})\quad\text{and}$$

$$F_p\mathcal{H}^i\big(f_*(M)\big)=\mathrm{Im}\big(\mathbf{R}^if_*(F_p(\widetilde{C}^\bullet\otimes_{\mathcal{O}_X}D_{X\to Y}))\to\mathbf{R}^if_*(\widetilde{C}^\bullet\otimes_{\mathcal{O}_X}D_{X\to Y})\big).\tag{4.4}$$

Note that the filtration on the terms of $\widetilde{C}^\bullet\otimes_{\mathcal{O}_X}D_{X\to Y}$ is the tensor product filtration, where $D_{X\to Y}$ carries the filtration induced by the order filtration on $\mathcal{D}_Y$. The reason why this is meaningful and useful is a deep result of Saito, his Strictness Theorem, which says that the maps in (4.4) are injective.

EXERCISE 4.8. Show that if X is proper, f is the morphism to a point, and $M = \mathbf{Q}_X^H[n]$, then the assertion of the Strictness Theorem is equivalent to the degeneration of the Hodge-to-de Rham spectral sequence for X.

An important consequence of the Strictness Theorem is the compatibility of the graded de Rham complex with proper push-forward: for every $M \in D^b\big(\mathrm{MHM}(X)\big)$ and every $p \in \mathbf{Z}$, then is a canonical isomorphism

$$\mathrm{Gr}_p^F \mathrm{DR}_Y\big(f_*(M)\big) \simeq \mathbf{R}f_* \mathrm{Gr}_p^F \mathrm{DR}_X(M). \tag{4.5}$$

An easy case to understand is that of a closed immersion: suppose that $i\colon X \hookrightarrow Y$ is a closed immersion, with $\dim(Y) - \dim(X) = r$. Working locally, we may assume that we have global coordinates $x_1, \dots, x_n, y_1, \dots, y_r$ on Y such that X is defined by $(y_1, \dots, y_r)$. As we discussed in Section 2.5.2, if $\mathcal{M}$ is a left $\mathcal{D}_X$-module, then $i_+(\mathcal{M}) = \bigoplus_{\alpha \in \mathbf{Z}^r_{\geq 0}} \mathcal{M}\partial_y^\alpha$. If $(\mathcal{M}, F_\bullet\mathcal{M})$ underlies a mixed Hodge module M, then the corresponding filtration on the $\mathcal{D}_Y$-module $i_+(\mathcal{M})$ underlying $i_*(M)$ is given by

$$F_{p+r} i_+(\mathcal{M}) = \bigoplus_{|\alpha| \leq p} (F_{p-|\alpha|}\mathcal{M})\partial_y^\alpha$$

(the shift by r is due to the difference in the shifts when going between left and right $\mathcal{D}$-modules on X and Y). In this case we also have $W_q i_+(\mathcal{M}) = i_+(W_q\mathcal{M})$ for all q.

We have in this setting a version of Kashiwara's Equivalence Theorem: if $i\colon X \hookrightarrow Y$ is a closed immersion, then i_* gives an equivalence of categories between Hodge modules on X and Hodge modules on Y whose support is contained in X.

4.2.4. Nearby and vanishing cycles. We next describe the nearby/vanishing cycle functors for mixed Hodge modules. Suppose that X is a smooth, irreducible, complex algebraic variety, $f \in \mathcal{O}_X(X)$ is nonzero, and $\iota\colon X \hookrightarrow X \times \mathbf{A}^1$ is the graph embedding corresponding to f. Let $(\mathcal{M}, F_\bullet\mathcal{M})$ be the filtered $\mathcal{D}_X$-module underlying a Hodge module M. Note that, by assumption, $\mathcal{M}$ has a V-filtration with respect to f. Recall from the previous section that the filtration $F_\bullet\mathcal{M}$ induces a filtration on $\iota_+(\mathcal{M})$ given by

$$F_{p+1}\iota_+(\mathcal{M}) = \bigoplus_{0 \leq j \leq p} F_{p-j}\mathcal{M}\partial_t^j\delta.$$

We thus have induced filtrations on each $\mathrm{Gr}_V^\alpha(\iota_+\mathcal{M})$.

The filtered $\mathcal{D}_X$-modules underlying the nearby cycles, respectively vanishing cycles, of M are[2]

$$\psi_f(\mathcal{M}) := \bigoplus_{\alpha \in (0,1] \cap \mathbf{Q}} \mathrm{Gr}_V^\alpha(\iota_+\mathcal{M}), \quad \text{respectively,}$$

$$\varphi_f(\mathcal{M}) := \mathrm{Gr}_V^0(\iota_+\mathcal{M})(-1) \oplus \bigoplus_{\alpha \in (0,1) \cap \mathbf{Q}} \mathrm{Gr}_V^\alpha(\iota_+\mathcal{M}).$$

Note that the filtrations on $\psi_f(\mathcal{M})$ and $\varphi_f(\mathcal{M})$ are direct sum filtrations. We also write

$$\psi_{f,\alpha}(\mathcal{M}) = \mathrm{Gr}_V^\alpha(\iota_+\mathcal{M}) \quad \text{and} \quad \varphi_{f,1}(\mathcal{M}) = \mathrm{Gr}_V^0(\iota_+\mathcal{M})(-1).$$

[2]It is worth pointing out that there is a shift by (-1) when compared to the definition in terms of right $\mathcal{D}$-modules.

This is compatible with the topological picture via the Riemann-Hilbert correspondence: there are canonical isomorphisms

$$\mathrm{DR}_X^{\mathrm{an}}\big(\psi_f(\mathcal{M})\big) \simeq \psi_f^p(\mathcal{P}) \otimes_{\mathbf{Q}} \mathbf{C} \quad \text{and} \quad \mathrm{DR}_X^{\mathrm{an}}\big(\varphi_f(\mathcal{M})\big) \simeq \varphi_f^p(\mathcal{P}) \otimes_{\mathbf{Q}} \mathbf{C},$$

where ψ_f^p and φ_f^p are the (perverse) topological nearby and vanishing cycle functors associated to f. It turns out that for a suitable weight filtration, we get mixed Hodge modules $\psi_f(M)$ and $\varphi_f(M)$, whose underlying filtered $\mathcal{D}_X$-modules are the ones described above.

Example 4.9. It follows from Remark 3.36 that if f defines a smooth hypersurface Z in X, and $i\colon Z \hookrightarrow X$ is the inclusion, then

$$\psi_f\big(\mathbf{Q}_X^H[n]\big) \simeq i_*\big(\mathbf{Q}_Z^H[n-1]\big) \quad \text{and} \quad \varphi_f\big(\mathbf{Q}_X^H[n]\big) = 0.$$

Remark 4.10. Note that $\psi_{f,1}(M)$ and $\varphi_{f,1}(M)$ are direct summands of $\psi_f(M)$ and $\varphi_f(M)$, respectively, as Hodge modules. However, the summands $\mathrm{Gr}_V^\alpha(\iota_+\mathcal{M})$ for $\alpha \in (0,1)$ are not Hodge modules on their own since they don't have, in general, a $\mathbf{Q}$-structure (it is only the direct sum of the corresponding perverse sheaves that is defined over $\mathbf{Q}$).

Let us describe the weight filtration on $\psi_f(M)$ and $\varphi_f(M)$ when M is a pure Hodge module of weight w. Note that we have a morphism of filtered $\mathcal{D}_X$-modules

$$N\colon \psi_f(\mathcal{M}) \to \psi_f(\mathcal{M})(-1)$$

which is given by multiplication with $\partial_t t - \alpha$ on $\psi_{f,\alpha}(\mathcal{M})$ for $0 < \alpha \le 1$. Ignoring the Tate twist, this is a nilpotent endomorphism of the $\mathcal{D}_X$-module $\psi_f(\mathcal{M})$. In every abelian category, associated to a nilpotent endomorphism $N\colon A \to A$, there is a canonical filtration $W(N)_\bullet A$ (the *monodromy filtration*), which is a finite, exhaustive and increasing filtration, characterized by the following properties:

i) $N\big(W(N)_i A\big) \subseteq W(N)_{i-2}A$ for all $i \in \mathbf{Z}$, and

ii) $N^i\colon \mathrm{Gr}_i^{W(N)}(A) \to \mathrm{Gr}_{-i}^{W(N)}(A)$ is an isomorphism for all $i \ge 0$.

For example, if $N = 0$, then $\mathrm{Gr}_0^{W(N)}(A) = A$, while if $N^2 = 0$, then

$$W(N)_{-2}A = 0,\ W(N)_{-1}A = \mathrm{Im}(N),\ W(N)_0 A = \mathrm{Ker}(N),\ W(N)_1 A = A.$$

With this notation, part of the definition of a pure Hodge module of weight w is that $\mathrm{Gr}_i^{W(N)}\big(\psi_f(\mathcal{M})\big)$ is a pure Hodge module of weight $w - 1 + i$ for all $i \in \mathbf{Z}$. One can show that this condition, in conjunction with the compatibility between the Hodge and V-filtration, implies that $\mathrm{Gr}_i^{W(N)}\big(\varphi_{f,1}(\mathcal{M})\big)$ is a pure Hodge module of weight $w + i$ for every $i \in \mathbf{Z}$ (here N is given by multiplication with $\partial_t t$).

Remark 4.11. In order to handle polarizations, one has to use a result of Saito concerning the compatibility of duality and nearby/vanishing cycles, which is interesting in its own right. Suppose that $(\mathcal{M}, F_\bullet\mathcal{M})$ is a filtered $\mathcal{D}_X$-module for which the dual is defined (for example, one that underlies a Hodge module). In this case, it is shown in [**Sai89**, Theorem 1.6] that there are filtered isomorphisms

$$\mathbf{D}\big(\psi_{f,1}(\mathcal{M})\big)(1) \simeq \psi_{f,1}\big(\mathbf{D}(\mathcal{M})\big) \quad \text{and} \quad \mathbf{D}\big(\varphi_{f,1}(\mathcal{M})\big) \simeq \varphi_{f,1}\big(\mathbf{D}(\mathcal{M})\big), \tag{4.6}$$

as well as

$$\mathbf{D}\big(\psi_{f,\alpha}(\mathcal{M})\big)(1) \simeq \psi_{f,1-\alpha}\big(\mathbf{D}(\mathcal{M})\big) \quad \text{for all} \quad \alpha \in (0,1) \cap \mathbf{Q}. \tag{4.7}$$

Moreover, via these isomorphisms, the duals of

$$\varphi_{f,1}(\mathcal{M}) \xrightarrow{t\cdot} \psi_{f,1}(\mathcal{M})(-1) \quad \text{and} \quad \psi_{f,1}(\mathcal{M}) \xrightarrow{\partial_t\cdot} \varphi_{f,1}(\mathcal{M})$$

get identified with

$$\psi_{f,1}\big(\mathbf{D}(\mathcal{M})\big) \xrightarrow{\partial_t\cdot} \varphi_{f,1}\big(\mathbf{D}(\mathcal{M})\big) \quad \text{and} \quad \varphi_{f,1}\big(\mathbf{D}(\mathcal{M})\big) \xrightarrow{(-t)\cdot} \psi_{f,1}\big(\mathbf{D}(\mathcal{M})\big)(-1),$$

respectively (note that there are some sign changes with respect to *loc. cit.* due to the fact that we work with left, as opposed to right, $\mathcal{D}$-modules).

REMARK 4.12. As a reality check, note that if $(\mathcal{M}, F_\bullet\mathcal{M})$ underlies a pure, polarizable Hodge module of weight w, then the polarization gives an isomorphism $\mathbf{D}(\mathcal{M}) \simeq \mathcal{M}(w)$, hence the isomorphism (4.6) induces an isomorphism

$$\mathbf{D}\big(\psi_{f,1}(\mathcal{M})\big) \simeq \psi_{f,1}(\mathcal{M})(w-1)$$

such that $\mathbf{D}(N)$ corresponds to $-N$. Since $\mathbf{D}$ is a duality, it is straightforward to see that

$$\mathrm{Gr}_i^{W(N)}\big(\mathbf{D}(\psi_{f,1}(\mathcal{M})\big) \simeq \mathbf{D}\big(\mathrm{Gr}_{-i}^{W(N)}(\psi_{f,1}(\mathcal{M}))\big).$$

As we have seen, the left-hand side is isomorphic to $\mathrm{Gr}_i^{W(N)}\big(\psi_{f,1}(\mathcal{M})\big)(w-1)$, while the right-hand side is isomorphic via N^i to $\mathbf{D}\big(\mathrm{Gr}_i^{W(N)}(\psi_{f,1}(\mathcal{M}))\big)(-i)$. This is compatible, indeed, with the fact that $\mathrm{Gr}_i^{W(N)}\big(\psi_{f,1}(\mathcal{M})\big)$ is a pure Hodge module of weight $w-1+i$.

4.3. Pure Hodge modules: decomposition by singular support

The category of pure Hodge modules of a given weight turns out to have properties that are similar to that of the category of perverse sheaves. We now describe this. Let X be a smooth, irreducible, complex algebraic variety.

DEFINITION 4.13. If Z is an irreducible closed subset of X, then we say that a coherent $\mathcal{D}_X$-module $\mathcal{M}$ has *strict support* Z if every nonzero subobject or quotient object of $\mathcal{M}$ has support Z. We say that a Hodge module has strict support Z if its underlying $\mathcal{D}_X$-module satisfies this property.

REMARK 4.14. If $\mathcal{M}$ is a holonomic $\mathcal{D}_X$-module, then it is known that $\mathcal{M}$ has strict support Z if and only if it is isomorphic to the *intermediate extension* $j_{!*}(\mathcal{E})$, where $\mathcal{E}$ is a vector bundle with an integrable connection on a nonempty smooth open subset U of Z and $j\colon U \hookrightarrow X$ is the inclusion. We recall that this can be characterized as being the smallest $\mathcal{D}_X$-submodule of $\mathcal{H}^0\big(j_+(\mathcal{E})\big)$ whose restriction to U is $\mathcal{E}$. If $\mathcal{M}$ is regular holonomic, then it follows from the Riemann-Hilbert correspondence that $\mathcal{M}$ has strict support Z if and only if every nonzero subobject or quotient object of $\mathrm{DR}_X^{\mathrm{an}}(\mathcal{M})$ in the category of perverse sheaves on X^{an} has support Z. We note that the perverse sheaf corresponding to $j_{!*}(\mathcal{E})$ is the *intersection cohomology* perverse sheaf $\mathrm{IC}(\mathcal{L})$, where $\mathcal{L}$ is the local system corresponding to $\mathcal{E}$.

An important property of pure Hodge modules is that they have a canonical decomposition by strict support: every pure Hodge module M can be canonically written as

$$M = \bigoplus_{Z\subseteq X} M_Z,$$

where Z runs over the irreducible closed subsets of X and M_Z is a pure Hodge module with strict support Z. The key for this decomposition is the compatibility

between the Hodge filtration on the $\mathcal{D}$-module underlying $\mathcal{M}$ and the V-filtration with respect to regular functions on X which is imposed as part of the definition of Hodge modules (see [**Sch19**, Section 11] for a detailed discussion).

Another crucial result concerning (polarizable) pure Hodge modules is the structure theorem relating them to variations of Hodge structures. This says that the following holds for every irreducible closed subset Z of X:

i) For every polarizable variation of Hodge structures $\mathcal{V}$ of weight w on some smooth open subset U of Z (viewed as a pure Hodge module of weight $w+\dim(Z)$), there is a unique polarizable pure Hodge module M on X, having strict support on Z, and of weight $w+\dim(Z)$, that extends $\mathcal{V}$ (in other words, we have an open subset U' in X with $U'\cap Z = U$ such that if $i\colon U\hookrightarrow U'$ is the inclusion, then $M|_{U'}\simeq i_*(\mathcal{V})$).
ii) Every polarizable pure Hodge module M on X with strict support Z extends some polarizable variation of Hodge structures defined on some smooth open subset of Z.

A consequence of the above results is that the abelian category of pure Hodge modules of fixed weight is semisimple.

4.4. The 6-functor formalism

So far, we have only considered Hodge modules on smooth varieties. However, Kashiwara's Equivalence Theorem for Hodge modules allows us to talk about such modules on arbitrary complex algebraic varieties, which is convenient for the 6-functor formalism. More precisely, suppose that X is an arbitrary complex algebraic variety (reduced and separated, but possibly not irreducible). If X admits a closed embedding in a smooth, irreducible variety Y, then we take a (pure) Hodge module on X to be a (pure) Hodge module on Y with support on X. Kashiwara's Equivalence Theorem for Hodge modules shows that this notion is independent of the chosen embedding. Moreover, such an embedding always exists locally and by considering a suitable cover for X, one can make sense of this notion in general (however, we will not need this level of generality). It follows that for every algebraic variety X, we have an abelian category of Hodge modules and a corresponding derived category $D^b\big(\mathrm{MHM}(X)\big)$.

Remark 4.15. Even if X is a singular variety, we have an exact functor $\mathrm{Gr}_p^F\mathrm{DR}_X\colon D^b\big(\mathrm{MHM}(X)\big)\to D^b_{\mathrm{coh}}(X)$. If X admits a closed immersion in a smooth, irreducible variety Y, this follows from the fact that if $(\mathcal{M},F_\bullet\mathcal{M})$ is the filtered $\mathcal{D}_X$-module underlying a Hodge module supported on X, then the complex $\mathrm{Gr}_p^F\mathrm{DR}_Y(\mathcal{M})$ is a complex of $\mathcal{O}_X$-modules: this follows from the assertion in Remark 4.1.

These derived categories satisfy a 6-functor formalism: given any morphism of algebraic varieties $f\colon X\to Y$, we have exact functors

$$f_*, f_!\colon D^b\big(\mathrm{MHM}(X)\big)\to D^b\big(\mathrm{MHM}(Y)\big)\quad\text{and}$$

$$f^*, f^!\colon D^b\big(\mathrm{MHM}(Y)\big)\to D^b\big(\mathrm{MHM}(X)\big),$$

compatible with composition of morphisms and with Tate twists, as well as an anti-equivalence of categories

$$\mathbf{D}=\mathbf{D}_X\colon D^b\big(\mathrm{MHM}(X)\big)\to D^b\big(\mathrm{MHM}(X)\big)$$

such that $\mathbf{D} \circ \mathbf{D} \simeq \mathrm{Id}$ (so $\mathbf{D}$ is a *duality*). These functors satisfy several compatibilities for every morphism $f\colon X \to Y$:

i) The following pairs of functors are adjoint:

$$(f^*, f_*) \quad \text{and} \quad (f_!, f^!), \quad \text{and} \tag{4.8}$$

ii) There are canonical isomorphisms of functors

$$\mathbf{D}_Y \circ f_* \simeq f_! \circ \mathrm{D}_X \quad \text{and} \quad f^* \circ \mathbf{D}_Y \simeq \mathbf{D}_X \circ f^!. \tag{4.9}$$

We have already discussed duality for smooth varieties and this induces the duality functor in the general case. We next discuss briefly some of the other functors.

4.4.1. The functors f_* and $f_!$. If $f\colon X \to Y$ is an arbitrary morphism, it follows from a theorem of Nagata and Deligne that we can factor f as $X \overset{j}{\hookrightarrow} W \xrightarrow{g} Y$, with j an open immersion and g proper. Since we have isomorphisms of functors

$$f_* \simeq g_* \circ j_* \quad \text{and} \quad f_! \simeq g_! \circ j_!,$$

it follows that, in order to describe f_* and $f_!$, it is enough to understand the cases when f is proper and when it is an open immersion.

If f is a proper morphism, then $f_! = f_*$. We have discussed this functor when X and Y are smooth and the general case can be easily reduced to this setting.

On the other hand, the functor $j_*\colon D^b\big(\mathrm{MHM}(U)\big) \to D^b\big(\mathrm{MHM}(Y)\big)$ for an open immersion $j\colon U \hookrightarrow Y$ is one of the most subtle ones in the theory of Hodge modules. At the level of $\mathcal{D}$-modules, this coincides with j_+, which itself coincides with (derived) push-forward for $\mathcal{O}$-modules. On the other hand, the description of the filtration is quite subtle. In the next chapter, we will discuss in more detail the case of $j_*\mathbf{Q}^H_U[n]$, when Y is smooth, irreducible, of dimension n. We will explain shortly that one can say more when U is the complement of a hypersurface defined by a nonzero $f \in \mathcal{O}_X(X)$. However, we will see later that even in this case, the Hodge filtration on $\mathcal{O}_X[1/f]$ encodes subtle information about the singularities of the hypersurface defined by f.

The functor $j_!$ can then be understood via $j_! \simeq \mathbf{D}_Y \circ j_* \circ \mathbf{D}_U$ by (4.9). For example, if Y is smooth, irreducible, of dimension n, then $\mathbf{D}_U\big(\mathbf{Q}^H_U[n]\big) \simeq \mathbf{Q}^H_U[n](n)$, and thus

$$j_!\big(\mathbf{Q}^H_U[n]\big) \simeq \mathbf{D}_Y\big(j_*\mathbf{Q}^H_U[n]\big)(-n).$$

4.4.2. The functors f^* and $f^!$. In order to describe f^* and $f^!$ for a morphism $f\colon X \to Y$, we may assume that Y is smooth. First, we factor f as

$$X \overset{i}{\hookrightarrow} X \times Y \xrightarrow{p} Y,$$

with i the graph embedding and p the projection. Assuming that X admits a closed embedding $X \hookrightarrow W$, with W smooth, we then factor p as $X \times Y \overset{i'}{\hookrightarrow} W \times Y \xrightarrow{q} Y$. Since we have the isomorphisms of functors

$$f^* \simeq (i' \circ i)^* \circ q^* \quad \text{and} \quad f^! \simeq (i' \circ i)^! \circ q^!,$$

we see that it is enough to understand f^* and $f^!$ when f is a closed immersion and when it is a smooth morphism.

Let's consider first the case when $f\colon X \to Y$ is a smooth morphism of relative dimension r between smooth, irreducible varieties. In this case, if $(\mathcal{M}, F_\bullet\mathcal{M})$ is the filtered $\mathcal{D}_Y$-module underlying the Hodge module M on Y, then $f^*(M)[r]$ is a Hodge module with underlying filtered $\mathcal{D}_X$-module $\big(f^*(\mathcal{M}), f^*(F_\bullet\mathcal{M})\big)$. Moreover,

the weight filtration is determined by $W_{p+r}f^*(\mathcal{M}) = f^*(W_p\mathcal{M})$ for all $p \in \mathbf{Z}$. The functor $f^!$ is then described via (4.9) by $f^! \simeq \mathbf{D}_X \circ f^* \circ \mathbf{D}_Y$. For example, we see that in this case

$$f^*(\mathbf{Q}_Y^H) \simeq \mathbf{Q}_X^H \quad \text{and} \quad f^!(\mathbf{Q}_Y^H) \simeq \mathbf{Q}_X^H[2r](r). \tag{4.10}$$

In general, if $(\mathcal{M}, F_\bullet\mathcal{M})$ is the filtered $\mathcal{D}_Y$-module underlying the Hodge module M, then $f^!(M)[-r]$ is a Hodge module with underlying $\mathcal{D}_X$-module $f^*(\mathcal{M})$ and filtrations given by

$$F_pf^*(\mathcal{M}) = f^*(F_{p-r}\mathcal{M}) \quad \text{and} \quad W_{p-r}f^*(\mathcal{M}) = f^*(W_p\mathcal{M}) \quad \text{for all} \quad p \in \mathbf{Z}.$$

Let us consider now a closed immersion $i\colon Z \hookrightarrow X$, with X smooth and irreducible and let $U = X \smallsetminus Z$, with $j\colon U \hookrightarrow X$ being the corresponding open immersion. In this case, for every $M \in D^b\big(\mathrm{MHM}(X)\big)$, we have an exact triangle

$$i_*i^!(M) = i_!i^!(M) \to M \to j_*j^*(M) \xrightarrow{+1}, \tag{4.11}$$

in which the two maps are induced by the adjunctions (4.8). Applying this for $\mathbf{D}_X(M)$ and taking the dual, gives another exact triangle

$$j_!j^*(M) = j_!j^!(M) \to M \to i_*i^*(M) \xrightarrow{+1}. \tag{4.12}$$

It follows from (4.11) that given a Hodge module M on X, understanding $i_*i^!(M)$ is essentially equivalent to understanding $j_*j^*(M)$. We also see from this exact triangle that if $\mathcal{M}$ is the $\mathcal{D}_X$-module underlying M, then the $\mathcal{D}_X$-module underlying $\mathcal{H}^q\big(i_*i^!(M)\big)$ is the local cohomology sheaf $\mathcal{H}^q_Z(\mathcal{M})$.

Remark 4.16. In order to justify the upcoming definition of $i_*i^!(M)$ and $i_*i^*(M)$ when Z is the hypersurface in X defined by a nonzero $f \in \mathcal{O}_X(X)$ and M is a Hodge module on X, suppose first that Z is smooth, defined by t. Let $i\colon Z \hookrightarrow X$ be the inclusion and $\mathcal{M}$ a holonomic $\mathcal{D}_X$-module that has a V-filtration with respect to t. In this case, the standard 6-functor formalism for holonomic $\mathcal{D}_X$-modules is such that $i^!(\mathcal{M}) = \mathbf{L}i^*[-1]$, cf. (2.9). Moreover, we have a canonical isomorphism $i_+\big(i^!(\mathcal{M})\big) \simeq \mathbf{R}\Gamma_Z(\mathcal{M})$, hence an exact triangle

$$i_+\big(i^!(\mathcal{M})\big) \to \mathcal{M} \to j_+\big(j^*(\mathcal{M})\big) \xrightarrow{+1},$$

where $j\colon U = X \smallsetminus Z \hookrightarrow X$ is the inclusion. Furthermore, as mentioned in Remark 3.32, we have an isomorphism between $i^!(\mathcal{M})$ and the complex

$$\mathrm{Gr}_V^0(\mathcal{M}) \xrightarrow{t\cdot} \mathrm{Gr}_V^1(\mathcal{M}),$$

placed in cohomological degrees 0 and 1. Saito's idea is to use this identification in order to endow $i^!(\mathcal{M})$ with a Hodge filtration when $\mathcal{M}$ underlies a Hodge module on X.

When Z is not necessarily smooth, we consider the graph embedding $\iota\colon X \hookrightarrow X \times \mathbf{A}^1$ corresponding to f and let $i_0\colon X \hookrightarrow X \times \mathbf{A}^1$ be given by $i_0(x) = (x, 0)$, so its image is the hypersurface H defined by t. Applying ι_+ to the exact triangle

$$\mathbf{R}\Gamma_Z(\mathcal{M}) \to \mathcal{M} \to j_+j^*(\mathcal{M}) \xrightarrow{+1}$$

and using the fact that $\iota^{-1}(H) = Z$, we see easily that we have an isomorphism

$$\iota_+\mathbf{R}\Gamma_Z(\mathcal{M}) \simeq \mathbf{R}\Gamma_H\big(\iota_+(\mathcal{M})\big).$$

If $\mathcal{M}$ has a V-filtration with respect to f, we thus conclude that

$$\iota_+\mathbf{R}\Gamma_Z(\mathcal{M}) \simeq (i_0)_+\big(\mathrm{Gr}_V^0(\iota_+\mathcal{M}) \xrightarrow{t\cdot} \mathrm{Gr}_V^1(\iota_+\mathcal{M})\big),$$

with the complex being placed in cohomological degrees 0 and 1. Finally, we note that if $\mathcal{M}_1$ and $\mathcal{M}_2$ are $\mathcal{D}_X$-modules, with $\mathcal{M}_1$ supported on Z, such that $\iota_+(\mathcal{M}_1) \simeq (i_0)_+(\mathcal{M}_2)$, then $\mathcal{M}_1 \simeq \mathcal{M}_2$: indeed, we have

$$\mathcal{M}_2 \simeq \big\{u \in \iota_+(\mathcal{M}_1) \mid tu = 0\big\} \simeq \mathcal{M}_1,$$

where the second isomorphism maps $w = \sum_{j\geq 0} w_j\partial_t^j\delta$ to w_0 and its inverse maps w_0 to $\sum_{j\geq 0} \frac{f^j}{j!}w_0\partial_t^j\delta \in \iota_+(\mathcal{M}_1)$ (see Example 3.35). We thus conclude that $\mathbf{R}\Gamma_Z(\mathcal{M})$ is represented by the complex

$$\mathrm{Gr}_V^0(\iota_+\mathcal{M}) \xrightarrow{t\cdot} \mathrm{Gr}_V^1(\iota_+\mathcal{M}).$$

We can now give the definition of $i_*i^!$ and i_*i^* in the setting of Hodge modules, when $i\colon Z \hookrightarrow X$ is a closed immersion, with X smooth and irreducible, and Z defined by a nonzero $f \in \mathcal{O}_X(X)$. In this case, by definition $i_*i^!(M)$ is represented by the complex of Hodge modules

$$0 \to \varphi_{f,1}(M) \xrightarrow{t\cdot} \psi_{f,1}(M)(-1) \to 0, \tag{4.13}$$

placed in cohomological degrees 0 and 1. Applying this for $\mathbf{D}(M)$, taking the dual, and using Remark 4.11, it follows that $i_*i^*(M)$ is represented by the complex of Hodge modules

$$0 \to \psi_{f,1}(M) \xrightarrow{\partial_t\cdot} \varphi_{f,1}(M) \to 0, \tag{4.14}$$

placed in cohomological degrees -1 and 0.

Remark 4.17. It is a consequence of Remark 3.36 that if Y is, in fact, a smooth hypersurface defined by f, and $\mathcal{M}$ is the filtered $\mathcal{D}_X$-module underlying M, then the complex of filtered $\mathcal{D}_Y$-modules underlying $i^!(M)$ is

$$0 \to \mathrm{Gr}_V^0(\mathcal{M})(-1) \xrightarrow{f\cdot} \mathrm{Gr}_V^1(\mathcal{M})(-1) \to 0,$$

where the V-filtration is on $\mathcal{M}$ with respect to f.

Example 4.18. If Z is smooth, then it follows from Example 4.9 that

$$i^*\big(\mathbf{Q}_X^H[n]\big) \simeq \mathbf{Q}_Z^H[n]. \tag{4.15}$$

Remark 4.19. By combining (4.15) with the description of pull-back for smooth morphisms, we see that for every morphism $f\colon Y \to X$ of smooth varieties, we have a canonical isomorphism

$$f^*\big(\mathbf{Q}_X^H\big) \simeq \mathbf{Q}_Y^H. \tag{4.16}$$

Motivated by this, for every algebraic variety X, possibly singular, we put

$$\mathbf{Q}_X^H = a_X^*(\mathbf{Q}_{\mathrm{pt}}^H) \in D^b\big(\mathrm{MHM}(X)\big),$$

where a_X is the morphism from X to a point. As we have seen, this is compatible with our previous definition in the case when X is smooth. In general, compatibility of pull-back with composition of morphisms implies that the isomorphism (4.16) holds for every morphism of algebraic varieties $f\colon X \to Y$.

In particular, if Z is a closed subvariety of the smooth, irreducible, n-dimensional variety X, and $i\colon Z \hookrightarrow X$ is the inclusion, since $\mathbf{D}_X\big(\mathbf{Q}_X^H[n]\big) \simeq \mathbf{Q}_X^H[n](n)$, it follows that

$$\mathbf{D}(\mathbf{Q}_Z^H) \simeq i^!\mathbf{Q}_X^H[2n](n). \tag{4.17}$$

It is elementary to see that if $r = \operatorname{codim}_X(Z)$, then $\mathcal{H}^i_Z(\mathcal{O}_X) = 0$ for $i < r$ and for $i > n$. We conclude from (4.17) that $\mathbf{D}(\mathbf{Q}_Z^H)$ has nonzero entries only in cohomological degrees i, with $r \le i + n \le n$, and thus $\mathcal{H}^i(\mathbf{Q}_Z^H) = 0$, unless $0 \le i \le \dim(Z)$. Of course, the same holds regardless of whether Z can be embedded in X as above. Note also that $\mathcal{H}^i(\mathbf{Q}_Z^H) \neq 0$ for $i = \dim(Z)$, since this is the case on the smooth locus of an irreducible component of Z of maximal dimension.

4.4.3. Base-change. Suppose now that we have the following Cartesian diagram

$$\begin{array}{ccc} W & \xrightarrow{v} & X \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{u} & Y \end{array}$$

in the category of complex algebraic varieties. In this case we have natural isomorphisms of functors

$$u^! \circ f_* \simeq g_* \circ v^! \quad \text{and} \quad u^* \circ f_! \simeq g_! \circ v^*, \tag{4.18}$$

see [**Sai90**, 4.4.3].

4.4.4. The intersection complex Hodge module. If Z is any algebraic variety, then it is shown in [**Sai90**, Section 4.5] that $\operatorname{Gr}_i^W\big(\mathcal{H}^j(\mathbf{Q}_Z^H)\big) = 0$ for all $i > j$ (in the terminology introduced in Section 4.5 below, this says that $\mathbf{Q}_Z^H$ has weight ≤ 0). Suppose now that Z has pure dimension d, so $\mathcal{H}^i(\mathbf{Q}_X^H) = 0$ for $i > d$, as we have seen in Remark 4.19. We thus have a canonical morphism in $D^b\big(\mathrm{MHM}(Z)\big)$

$$\tau_Z\colon \mathbf{Q}_Z^H[d] \to \mathcal{H}^d(\mathbf{Q}_Z^H) \to \operatorname{Gr}_d^W\mathcal{H}^d(\mathbf{Q}_Z^H) =: \mathrm{IC}_Z\mathbf{Q}^H.$$

By definition, $\mathrm{IC}_Z\mathbf{Q}^H$ is a pure Hodge module of weight d, called the *intersection complex Hodge module* of Z. This can be characterized as the unique Hodge module whose restriction to $U = Z \smallsetminus Z_{\mathrm{sing}}$ is $\mathbf{Q}_U^H[d]$ and which has no subobject or quotient supported on Z_{sing}.

Suppose now that Z is a closed irreducible subset of the smooth, irreducible variety X. In the terminology introduced in Section 4.3, $\mathrm{IC}_Z\mathbf{Q}^H$ has strict support Z. If $j\colon U \hookrightarrow X$ is the inclusion, then the $\mathcal{D}_X$-module underlying $\mathrm{IC}_Z\mathbf{Q}^H$ is the intermediate extension $j_{!*}(\mathcal{O}_U)$, and the associated $\mathbf{Q}$-perverse sheaf is the intersection complex $\mathrm{IC}_Z\mathbf{Q}$. This is a simple perverse sheaf, hence $\mathrm{IC}_Z\mathbf{Q}^H$ is a simple Hodge module, and $\mathbf{Q} = \operatorname{End}(\mathrm{IC}_Z\mathbf{Q}^H)$. If Z has irreducible components $Z_1, \dots, Z_r$, then

$$\mathrm{IC}_Z\mathbf{Q}^H \simeq \bigoplus_{i=1}^r \mathrm{IC}_{Z_i}\mathbf{Q}^H.$$

By taking duals and using (4.17), we see that if Z is a closed subvariety of the smooth, irreducible, n-dimensional variety X, with $r = n - d$, then $\operatorname{Gr}_i^W\mathcal{H}^r_Z(\mathcal{O}_X) = 0$ for $i < n + r$ and

$$\operatorname{Gr}_{n+r}^W\mathcal{H}^r_Z(\mathcal{O}_X) \simeq \mathrm{IC}_Z\mathbf{Q}^H(-r).$$

The underlying $\mathcal{D}_X$-module is the *intersection cohomology $\mathcal{D}$-module* introduced by Brylinski and Kashiwara in [**BK81**]; when Z is irreducible, it can be characterized as the unique simple $\mathcal{D}_X$-submodule of $\mathcal{H}^r_Z(\mathcal{O}_X)$.

4.5. The Decomposition Theorem and Saito's Vanishing Theorem

While we will not need these results, we can't end this overview without mentioning two fundamental results of the theory of Hodge modules. In order to state the Decomposition Theorem, it is convenient to introduce first some terminology. Let X be a complex algebraic variety. As usual, all Hodge modules in this section are polarizable.

Definition 4.20. If M is an object in $D^b\big(\mathrm{MHM}(X)\big)$ and $q \in \mathbf{Z}$, we say that M is of weight $\leq w$ (respectively, $\geq w$) if $\mathrm{Gr}^W_j \mathcal{H}^i(M) = 0$ for all $i \in \mathbf{Z}$ and all $j > i + w$ (respectively, $j < i + w$). We say that M is *pure, of weight* w, if it is both of weight $\leq w$ and $\geq w$, that is, $\mathcal{H}^i(M)$ is pure, of weight $i + w$ for all $i \in \mathbf{Z}$ (note that this is compatible with our previous definition in the case when M is a Hodge module).

An important result of Saito (see [**Sai90**, (4.5.2)]) says that for every morphism $f\colon X \to Y$, if $M \in D^b\big(\mathrm{MHM}(X)\big)$ has weight $\leq w$ (respectively, $\geq w$), then $f_!(\mathcal{M})$ is of weight $\leq w$ (respectively, $f_*(\mathcal{M})$ is of weight $\geq w$). In particular, if f is proper, so $f_! = f_*$, then for every $M \in D^b\big(\mathrm{MHM}(X)\big)$ which is pure, of weight w, we have that $f_*(M)$ is pure, of weight w.

We also have the following decomposition result for pure objects in $D^b\big(\mathrm{MHM}(X)\big)$ (see [**Sai90**, (4.5.4)]): if $M \in D^b\big(\mathrm{MHM}(X)\big)$ is pure, of weight w, then there is a (non-canonical) isomorphism

$$M \simeq \bigoplus_{i \in \mathbf{Z}} \mathcal{H}^i(M)[-i].$$

By combining the above results, we obtain Saito's version of the Decomposition Theorem: if $f\colon X \to Y$ is a proper morphism of algebraic varieties and if $M \in D^b\big(\mathrm{MHM}(X)\big)$ is pure, of weight w, then each $\mathcal{H}^i\big(f_*(M)\big)$ is a pure Hodge module, of weight $i + w$, and there is an isomorphism

$$f_*(\mathcal{M}) \simeq \bigoplus_{i \in \mathbf{Z}} \mathcal{H}^i\big(f_*(M)\big)[-i].$$

Another important result is Saito's Vanishing Theorem, extending the classical vanishing result of Kodaira-Akizuki-Nakano (see [**Sai90**, Section 2.g]). This says that if X is a projective complex algebraic variety and $\mathcal{L}$ is an ample line bundle on X, then for every Hodge module M on X and every $p \in \mathbf{Z}$, we have

$$H^i\big(X, \mathrm{Gr}^F_p \mathrm{DR}_X(M) \otimes \mathcal{L}\big) = 0 \quad \text{for} \quad i > 0 \quad \text{and}$$

$$H^i\big(X, \mathrm{Gr}^F_p \mathrm{DR}_X(M) \otimes \mathcal{L}^{-1}\big) = 0 \quad \text{for} \quad i < 0.$$

Note that the second vanishing assertion follows by applying the first one to $\mathbf{D}(M)$ and using Grothendieck-Serre duality and the isomorphism (4.2).

Example 4.21. If we take X to be smooth, irreducible, of dimension n, and $M = \mathbf{Q}^H_X[n]$, then it follows from Example 4.5 that the above vanishing result says that

$$H^p(X, \Omega^q_X \otimes L) = 0 \quad \text{if} \quad p + q > n,$$

which is the statement of the Kodaira-Akizuki-Nakano Vanishing Theorem.

CHAPTER 5

The Hodge filtration on local cohomology

Our goal in this chapter is to discuss the Hodge filtration on local cohomology. In order to fix the ideas, we first discuss the case of hypersurfaces. We begin by explicitly describing the Hodge filtration on $\mathcal{O}_X(*Z)$, when Z is a hypersurface in X defined by f, in terms of the V-filtration of $\mathcal{O}_X$ with respect to f. As a consequence, we obtain an explicit formula for the filtration in the simple normal crossing case and use it to show that for every Z, the Hodge filtration is contained in the pole order filtration. The difference between these two filtrations is measured by the Hodge ideals introduced and studied in [MP19] and which we discuss next. We show that how far the Hodge and the pole filtrations agree is governed by the minimal exponent.

We then turn to the discussion of the Hodge filtration on the local cohomology on $\mathcal{H}^q_Z(\mathcal{O}_X)$, for any closed subset Z of the smooth variety X. We compute the first piece of the Hodge filtration in terms of a log resolution of (X, Z) and give a complex that computes each graded piece. We focus on describing the generation level of the Hodge filtration on the top nonzero local cohomology. As an application, we obtain a characterization of local cohomological dimension in terms of the vanishing of the higher direct image of suitable sheaves of differentials with log poles on the log resolution of (X, Z).

In the thirs section, we return to the codimension 1 setting and discuss some basic properties of Hodge ideals from [MP18]. We prove the Restriction Theorem, which extends a basic result for multiplier ideals. We also discuss the case of the restriction to a general hypersurface, as well as some consequences concerning the behavior in families and a Subadditivity Theorem.

Finally, in the last section we use these results on Hodge ideals to deduce some basic properties of the minimal exponent, following [MP20]. However, unlike in *loc. cit.*, we do not use Hodge ideals for $\mathbf{Q}$-divisors (which we do not treat in these notes), but use instead a Thom-Sebastiani type result for the minimal exponent from [MSS20] that allows us to only use Hodge ideals for reduced divisors.

5.1. The Hodge filtration on $\mathcal{O}_X(*Z)$

Let X be a smooth, irreducible, n-dimensional complex algebraic variety and Z a hypersurface in X. Consider the open immersion $j\colon U = X \smallsetminus Z \hookrightarrow X$ and the closed immersion $i\colon Z \hookrightarrow X$. Note that in this case $\mathcal{H}^q_Z(\mathcal{O}_X) = 0$ for all $q \neq 1$, and we have a short exact sequence of filtered $\mathcal{D}_X$-modules

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(*Z) \to \mathcal{H}^1_Z(\mathcal{O}_X) \to 0 \tag{5.1}$$

underlying the short exact sequence of Hodge modules

$$0 \to \mathbf{Q}^H_X[n] \to j_*j^*\big(\mathbf{Q}^H_X[n]\big) \to i_*i^!\big(\mathbf{Q}^H_X[n]\big)[1] \to 0$$

(see the exact triangle (4.11)).

REMARK 5.1. If $f \in \mathcal{O}_X(X)$ is nonzero and $\iota\colon X \hookrightarrow X \times \mathbf{A}^1$ is the corresponding closed immersion, then it follows from the definition of the Hodge filtration on $\iota_+(\mathcal{M})$ that if $\mathcal{M}$ underlies a Hodge module on X and $F_p\mathcal{M} = 0$ for all $p < 0$, then $F_p\mathrm{Gr}_V^1(\iota_+\mathcal{M}) = 0$ for all $p < -1$. Therefore, it follows from the description of $i_*i^!\big(\mathbf{Q}_X^H[n]\big)$ via the complex (4.13) that for Z and X as above, we have $F_p\mathcal{H}^1_Z(\mathcal{O}_X) = 0$ for all $p < 0$. On the other hand, since the morphisms in (5.1) are strict, it follows that for all $p \geq 0$, we have

$$F_p\mathcal{H}^1_Z(\mathcal{O}_X) = F_p\mathcal{O}_X(*Z)/\mathcal{O}_X.$$

The above remark shows that understanding the Hodge filtration on $\mathcal{H}^1_Z(\mathcal{O}_X)$ is equivalent to understanding the Hodge filtration on $\mathcal{O}_X(*Z)$.

In order to begin understanding the filtration on $\mathcal{H}^1_Z(\mathcal{O}_X)$, we begin with the following description via the V-filtration. In fact, we do this, more generally, for every $\mathcal{D}_X$-module $\mathcal{M}$ underlying a Hodge module.

Suppose that Z is defined in X by $f \in \mathcal{O}_X(X)$ and let $\iota\colon X \hookrightarrow X \times \mathbf{A}^1$ be the corresponding graph embedding. We define the map $\sigma\colon \mathcal{M}_f[s]f^s \to \mathcal{M}_f$ to be given by evaluating at $s = -1$:

$$\sigma\big(P(s)uf^s\big) = P(-1)\tfrac{1}{f}u \quad \text{for every} \quad P \in \mathbf{C}[s], u \in \mathcal{M}.$$

Via the isomorphism in Proposition 3.38, we identify σ with a map $\iota_+\mathcal{M}_f \to \mathcal{M}_f$ and denote by $\overline{\sigma}$ the composition of the canonical projection $\mathcal{M}_f \to \mathcal{H}^1_Z(\mathcal{M})$ with σ.

REMARK 5.2. It is useful to also write the map σ more explicitly, in terms of the decomposition $\iota_+(\mathcal{M}_f) = \bigoplus_{j\geq 0}\mathcal{M}_f\partial_t^j\delta$. Since for every $u \in \mathcal{M}_f$, we can write using Lemma 3.22

$$u\partial_t^q\delta = \tfrac{1}{f^q}(\partial_t^q t^q)u\delta = (-1)^q\tfrac{1}{f^q}\prod_{j=0}^{q-1}(s-j)u\delta,$$

it follows that for $u_0, \ldots, u_m \in \mathcal{M}_f$, we have

$$\sigma\left(\sum_{q=0}^m u_q\partial_t^q\delta\right) = \sum_{q=0}^m \tfrac{q!u_q}{f^{q+1}}.$$

PROPOSITION 5.3. With the above notation, the isomorphism of filtered $\mathcal{D}_X$-modules underlying Hodge modules

$$\mathrm{Coker}\big(\mathrm{Gr}_V^0(\iota_+\mathcal{M}) \xrightarrow{t} \mathrm{Gr}_V^1(\iota_+\mathcal{M})\big)(-1) \simeq \mathcal{H}^1_Z(\mathcal{M})$$

is given by $\overline{u} \mapsto \overline{\sigma}(u)$.

PROOF. We need the explicit description of the isomorphisms that appear in Remark 4.16. Let $i_0\colon X \hookrightarrow X \times \mathbf{A}^1$ be the closed immersion given by $i_0(x) = (x, 0)$. Recall that if we put

$$C = \mathrm{Coker}\big(\mathrm{Gr}_V^0(\iota_+\mathcal{M}) \xrightarrow{t\cdot} \mathrm{Gr}_V^1(\iota_+\mathcal{M})\big),$$

then we have a chain of isomorphisms

(5.2)

$$(i_0)_+C \xrightarrow{\gamma_1} (i_0)_+(\iota_+\mathcal{M}/t\cdot\iota_+\mathcal{M}) \xrightarrow{\gamma_2} \mathrm{Coker}\big(\iota_+\mathcal{M} \to (\iota_+\mathcal{M})_t\big) \xrightarrow{\gamma_3} \iota_+\mathrm{Coker}(\mathcal{M} \to \mathcal{M}_f),$$

where γ_1 is obtained by applying $(i_0)_+$ to the isomorphism provided by Remark 3.32, γ_2 is the isomorphism described in Remark 2.28, and γ_3 is induced by the exactness of ι_+ and the isomorphism

$$\gamma_3' \colon (\iota_+\mathcal{M})_t \simeq \iota_+(\mathcal{M}_f),$$

which is induced by the fact that $t\circ\iota = f$. Let us put $\gamma = \gamma_3\circ\gamma_2\circ\gamma_1$. As explained in Remark 4.16, this induces the isomorphism

$$\widetilde{\gamma}\colon C \to \mathrm{Coker}(\mathcal{M}\to\mathcal{M}_f)$$

given by the composition

$$C \hookrightarrow (i_0)_+C \xrightarrow{\gamma} \iota_+\mathrm{Coker}(\mathcal{M}\to\mathcal{M}_f) \xrightarrow{p} \mathrm{Coker}(\mathcal{M}\to\mathcal{M}_f),$$

where we identify $(i_0)_+C = \bigoplus_{j\geq 0} C\partial_t^j$ and $p\left(\sum_j v_j\partial_t^j\delta\right) = v_0$. Note that p is an additive map and if $Q(s)\in\mathbf{C}[s]$ and $w\in\mathrm{Coker}(\mathcal{M}\to\mathcal{M}_f)$, then $p\big(Q(s)w\delta\big) = Q(0)w$. Indeed, since $s = -\partial_t t$, it follows that

$$s^m\cdot\iota_+\mathrm{Coker}(\mathcal{M}\to\mathcal{M}_f) \subseteq \bigoplus_{j\geq 1}\mathrm{Coker}(\mathcal{M}\to\mathcal{M}_f)\partial_t^j\delta \quad \text{for all} \quad m\geq 1.$$

We next give the explicit description of γ_3'. Given an element $a\in\mathcal{M}$ and $j,m\in\mathbf{Z}_{\geq 0}$, in order to compute $\gamma_3'\big(\frac{a\partial_t^j\delta}{t^m}\big)$, we write in $\iota_+(\mathcal{M}_f)$, using Lemma 3.22

$$t^{-m}a\partial_t^j\delta = t^{-m-j}t^j\partial_t^j a\delta = (-1)^j t^{-m-j}(s+1)\cdots(s+j)a\delta$$

$$= (-1)^j(s+1-m-j)\cdots(s-m)t^{-m-j}a\delta = (-1)^j(s+1-m-j)\cdots(s-m)\tfrac{a}{f^{m+j}}\delta,$$

hence

$$\gamma_3'\big(\tfrac{a\partial_t^j\delta}{t^m}\big) = (-1)^j(s+1-m-j)\cdots(s-m)\big[\tfrac{a}{f^{m+j}}\big]\delta,$$

where we denote by $[v]$ the class of $v\in\mathcal{M}_f$ in $\mathrm{Coker}(\mathcal{M}\to\mathcal{M}_f)$.

Suppose now that $\overline{u}$ is the class of $u = \sum_{j=0}^N u_j\partial_t^j\delta\in V^1(\iota_+\mathcal{M})$ in C and let us compute $\gamma(\overline{u})$. As explained in Remark 3.32, we have

$$\gamma_1(u) = \widetilde{u},$$

the class of u in $\iota_+\mathcal{M}/t\cdot\iota_+\mathcal{M}$. Furthermore, it follows from Remark 2.28 that this is mapped by γ_2 to the class of $\frac{u}{t}$ and the above formula for γ_3' (with $m=1$) implies that

$$\gamma(\overline{u}) = \sum_{j=0}^N (-1)^j(s-j)\cdots(s-1)\big[\tfrac{u_j}{f^{j+1}}\big]\delta.$$

We thus conclude that

$$\widetilde{\gamma}(\overline{u}) = p\big(\gamma(\overline{u})\big) = \sum_{j=0}^N\big[\tfrac{j!u_j}{f^{j+1}}\big],$$

and the assertion in the proposition follows from the description of σ in Remark 5.2. □

From now on we focus on the case when $\mathcal{M} = \mathcal{O}_X$. We note that since $\mathcal{O}_X(*Z)$ only depends on the reduced structure of Z, from now on we assume that Z is reduced.

As a first consequence of Proposition 5.3, we obtain the following description of the Hodge filtration on $\mathcal{O}_X(*Z)$, when Z has simple normal crossings.

PROPOSITION 5.4. If the reduced hypersurface Z in X has simple normal crossings, then

$$F_p\mathcal{O}_X(*Z) = F_p\mathcal{D}_X \cdot \mathcal{O}_X(Z) \quad \text{for all} \quad p \geq 0.$$

PROOF. Working locally, we may assume that we have algebraic coordinates $x_1, \ldots, x_n$ on X such that Z is defined by $f = x_1 \cdots x_r$. The case $r = 0$ is trivial to check, hence from now on we assume that $r \geq 1$. In this case, it follows from Example 3.37 that

$$V^1\iota_+(\mathcal{O}_X) = \sum_{j\geq 0} \mathcal{D}_X \cdot f^j\partial_t^j\delta = \mathcal{D}_X[s] \cdot \delta = \mathcal{D}_X \cdot \delta, \tag{5.3}$$

where the second equality follows from the fact that $\partial_t^j t^j \in \mathbf{C}[s]$ and the last equality follows from the fact that

$$x_i\partial_{x_i} \cdot \delta = s\delta \quad \text{for} \quad 1 \leq i \leq r.$$

Our goal is to deduce from (5.3) the formula in the proposition.

Since $(x_1, \ldots, x_n)$ gives an étale morphism $X \to \mathbf{A}^n$, it is easy to see that we may assume that $X = \mathbf{A}^n$. We consider the $\mathbf{Z}^n$-grading on $S = \mathbf{C}[x_1, \ldots, x_n]$, with $\deg(x_i) = e_i$, the i^{th} element of the standard basis. We get a corresponding $\mathbf{Z}^n$-grading on $\iota_+(\mathcal{O}_X)$, with $\deg(\delta) = 0$, on $A = S\langle\partial_{x_1}, \ldots, \partial_{x_n}\rangle$, and on $A\langle t, \partial_t\rangle$, with $\deg(\partial_{x_i}) = -e_i$ and $\deg(t) = \mathbf{e} = -\deg(\partial_t)$, where $\mathbf{e} = e_1 + \ldots + e_r$, and such that the action of A on $\iota_+(\mathcal{O}_X)$ is compatible with the gradings. Since f is homogeneous with respect to this grading, hence preserved (up to multiples) by the standard action of $\mathbf{C}^*$ on $\mathbf{A}^n$, it is easy to deduce using the uniqueness of the V-filtration (see Proposition 3.27) that each $V^\alpha\iota_+(\mathcal{O}_X)$ is a graded submodule of $\iota_+(\mathcal{O}_X)$.

Up to multiplication by a nonzero scalar, a homogeneous element that lies in $F_q\iota_+(\mathcal{O}_X) \smallsetminus F_{q-1}\iota_+(\mathcal{O}_X)$ is of the form $w = x^u\partial_t^q\delta + \sum_{j=1}^q a_j x^{u-j\mathbf{e}}\partial_t^{q-j}\delta$, for suitable $a_1, \ldots, a_q \in \mathbf{C}$ (of course, we need to have $a_j = 0$ if $u - j\mathbf{e} \notin \mathbf{Z}^n_{\geq 0}$). This element lies in $V^1\iota_+(\mathcal{O}_X)$ if and only if we can write $w = P(s)\delta$ for some $P \in A$. Note that $\deg(s) = 0$, hence we may and will assume that all coefficients of P are homogeneous, of degree $u - q\mathbf{e}$. Furthermore, since we can replace any $x_i\partial_{x_i}$, with $i \leq r$ by s, and since $\partial_{x_i}\delta = 0$ for $i > r$, we may and will assume that every nonzero coefficient of $P(s)$ is equal to a scalar multiple of

$$\prod_{i;u_i\geq q} x_i^{u_i-q} \cdot \prod_{i;u_i\leq q} \partial_{x_i}^{q-u_i}$$

and that $u_i \geq q$ for $i > r$. Note that $v \in F_k\iota_+(\mathcal{O}_X)$ if and only if $sv \in F_{k+1}\iota_+(\mathcal{O}_X)$ and if $b \in \mathbf{Z}_{\geq 0}$ is such that $b_i = 0$ for $i > r$, then $\partial_x^b\delta \in F_{|b|}\iota_+(\mathcal{O}_X) \smallsetminus F_{|b|-1}\iota_+(\mathcal{O}_X)$. By putting all these together, we conclude that there is w as above in $V^1\iota_+(\mathcal{O}_X)$ if and only if $u_i \geq q$ for all $i > r$ and $\sum_{i;u_i\leq q}(u_i - q) \leq q$. We now use Proposition 5.3: we see that the $\mathcal{O}_X$-submodule of $\mathcal{O}_X[1/f]$ generated by the images of such w after applying $\overline{\sigma}$ is the submodule generated by $\partial_x^\alpha \cdot \frac{1}{f}$, where $\alpha \in \mathbf{Z}^n_{\geq 0}$ is such that $|\alpha| \leq q$. This gives the assertion in the proposition. □

EXAMPLE 5.5. An important special case of the above proposition is when the hypersurface Z in X is smooth: in this case, by choosing local coordinates $x_1, \ldots, x_n$ on X such that Z is defined by x_1, it is easy to see that the assertion in the proposition implies that $F_p\mathcal{O}_X(*Z) = \mathcal{O}_X\big((p+1)Z\big)$ for all $p \geq 0$.

Recall that for every reduced hypersurface Z in X, the *pole order filtration* $P_\bullet\mathcal{O}_X(*Z)$ is given by

$$P_k\mathcal{O}_X(*Z) = \mathcal{O}_X\big((k+1)Z\big) \quad \text{for} \quad k \geq 0$$

and $P_k\mathcal{O}_X(*Z) = 0$ for $k < 0$. Therefore the assertion in Example 5.5 says that for smooth hypersurfaces, the Hodge filtration on $\mathcal{O}_X(*Z)$ agrees with the pole order filtration. In general, we still have an inclusion, as follows:

PROPOSITION 5.6. For every reduced hypersurface Z of X, we have

$$F_k\mathcal{O}_X(*Z) \subseteq P_k\mathcal{O}_X(*Z) \quad \text{for all} \quad k \in \mathbf{Z}.$$

PROOF. Since X is smooth, it is normal. Since $\mathcal{O}_X\big((k+1)Z\big)$ is a line bundle on X, in order to prove the inclusion in the proposition, it is enough to find an open subset U of X, with $\mathrm{codim}_X(X \smallsetminus U) \geq 2$, such that the inclusion holds on U. However, since Z is reduced, it is generically smooth, and therefore there is such U with $Z \cap U$ smooth, in which case the assertion follows from Example 5.5. □

We now introduce some ideals that measure the difference between the Hodge filtration and the pole order filtration. Since $F_k\mathcal{O}_X(*Z) \subseteq \mathcal{O}_X\big((k+1)Z\big)$ for all $k \geq 0$ and $\mathcal{O}_X\big((k+1)Z\big)$ is a line bundle on X, it follows that there is a unique coherent ideal $I_k(Z)$ on X, the k^{th} *Hodge ideal of* Z, such that

$$F_k\mathcal{O}_X(*Z) = I_k(Z) \cdot \mathcal{O}_X\big((k+1)Z\big).$$

EXAMPLE 5.7. We can now reformulate the assertion in Example 5.5 as saying that if Z is smooth, then $I_k(Z) = \mathcal{O}_X$ for all $k \geq 0$. More generally, the description of the Hodge filtration in Example 5.4 implies that if we have local coordinates $x_1, \ldots, x_n$ on X such that Z is defined by $x_1 \cdots x_r$, then $F_k\mathcal{O}_X(*Z)$ is generated by $\frac{1}{x_1^{a_1} \cdots x_r^{a_r}}$, with all $a_i \geq 1$ and such that $\sum_{i=1}^r a_i \leq k + r$. This implies that

$$I_k(Z) = \big(x_1^{b_1} \cdots x_r^{b_r} \mid 0 \leq b_i \leq k \text{ for all } i,\ b_1 + \ldots + b_r \geq k(r-1)\big).$$

In particular, we see that if $r \geq 2$, then $I_k(Z) = \mathcal{O}_X$ if and only if $k = 0$.

REMARK 5.8. For every reduced hypersurface Z in X, we have

$$I_{k+1}(Z) \subseteq I_k(Z) \quad \text{for all} \quad k \geq 0.$$

Indeed, since $\mathcal{H}^1_Z(\mathcal{O}_X)$ is supported on Z, it follows from Remark 4.1 that

$$\mathcal{O}_X(-Z) \cdot F_{k+1}\mathcal{H}^1_Z(\mathcal{O}_X) \subseteq F_k\mathcal{H}^1_Z(\mathcal{O}_X).$$

By Remark 5.1, this implies

$$\mathcal{O}_X(-Z) \cdot F_{k+1}\mathcal{O}_X(*Z) \subseteq F_k\mathcal{O}_X(*Z),$$

and the inclusion in the statement then follows from the definition of Hodge ideals.

REMARK 5.9. A trivial consequence of the definition of Hodge ideals and of the fact that $F_k\mathcal{O}_X(*Z) \subseteq F_{k+1}\mathcal{O}_X(*Z)$ is that $\mathcal{O}_X(-Z) \cdot I_k(Z) \subseteq I_{k+1}(Z)$ for all $k \geq 0$.

We have the following description of Hodge ideals in terms of the Hodge filtration on $V^1\iota_+(\mathcal{O}_X)$:

PROPOSITION 5.10. If Z is a reduced hypersurface in X defined by $f \in \mathcal{O}_X(X)$ and $\iota\colon X \hookrightarrow X \times \mathbf{A}^1$ is the corresponding graph embedding, then for every $k \geq 0$, we have

$$I_k(Z) = \left\{ \sum_{i=0}^{k} i! h_i f^{k-i} \mid \sum_{i=0}^{k} h_i \partial_t^i \delta \in V^1\iota_+(\mathcal{O}_X) \right\}.$$

PROOF. The assertion follows directly from Proposition 5.3 (see also Remark 5.2), using the definition of Hodge ideals. □

This implies that the first Hodge ideal is a multiplier ideal:

COROLLARY 5.11. For every reduced hypersurface Z in X, we have

$$I_0(Z) = \mathcal{J}\big((1-\epsilon)Z\big) \quad \text{for} \quad 0 < \epsilon \ll 1.$$

PROOF. Working locally, we may assume that Z is defined by some $f \in \mathcal{O}_X(X)$. In this case, the assertion follows from the proposition and the description of $F_1V^1\iota_+(\mathcal{O}_X)$ in Theorem 3.50. □

For higher k, we only have the following weaker assertion:

COROLLARY 5.12. For every reduced hypersurface Z in X and every $k \in \mathbf{Z}_{\geq 0}$, we have

$$I_k(Z) + \mathcal{O}_X(-Z) = \widetilde{\mathcal{J}}\big((k+1-\epsilon)Z\big) + \mathcal{O}_X(-Z) \quad \text{for} \quad 0 < \epsilon \ll 1.$$

PROOF. Working locally, we may assume that Z is defined by some $f \in \mathcal{O}_X(X)$ and let $\iota\colon X \hookrightarrow X \times \mathbf{A}^1$ be the corresponding graph embedding. The assertion then follows from the description of $I_k(Z)$ in Proposition 5.10 and the fact that, by definition, we have

$$\widetilde{\mathcal{J}}\big((k+1-\epsilon)Z\big) = \left\{ h_k \mid \sum_{i=0}^{k} h_i \partial_t^i \delta \in V^1\iota_+(\mathcal{O}_X) \right\}.$$

□

COROLLARY 5.13. For every reduced hypersurface Z in X and every $k \in \mathbf{Z}_{\geq 0}$, we have $\widetilde{\alpha}(Z) \geq k+1$ if and only if $I_k(Z) = \mathcal{O}_X$.

PROOF. Note that on $X \smallsetminus Z$, we have $I_k(Z) = \widetilde{\mathcal{J}}\big((k+1-\epsilon)Z\big) = \mathcal{O}_X$. Therefore, it follows from Corollary 5.12 and Nakayama's Lemma that for every $x \in X$, we have $I_k(Z)_x = \mathcal{O}_{X,x}$ if and only if $\widetilde{\mathcal{J}}\big((k+1-\epsilon)Z\big) = \mathcal{O}_{X,x}$. The assertion in the corollary thus follows from Proposition 3.67. □

It is possible to give a description of $I_k(Z)$ in terms of a log resolution of (X, Z) that is an isomorphism over $X \smallsetminus Z$. However, we will do this more generally in the next section, in the context of closed subvarieties Z of arbitrary codimension.

Our final result in this section concerns the characterization of smoothness in terms of the minimal exponent and Hodge ideals.

PROPOSITION 5.14. For every reduced hypersurface Z in X, the following are equivalent:

i) Z is smooth.
ii) $\widetilde{\alpha}(Z) = \infty$.
iii) $I_k(Z) = \mathcal{O}_X$ for all k.

PROOF. The implication i)⇒ii) is clear (see Example 3.12), while the implication ii)⇒iii) follows from Corollary 5.13. In order to prove iii)⇒i), let us assume that Z is not smooth and show that $I_k(Z) \neq \mathcal{O}_X$ for all $k \gg 0$. Since the filtration $F_\bullet\mathcal{O}_X(*Z)$ is a good filtration, there is p such that $F_{q+1}\mathcal{O}_X(*Z) = F_1\mathcal{D}_X \cdot F_q\mathcal{O}_X(*Z)$ for all $q \geq p$. We show that if $P \in Z$ is a singular point, then $I_k(Z)_P \neq \mathcal{O}_{X,P}$ for all $k \geq p+1$.

Working locally around P, we may assume that X is affine, f is an equation of Z, and $x_1, \ldots, x_n$ is an algebraic system of coordinates centered at P. By assumption, we have

$$I_k(Z) = f^{k+1} \cdot F_k\mathcal{O}_X(*Z) = f^{k+1} \cdot \big(F_1\mathcal{D}_X \cdot F_{k-1}\mathcal{O}_X(*Z)\big)$$

$$= f^{k+1} \cdot \big(\tfrac{1}{f^k} I_{k-1}(Z) + \sum_{i=1}^n \mathcal{O}_X\partial_{x_i} \cdot \tfrac{1}{f^k} I_{k-1}(Z)\big) \subseteq \big(f, \tfrac{\partial f}{\partial x_1}, \ldots, \tfrac{\partial f}{\partial x_n}\big).$$

Since P is a singular point of Z, the right-hand side vanishes at P, hence $I_k(Z)_P \neq \mathcal{O}_{X,P}$. □

5.2. The Hodge filtration on the local cohomology sheaves $\mathcal{H}^q_Z(\mathcal{O}_X)$

In this section, we keep X to be a smooth, irreducible, n-dimensional complex algebraic variety, but Z will now denote an arbitrary proper closed subset of X. We typically consider it as a reduced closed subscheme of X and denote by $\mathcal{I}_Z$ the corresponding coherent ideal. We put $U = X \smallsetminus Z$ and denote by $i\colon Z \hookrightarrow X$ and $j\colon U \hookrightarrow X$ the two embeddings. Our goal is to study the Hodge filtration on the local cohomology sheaves $\mathcal{H}^q_Z(\mathcal{O}_X)$. We note that the right $\mathcal{D}_X$-module corresponding to $\mathcal{H}^q_Z(\mathcal{O}_X)$ is $\mathcal{H}^q_Z(\omega_X)$. The results in this section are based on [**MP22**].

Note that the exact triangle

$$i_*i^!\big(\mathbf{Q}^H_X[n]\big) \to \mathbf{Q}^H_X[n] \to j_*j^*\big(\mathbf{Q}^H_X[n]\big) \xrightarrow{+1}$$

induces isomorphisms of filtered $\mathcal{D}_X$-modules underlying Hodge modules

$$R^q j_*(\mathcal{O}_U) \simeq \mathcal{H}^{q+1}_Z(\mathcal{O}_X) \quad \text{for} \quad q \geq 1 \quad \text{and} \tag{5.4}$$

$$\operatorname{Coker}\big(\mathcal{O}_X \hookrightarrow j_*(\mathcal{O}_U)\big) \simeq \mathcal{H}^1_Z(\mathcal{O}_X). \tag{5.5}$$

Of course, we have $\mathcal{H}^0_Z(\mathcal{O}_X) = 0$.

EXERCISE 5.15. Show that if $\mathcal{I}_Z = (f_1, \ldots, f_r)$, then the complex

$$0 \to \mathcal{O}_X \to \bigoplus_i \mathcal{O}_X[1/f_i] \to \bigoplus_{i_1<i_2} \mathcal{O}_X[1/f_{i_1}f_{i_2}] \to \ldots \to \mathcal{O}_X[1/f_1 \cdots f_r] \to 0,$$

placed in cohomological degrees $0, \ldots, r$, is a complex of filtered $\mathcal{D}_X$-modules underlying a complex of Hodge modules that represents $i_*i^!\big(\mathbf{Q}^H_U[n]\big)$.

EXAMPLE 5.16. If Z is smooth, of pure codimension r, then $\mathcal{H}^i_Z(\mathcal{O}_X) = 0$ for $i \neq r$. In order to describe the filtration on $\mathcal{H}^r_Z(\mathcal{O}_X)$, working locally, we can find an algebraic system of coordinates $x_1, \ldots, x_n$ such that $\mathcal{I}_Z = (x_1, \ldots, x_r)$. In this case, it follows from Example 5.15 and Proposition 5.4 that $F_k\mathcal{H}^r_Z(\mathcal{O}_X) = 0$ for $k < 0$, $F_0\mathcal{H}^r_Z(\mathcal{O}_X) = \mathcal{O}_X \cdot [1/x_1 \cdots x_r]$, and

$$F_k\mathcal{H}^r_Z(\mathcal{O}_X) = F_k\mathcal{D}_X \cdot F_0\mathcal{H}^r_Z(\mathcal{O}_X) = \bigoplus_{a_1,\ldots,a_r} \mathcal{O}_X \cdot [1/x_1^{a_1} \cdots x_r^{a_r}],$$

where the sum is over the integers $a_1, \ldots, a_r \geq 1$ such that $\sum_i a_i = k + r$.

REMARK 5.17. We recall the description of local cohomology sheaves via Ext sheaves. If $\mathcal{M}$ is any coherent $\mathcal{O}_X$-module and $\mathcal{I}$ is any sheaf of ideals co-supported on Z (not necessarily radical), then for every $q \geq 0$, we have a canonical isomorphism

$$\varinjlim_{j \geq 0} \mathcal{E}xt^q_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}^j, \mathcal{M}) \simeq \mathcal{H}^q_Z(\mathcal{M}).$$

Indeed, this is clear for $q = 0$ and then it is straightforward to see that we get, in fact, an isomorphism of δ-functors.

5.2.1. Description via log resolutions. Our first goal is to describe the Hodge filtration on $\mathcal{H}^q_Z(\mathcal{O}_X)$ in terms of a log resolution of (X, Z). More precisely, we consider a proper, birational morphism $\pi\colon Y \to X$, with Y smooth, such that π is an isomorphism over U and $\pi^{-1}(Z)_{\rm red} = E$ is a reduced simple normal crossing divisor on Y. Let $j'\colon V = Y \smallsetminus E \hookrightarrow Y$ be the inclusion map. As we will see, the key computation is that of the Hodge filtration on the cohomology Hodge modules of $\pi_* j'_* \mathbf{Q}_V[n]$. For this, the main ingredient is the following:

PROPOSITION 5.18. If E is a simple normal crossing divisor on the n-dimensional smooth, irreducible variety Y, and if on $\omega_Y(*E)$ we consider the Hodge filtration[1], then $\omega_Y(*E)$ has a filtered resolution given by the log de Rham complex of the left $\mathcal{D}_Y$-module $\mathcal{D}_Y$:

$$0 \to \mathcal{D}_Y \to \Omega^1_Y(\log E) \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \to \ldots \to \Omega^{n-1}_Y(\log E) \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \to \omega_Y(E) \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \to 0,$$

with the map $\omega_Y(E) \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \to \omega_Y(*E)$ given by the right $\mathcal{D}_Y$-action on $\omega_Y(*E)$ and where

$$F_{k-n}\big(\Omega^p_Y(\log E) \otimes \mathcal{D}_Y\big) = \Omega^p_Y(\log E) \otimes F_{k-n+p}\mathcal{D}_Y \quad \text{for all} \quad k.$$

PROOF. It is straightforward to see that for every left $\mathcal{D}_Y$-module $\mathcal{M}$, the de Rham complex $(\Omega^\bullet_Y \otimes_{\mathcal{O}_Y} \mathcal{M}, \nabla)$ of $\mathcal{M}$ induces a complex $(\Omega^\bullet_{Y,{\rm rat}} \otimes_{\mathcal{O}_Y} \mathcal{M}, \nabla)$, where $\Omega^p_{Y,{\rm rat}}$ is the sheaf of *rational* p-forms on Y, given by

$$\nabla\big(\varphi(\omega \otimes u)\big) = \varphi\nabla(\omega \otimes u) + d\varphi \wedge \nabla(\omega \otimes u)$$

for all $\omega \in \Omega^p_Y$, $u \in \mathcal{M}$, and any rational function φ on Y. Furthermore, we see that $\big(\Omega^\bullet_Y(\log E) \otimes_{\mathcal{O}_Y} \mathcal{M}, \nabla\big)$ is a subcomplex. Applying this for $\mathcal{M} = \mathcal{D}_Y$, we obtain the complex of right $\mathcal{D}_Y$-modules in the proposition. Note that if $x_1, \ldots, x_n$ are algebraic coordinates on an open subset U of Y such that $E|_U$ is defined by $(x_1 \cdots x_r)$, then the last map in the complex

$$\alpha\colon \Omega^{n-1}_Y(\log E) \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \to \omega_Y(E) \otimes \mathcal{D}_Y$$

has the image generated as a right $\mathcal{D}_Y$-module by

$$\eta \otimes x_i\partial_i \text{ for } 1 \leq i \leq r \text{ and } \eta \otimes \partial_j \text{ for } r+1 \leq j \leq n,$$

where $\eta = \frac{dx_1}{x_1} \wedge \ldots \wedge \frac{dx_r}{x_r} \wedge dx_{r+1} \wedge \ldots \wedge dx_n$. It is then clear that by composing this map with the right action map $\omega_Y(E) \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \to \omega_Y(*E)$ we get the 0 map. We thus get a complex $C^\bullet$ of right $\mathcal{D}_Y$-modules, placed in cohomological degrees $-n, \ldots, 1$. Furthermore, it is straightforward to check that the maps preserve the

[1]Note that $\omega_Y(*E)$ is the right $\mathcal{D}_Y$-module corresponding to $\mathcal{O}_Y(*E)$ and, therefore, it comes endowed with a Hodgle filtration.

filtrations. In order to conclude that this is filtered exact, it is thus enough to show that for every $p \in \mathbf{Z}$, the complex $\mathrm{Gr}^F_{p-n}(C^\bullet)$, given by

$$\ldots \to \Omega_Y^{n-1}(\log E) \otimes_{\mathcal{O}_Y} \mathrm{Sym}^{p-1}(\mathcal{T}_Y) \to \omega_Y(E) \otimes_{\mathcal{O}_Y} \mathrm{Sym}^p(\mathcal{T}_Y) \to \mathrm{Gr}^F_{p-n}\omega_Y(*E) \to 0,$$

is exact. Of course, we only need to consider $p \geq 0$. Furthermore, since we know that the filtration on $\omega_Y(*E)$ is generated at level 0 by Proposition 5.4, we have $\mathcal{H}^{-1}\big(\mathrm{Gr}^F_{p-n}(C^\bullet)\big) = 0$.

If we remove the right-most term of $\mathrm{Gr}^F_{p-n}(C^\bullet)$, we are left with the twist by $\omega_Y(E)$ of the following Eagon-Northcott type complex

$$\ldots \to \wedge^2\mathcal{T}_Y(-\log E) \otimes_{\mathcal{O}_Y} \mathrm{Sym}^{p-2}(\mathcal{T}_Y) \to \mathcal{T}_Y(-\log E) \otimes_{\mathcal{O}_Y} \mathrm{Sym}^{p-1}(\mathcal{T}_Y) \to \mathrm{Sym}^p(\mathcal{T}_Y) \to 0$$

associated to the inclusion of locally free sheaves $\beta\colon \mathcal{T}_Y(-\log E) \hookrightarrow \mathcal{T}_Y$. By [**Laz04**, Theorem B.2.2.(iii)], this complex is exact in cohomological degrees ≤ -1 if

$$\mathrm{codim}_Y\big(D_{n-\ell}(\beta)\big) \geq \ell \quad \text{for} \quad 1 \leq \ell \leq \min(p, n),$$

where $D_q(\beta) = \big\{y \in Y \mid \mathrm{rk}(\beta_y) \leq q\big\}$. In our case, with respect to a local system of coordinates $x_1, \ldots, x_n$ as above, β is given by the diagonal matrix $\mathrm{diag}(x_1, \ldots, x_r, 1, \ldots, 1)$. An easy computation shows that in this case $D_{n-\ell}(\beta)$ is empty if $\ell \geq r$ and, otherwise, it is the set on which $(\ell+1)$ of $x_1, \ldots, x_r$ vanish: therefore the condition for exactness is satisfied.

We are thus left with showing that $\mathcal{H}^0\big(\mathrm{Gr}^F_{p-n}(C^\bullet)\big) = 0$. After tensoring with ω_Y^{-1}, we see that we need to show that in local coordinates $x_1, \ldots, x_n$ as above, the kernel of the morphism

$$\mathrm{Sym}^p(\mathcal{T}_Y) \to \mathrm{Gr}^F_p\big(\mathcal{O}_Y(*E)\big), \quad Q \mapsto \big[Q \cdot \tfrac{1}{x_1 \cdots x_r}\big]$$

is equal to

$$\sum_{i=1}^{r} \mathrm{Sym}^{p-1}(\mathcal{T}_Y)\partial_i x_i + \sum_{j=r+1}^{n} \mathrm{Sym}^{p-1}(\mathcal{T}_Y)\partial_j.$$

By considering the étale morphism $U \to \mathbf{A}^n$ given by $(x_1, \ldots, x_n)$, we see that it is enough to check the assertion when $Y = \mathrm{Spec}\big(\mathbf{C}[x_1, \ldots, x_n]\big)$. Note that in this case, it follows easily from Proposition 5.4 that we have a k-vector space basis of $\mathrm{Gr}^F_p\big(\mathcal{O}_Y(*E)\big)$ given by $[x_1^{b_1} \cdots x_n^{b_n}]$, where $b_j \geq 0$ for all $j > r$ and

$$\#\{i \leq r \mid b_i \geq 0\} - \sum_{i \leq r, b_i < 0} b_i = p + r.$$

We use the $\mathbf{Z}^n$-grading as in the proof of Proposition 5.4. Suppose that $Q \in \Gamma\big(Y, \mathrm{Sym}^p(\mathcal{T}_Y)\big)$ is such that $Q \cdot \frac{1}{x_1 \cdots x_r} \in F_{p-1}\mathcal{O}_X(*E)$. We may assume that Q is homogeneous and, after removing the terms that involve $x_i\partial_i$ (with $i \leq r$) or ∂_j (with $j > r$), it follows that we may assume that $Q = cx^\alpha\partial^\beta$, where $c \in \mathbf{C}$, for every $i \leq r$ we have either $\alpha_i = 0$ or $\beta_i = 0$, for every $j > r$ we have $\beta_j = 0$, and $\sum_{i=1}^r \beta_i = p$. Since there is $c' > 0$ such that

$$Q \cdot \tfrac{1}{x_1 \cdots x_r} = cc' \cdot \prod_{i \leq r, \alpha_i > 0} x_i^{\alpha_i - 1} \cdot \prod_{i \leq r, \alpha_i = 0} x_i^{-\beta_i - 1} \cdot \prod_{j > r} x_j^{\alpha_j}$$

and the monomial on the right-hand side is an element of the basis of $\mathrm{Gr}^F_p\big(\mathcal{O}_Y(*E)\big)$ described above, we conclude that $c = 0$. This completes the proof of the proposition. □

For future reference, we record the following:

COROLLARY 5.19. If E is a simple normal crossing divisor on the n-dimensional, irreducible variety Y, then for every $p \in \mathbf{Z}$, we have

$$\mathrm{Gr}^F_{-p}\mathrm{DR}_Y\big(\mathcal{O}_Y(*E)\big) \simeq \Omega^p_Y(\log E)[n-p] \quad \text{in} \quad D^b_{\mathrm{coh}}(Y).$$

PROOF. On one hand, by Example 2.16, we have

$$\mathrm{Gr}^F_p\mathrm{DR}_Y(\mathcal{D}_Y) \simeq \mathrm{Gr}^F_p(\omega_Y) = \begin{cases} \omega_Y, & \text{if } p = -n; \\ 0, & \text{otherwise.} \end{cases}$$

On the other hand, it follows from the proposition (see also Remark 2.8), that after passing to left $\mathcal{D}_X$-modules, we have a filtered resolution on $\mathcal{O}_Y(*E)$ given by

$$0 \to \mathcal{D}_Y \otimes_{\mathcal{O}_Y} \omega_Y^{-1} \to \mathcal{D}_Y \otimes_{\mathcal{O}_Y} \Omega^1_Y(\log E) \otimes_{\mathcal{O}_Y} \omega_Y^{-1} \to \ldots \to \mathcal{D}_Y \otimes_{\mathcal{O}_Y} \Omega^n_Y(\log E) \otimes_{\mathcal{O}_Y} \omega_Y^{-1} \to 0,$$

where the filtration on $\mathcal{D}_Y \otimes_{\mathcal{O}_Y} \Omega^p_Y(\log E) \otimes_{\mathcal{O}_Y} \omega_Y^{-1}$ is given by

$$F_k\big(\mathcal{D}_Y \otimes_{\mathcal{O}_Y} \Omega^p_Y(\log E) \otimes_{\mathcal{O}_Y} \omega_Y^{-1}\big) = F_{k-n+p}\mathcal{D}_Y \otimes_{\mathcal{O}_Y} \Omega^p_Y(\log E) \otimes_{\mathcal{O}_Y} \omega_Y^{-1}$$

and therefore

$$\begin{aligned} \mathrm{Gr}^F_k\mathrm{DR}_Y\big(\mathcal{D}_Y \otimes_{\mathcal{O}_Y} \Omega^p_Y(\log E) \otimes_{\mathcal{O}_Y} \omega_Y^{-1}\big) &= \mathrm{Gr}^F_{k-n+p}\mathrm{DR}_Y(\mathcal{D}_Y) \otimes_{\mathcal{O}_Y} \Omega^p_Y(\log E) \otimes_{\mathcal{O}_Y} \omega_Y^{-1} \\ &= \begin{cases} \Omega^p_Y(\log E), & \text{if } k = -p; \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

The assertion in the corollary now follows from the fact that $\mathrm{Gr}^F_k\mathrm{DR}_Y(-)$ maps short exact sequences of filtered $\mathcal{D}_Y$-modules (with strict morphisms) to short exact sequences of complexes. □

We can now give the description of the filtration on the local cohomology sheaves along Z via the log resolution of (X, Z). Let $A^\bullet$ be the complex obtained by tensoring with $\mathcal{D}_{Y\to X}$ the filtered complex in Proposition 5.18:

$$0 \to \pi^*(\mathcal{D}_X) \to \Omega^1_Y(\log E) \otimes_{\mathcal{O}_Y} \pi^*(\mathcal{D}_X) \to \ldots \to \Omega^n_Y(\log E) \otimes_{\mathcal{O}_Y} \pi^*(\mathcal{D}_X) \to 0.$$

Note that this is again a filtered complex, with $F_{k-n}A^\bullet$ being given by

$$0 \to \pi^*(F_{k-n}\mathcal{D}_X) \to \ldots \to \Omega^n_Y(\log E) \otimes_{\mathcal{O}_Y} \pi^*(F_k\mathcal{D}_X) \to 0.$$

PROPOSITION 5.20. For every $q \geq 0$ and every $k \in \mathbf{Z}$, the map

$$R^q\pi_*(F_{k-n}A^\bullet) \to R^q\pi_*(A^\bullet)$$

is injective and it is canonically identified with the map

$$F_{k-n}R^qj_*(\omega_U) \hookrightarrow R^qj_*(\omega_U),$$

giving the $(k-n)^{\mathrm{th}}$ piece of the filtration on the right $\mathcal{D}_X$-module underlying $\mathcal{H}^q\big(j_*\mathbf{Q}^H_U[n]\big)$. Therefore, the image of this map is identified with $F_{k-n}\mathcal{H}^{q+1}_Z(\omega_X)$ for $q \geq 1$, and its quotient by ω_X is identified with $F_{k-n}\mathcal{H}^1_Z(\omega_X)$ for $q = 1$.

PROOF. Since π is an isomorphism over U, we have

$$j_*\mathbf{Q}^H_U[n] \simeq \pi_*\big(j'_*\mathbf{Q}_{U'}[n]\big).$$

Given the filtered resolution of $\omega_Y(*E)$ provided by Proposition 5.18, the first assertion in the proposition is a consequence of the computation of the filtration on $\mathcal{H}^q\big(\pi_*\omega_Y(*E)\big)$ discussed in Section 4.2.3 (in particular, the injectivity of the map in the statement is a consequence of Saito's Strictness Theorem). The last assertion in the proposition then follows from (5.4) and (5.5). □

COROLLARY 5.21. For every q, we have $F_k\mathcal{H}^q_Z(\mathcal{O}_X) = 0$ for $k < 0$.

PROOF. The assertion is equivalent to the fact that $F_{k-n}\mathcal{H}^q_Z(\omega_X) = 0$ for all $k < 0$, which follows directly from the proposition, since in this case $F_{k-n}A^\bullet = 0$. Alternatively, this can be deduced from Exercise 5.15, using the fact that we know the assertion when Z is a hypersurface. □

COROLLARY 5.22. For every $q \geq 1$, we have an isomorphism

$$F_{-n}\mathcal{H}^q_Z(\omega_X) \simeq R^{q-1}\pi_*(\omega_E).$$

PROOF. On Y we have the exact sequence

$$0 \to \omega_Y \to \omega_Y(E) \to \omega_E \to 0.$$

Since $R^i\pi_*(\omega_Y) = 0$ for $i \geq 1$ by Grauert-Riemenschneider vanishing, we get isomorphisms

$$R^{q-1}\pi_*\big(\omega_Y(E)\big) \simeq R^{q-1}\pi_*(\omega_E) \quad \text{for} \quad q \geq 2$$

and a short exact sequence

$$0 \to \pi_*(\omega_Y) = \omega_X \to \pi_*\big(\omega_Y(E)\big) \to \pi_*(\omega_E) \to 0.$$

The assertion in the corollary then follows using the proposition. □

REMARK 5.23. In fact, the map

$$R^{q-1}\pi_*(\omega_E) \hookrightarrow \mathcal{H}^q_Z(\mathcal{O}_X) \tag{5.6}$$

can be described explicitly as the composition

$$R^{q-1}\pi_*(\omega_E) \to \mathcal{E}xt^q_{\mathcal{O}_X}(\mathcal{O}_Z, \omega_X) \to \mathcal{H}^q_Z(\mathcal{O}_X). \tag{5.7}$$

Here, the second map is the canonical map coming from the description of local cohomology sheaves as a direct limit of Ext sheaves (see Remark 5.17). The first map is related to Grothendieck duality for the restriction $\mu\colon E \to Z$ of π. Recall that if we denote by $\omega^\bullet_W$ the dualizing complex of a scheme W, then $\omega^\bullet_X = \omega_X[n]$, $\omega^\bullet_E = \omega_E[n-1]$, and $\omega^\bullet_Z = \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_Z, \omega^\bullet_X)$. Moreover, if $\mu^!$ is the left adjoint of $\mathbf{R}\mu_*$, then $\omega_E \simeq \mu^!(\omega_Z)$. We have a canonical morphism $\tau\colon \mathbf{R}\mu_*\omega^\bullet_E \to \omega^\bullet_Z$ that corresponds via the adjunction isomorphism to the identity on $\omega^\bullet_E$. The first map in (5.7) is $\mathcal{H}^{q-n}(\tau)$. Checking that the composition in (5.7) is indeed our map in (5.6) is somewhat painful, so we leave it as an exercise; for details, see [**MP22**, Proposition 2.12].

The approach in the proof of Proposition 5.20 gives us a formula for the graded de Rham complex of $j_*\big(\mathbf{Q}^H_U[n]\big)$.

PROPOSITION 5.24. For every $p \in \mathbf{Z}$, we have an isomorphism

$$\mathrm{Gr}^F_{-p}\mathrm{DR}_X\big(j_*\mathbf{Q}^H_U[n]\big) \simeq \mathbf{R}\pi_*\Omega^p_Y(\log E)[n-p].$$

PROOF. Since $j_*\big(\mathbf{Q}^H_U[n]\big) \simeq \pi_*j'_*\big(\mathbf{Q}^H_U[n]\big)$, the assertion follows from the formula for $\mathrm{Gr}^F_{-p}\mathrm{DR}_Y\big(j'_*\mathbf{Q}^H_V[n]\big)$ in Corollary 5.19 and the compatibility of the graded de Rham complex with push-forward, see (4.5). □

5.2.2. Comparison with other filtrations. In the hypersurface case, we saw that the Hodge filtration on $\mathcal{O}_X(*Z)$ is always contained in the pole order filtration and the equality of the first few terms of the two filtrations provides a measure for the singularities of the hypersurface. We begin with an extension of this containment to the case when Z has arbitrary codimension.

PROPOSITION 5.25. For every proper closed subset Z of X and every k and q, we have

$$\mathcal{I}_Z \cdot F_k\mathcal{H}^q_Z(\mathcal{O}_X) \subseteq F_{k-1}\mathcal{H}^q_Z(\mathcal{O}_X).$$

In particular, $\mathcal{I}_Z^{k+1} \cdot F_k\mathcal{H}^q_Z(\mathcal{O}_X) = 0$.

PROOF. Since $\mathcal{H}^q_Z(\mathcal{O}_X)$ is supported on Z, the first assertion follows from Remark 4.1. The second assertion follows from the first one and the fact that $F_{-1}\mathcal{H}^q_Z(\mathcal{O}_X) = 0$ by Corollary 5.21. □

Motivated by our analysis in the case of hypersurfaces, one would like to compare the filtration $F_\bullet\mathcal{H}^q_Z(\mathcal{O}_X)$ with another filtration on $\mathcal{H}^q_Z(\mathcal{O}_X)$, of algebraic nature, in such a way that equality of some of the first few terms of the two filtrations would provide a measure of the singularities of Z. Unfortunately, it is not clear what the other filtration should be.

There are two natural choices. First, we could consider the *order filtration* on $\mathcal{H}^q_Z(\mathcal{O}_X)$ given by

$$O_k\mathcal{H}^q_Z(\mathcal{O}_X) = \{u \in \mathcal{H}^q_Z(\mathcal{O}_X) \mid \mathcal{I}_Z^{k+1}u = 0\} \quad \text{for} \quad k \geq 0.$$

We could also consider the *Ext filtration* given by

$$E_k\mathcal{H}^q_Z(\mathcal{O}_X) = \mathrm{Im}\big(\mathcal{E}xt^q_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}_Z^{k+1}, \mathcal{O}_X) \to \mathcal{H}^q_Z(\mathcal{O}_X)\big) \quad \text{for} \quad k \geq 0,$$

see Remark 5.17 (of course, one could also replace the sequence $(\mathcal{I}_Z^{k+1})_{k\geq 0}$ with another sequence of ideals, cofinal with this one). It is clear that we have

$$E_k\mathcal{H}^q_Z(\mathcal{O}_X) \subseteq O_k\mathcal{H}^q_Z(\mathcal{O}_X) \quad \text{for all} \quad k \geq 0.$$

By Proposition 5.25, we have $F_k\mathcal{H}^q_Z(\mathcal{O}_X) \subseteq O_k\mathcal{H}^q_Z(\mathcal{O}_X)$ for all k. The filtration $O_\bullet\mathcal{H}^q_Z(\mathcal{O}_X)$ also has the advantage that it is always compatible with the order filtration on $\mathcal{D}_X$. On the other hand, it has the disadvantage that, in general, $O_k\mathcal{H}^q_Z(\mathcal{O}_X)$ is not a coherent $\mathcal{O}_X$-module (see Remark 5.27 below). The filtration $E_\bullet\mathcal{H}^q_Z(\mathcal{O}_X)$ has the advantage that its terms are coherent $\mathcal{O}_X$-modules. However, we don't know whether it is compatible with the order filtration on $\mathcal{D}_X$ (there doesn't seem to be any reason why it should be). Moreover, we don't know whether we have $F_k\mathcal{H}^q_Z(\mathcal{O}_X) \subseteq E_k\mathcal{H}^q_Z(\mathcal{O}_X)$ for all k (we only know this for $k = 0$ by Remark 5.23).

Recall that the lowest nonzero local cohomology sheaf $\mathcal{H}^q_Z(\mathcal{O}_X)$ is for $q = r := \mathrm{codim}_X(Z)$. For this value of q, it turns out that $O_\bullet\mathcal{H}^q_Z(\mathcal{O}_X) = E_\bullet\mathcal{H}^q_Z(\mathcal{O}_X)$, so this is the correct filtration to compare the Hodge filtration with.

PROPOSITION 5.26. If Z is a reduced subscheme of X of codimension r, then

$$\mathcal{E}xt^r_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}_Z^{k+1}, \mathcal{O}_X) = E_k\mathcal{H}^r_Z(\mathcal{O}_X) = O_k\mathcal{H}^r_Z(\mathcal{O}_X) \quad \text{for all} \quad k \geq 0.$$

PROOF. It is well-known that if I is an ideal in a Noetherian ring R and M and N are finitely generated R-modules such that $IM = 0$ and $\mathrm{depth}(I, N) = d$, then $\mathrm{Ext}^i_R(M, N) = 0$ for $i < d$. Since X is Cohen-Macaulay, being regular, this

implies that $\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{I}_Z^{k+1}/\mathcal{I}_Z^{k+2}, \mathcal{O}_X) = 0$ for all $i < r$. By taking the long exact sequence associated to $\mathcal{H}om_{\mathcal{O}_X}(-, \mathcal{O}_X)$ and the exact sequence

$$0 \to \mathcal{I}_Z^{k+1}/\mathcal{I}_Z^{k+2} \to \mathcal{O}_X/\mathcal{I}_Z^{k+2} \to \mathcal{O}_X/\mathcal{I}_Z^{k+1} \to 0,$$

we conclude that each map

$$\mathcal{E}xt^r_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}_Z^{k+1}, \mathcal{O}_X) \to \mathcal{E}xt^r_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}_Z^{k+2}, \mathcal{O}_X)$$

is injective, and therefore each map

$$\mathcal{E}xt^r_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}_Z^{k+1}, \mathcal{O}_X) \to \mathcal{H}^r_Z(\mathcal{O}_X)$$

is injective. This gives the canonical isomorphism

$$\mathcal{E}xt^r_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}_Z^{k+1}, \mathcal{O}_X) = E_k\mathcal{H}^r_Z(\mathcal{O}_X).$$

In order to show that the inclusion $E_k\mathcal{H}^r_Z(\mathcal{O}_X) \subseteq O_k\mathcal{H}^r_Z(\mathcal{O}_X)$ is an equality, we consider the Grothendieck spectral sequence associated to the composition of functors

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}_Z^{k+1}, -) \circ \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}_Z^{\ell+1}, -) \simeq \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}_Z^{k+1}, -)$$

for $k \leq \ell$:

$$E_2^{p,q} = \mathcal{E}xt^p_{\mathcal{O}_X}\big(\mathcal{O}_X/\mathcal{I}_Z^{k+1}, \mathcal{E}xt^q_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}_Z^{\ell+1}, \mathcal{O}_X)\big) \underset{p}{\Rightarrow} \mathcal{E}xt^{p+q}_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}_Z^{k+1}, \mathcal{O}_X).$$

If $p + q = r$, then $E_2^{p,q} = 0$ unless $q = r$ and $p = 0$. This implies that

$$\mathcal{E}xt^r_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}_Z^{k+1}, \mathcal{O}_X) = \big\{u \in \mathcal{E}xt^r_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}_Z^{\ell+1}, \mathcal{O}_X) \mid \mathcal{I}_Z^{k+1}u = 0\big\}.$$

Since this holds for all $\ell \geq k$ and $\mathcal{H}^r_Z(\mathcal{O}_X) = \bigcup_{\ell \geq k} \mathcal{E}xt^r_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}_Z^{\ell+1}, \mathcal{O}_X)$, we conclude that $E_k\mathcal{H}^r_Z(\mathcal{O}_X) = O_k\mathcal{H}^r_Z(\mathcal{O}_X)$. □

Remark 5.27. For $q > r$, as long as $\mathcal{H}^q_Z(\mathcal{O}_X)$ is nonzero, $O_k\mathcal{H}^q_Z(\mathcal{O}_X)$ can't be equal to either $F_k\mathcal{H}^q_Z(\mathcal{O}_X)$ or $E_k\mathcal{H}^q_Z(\mathcal{O}_X)$. In fact, it is a result of Lyubeznik [**Lyu93**, Corollary 3.5] that in this case $\mathcal{O}_k\mathcal{H}^q_Z(\mathcal{O}_X)$ is not a coherent $\mathcal{O}_X$-module.

Remark 5.28. If Z is a hypersurface, then $O_k\mathcal{H}^1_Z(\mathcal{O}_X) = P_k\mathcal{O}_X(*Z)/\mathcal{O}_X$. Therefore, it follows from Theorem 5.14 that Z is smooth if and only if $F_k\mathcal{H}^1_Z(\mathcal{O}_X) = O_k\mathcal{H}^1_Z(\mathcal{O}_X)$ for all k. The corresponding result (with $\mathcal{H}^1_Z$ replaced by $\mathcal{H}^r_Z$) holds if Z is a complete intersection in X, of pure codimension r, see [**MP22**, Corollary 3.40] (note that in this case, $\mathcal{H}^r_Z(\mathcal{O}_X)$ is the only nonzero local cohomology sheaf, as can be seen from the computation of local cohomology via the Čech complex). On the other hand, it can happen that Z is a singular reduced subscheme of X, of pure codimension r, and $F_k\mathcal{H}^r_Z(\mathcal{O}_X) = O_k\mathcal{H}^r_Z(\mathcal{O}_X)$ for all $k \geq 0$ (this follows from [**Per24**, Corollary 1.6] when X is the affine space of $m \times n$ matrices, with $m < n$, and Z is the subvariety of matrices of rank $\leq p < m$, so $r = (m-p)(n-p)$).

5.2.3. A description of local cohomological dimension. Our goal in this section is two-fold: first, assuming that $\mathcal{H}^i_Z(\mathcal{O}_X) = 0$ for all $i > q$, we determine the level of generation for the Hodge filtration on $\mathcal{H}^q_Z(\mathcal{O}_X)$. Using this, we then give a description for the *local cohomological dimension* $\operatorname{lcd}(X, Z)$. By definition, this is $\sup\big\{i > 0 \mid \mathcal{H}^i_Z(\mathcal{O}_X) \neq 0\big\}$. Both descriptions will be in terms of higher direct images of sheaves of log differentials on a log resolution of (X, Z). We keep the notation in the previous section: in particular, $\pi\colon Y \to X$ is a log resolution of (X, Z) that is an isomorphism over $X \smallsetminus Z$ and $E = \pi^{-1}(Z)_{\mathrm{red}}$.

We begin with the following description for the generation level in terms of the graded de Rham complex:

LEMMA 5.29. *If $(\mathcal{M}, F_\bullet\mathcal{M})$ is a $\mathcal{D}_X$-module with a good filtration, then $F_\bullet\mathcal{M}$ is generated at level p if and only if $\mathcal{H}^0\big(\mathrm{Gr}^F_{i-n}\mathrm{DR}_X(\mathcal{M})\big) = 0$ for all $i \geq p+1$.*

PROOF. If $x_1, \ldots, x_n$ are local coordinates on X, then the map

$$\Omega_X^{n-1} \otimes_{\mathcal{O}_X} \mathrm{Gr}^F_{i-1}(\mathcal{M}) \to \Omega_X^n \otimes_{\mathcal{O}_X} \mathrm{Gr}^F_i(\mathcal{M})$$

in $\mathrm{Gr}^F_{i-n}\mathrm{DR}_X(\mathcal{M})$ is given by

$$dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_n \otimes \overline{u} \mapsto (-1)^{i-1} dx_1 \wedge \ldots \wedge dx_n \otimes \overline{\partial_{x_i} u}.$$

We thus see that $\mathcal{H}^0\big(\mathrm{Gr}^F_{i-n}\mathrm{DR}_X(\mathcal{M})\big) = 0$ if and only if $F_i\mathcal{M} \subseteq F_{i-1}\mathcal{M} + \sum_{i=1}^n \partial_{x_i} F_{i-1}\mathcal{M}$, that is, $F_i\mathcal{M} = F_1\mathcal{D}_X \cdot F_{i-1}\mathcal{M}$. This gives the assertion in the lemma. □

THEOREM 5.30. *Let $q \geq 1$ be such that $\mathcal{H}^i_Z(\mathcal{O}_X) = 0$ for all $i > q$. If $p \geq 0$, then the Hodge filtration on $\mathcal{H}^q_Z(\mathcal{O}_X)$ is generated at level p if and only if*

$$R^{q-1+i}\pi_*\Omega_Y^{n-i}(\log E) = 0 \quad \text{for all} \quad i \geq p+1.$$

Moreover, if $q \geq 2$, then the same equivalence holds also for $p < 0$.

PROOF. It follows from the isomorphisms of filtered $\mathcal{D}_X$-modules (5.4) and (5.5) that the filtration on $\mathcal{H}^q_Z(\mathcal{O}_X)$ is generated at level p if and only if the filtration on $R^{q-1}j_*(\mathcal{O}_U)$ is generated at level p (if $q = 1$, then we need $p \geq 0$, since $j_*(\mathcal{O}_U)$ contains $\mathcal{O}_X$, which is nonzero). By Lemma 5.29, the filtration on $R^{q-1}j_*(\mathcal{O}_U)$ is generated at level p if and only if $\mathcal{H}^0\big(\mathrm{Gr}^F_{i-n}\mathrm{DR}_X(R^{q-1}j_*\mathcal{O}_U)\big) = 0$ for all $i \geq p+1$.

Note now that we have a "hypercohomology" spectral sequence

$$E_2^{a,b} = \mathcal{H}^a\big(\mathrm{Gr}^F_{i-n}\mathrm{DR}_X(R^b j_*\mathcal{O}_U)\big) \underset{a}{\Rightarrow} \mathcal{H}^{a+b}\big(\mathrm{Gr}^F_{i-n}\mathrm{DR}_X(j_*\mathbf{Q}^H_U[n])\big).$$

Since the de Rham complex of a filtered $\mathcal{D}_X$-module only has entries in nonpositive degrees, we have $E_2^{a,b} = 0$ if $a > 0$. Our assumption on q, together with (5.4), implies $R^\ell j_*(\mathcal{O}_U) = 0$ for $\ell \geq q$, hence $E_2^{a,b} = 0$ if $b \geq q$. We conclude that if $a + b = q - 1$, then $E_2^{a,b} = 0$, unless $(a, b) = (0, q-1)$. Furthermore, we have $E_2^{0,q-1} = E_\infty^{0,q-1}$, hence the spectral sequence gives an isomorphism

$$\mathcal{H}^0\big(\mathrm{Gr}^F_{i-n}\mathrm{DR}_X(R^{q-1}j_*\mathcal{O}_U)\big) \simeq \mathcal{H}^{q-1}\big(\mathrm{Gr}^F_{i-n}\mathrm{DR}_X(j_*\mathbf{Q}^H_U[n])\big).$$

Finally, Corollary 5.24 gives

$$\mathcal{H}^{q-1}\big(\mathrm{Gr}^F_{i-n}\mathrm{DR}_X(j_*\mathbf{Q}^H_U[n])\big) \simeq R^{q-1+i}\pi_*\Omega_Y^{n-i}(\log E).$$

We thus obtain the assertion in the theorem. □

REMARK 5.31. In the setting of Theorem 5.30, we clearly have $R^{q-1+i}\pi_* = 0$ if $i \geq n - q + 1$, hence under the assumption of the theorem, we conclude that the Hodge filtration on $\mathcal{H}^q_Z(\mathcal{O}_X)$ is always generated at level $n - q$.

EXAMPLE 5.32. If Z is a hypersurface in X, then we can apply Theorem 5.30 with $q = 1$ to conclude that the Hodge filtration on $\mathcal{H}^1_Z(\mathcal{O}_X)$ (or, equivalently, on $\mathcal{O}_X(*Z)$) is generated at level $n-1$. In fact, one can do better: the theorem also says that the filtration is generated at level $n-2$ if and only if $R^{n-1}\pi_*\Omega_Y(\log E) = 0$. It is not hard to see that this is, indeed, the case: see, for example, [**MP19**, Theorem B]).

We deduce from Theorem 5.30 the following characterization of local cohomological dimension. Note that $\mathrm{lcd}(X,Z) \geq \mathrm{codim}_X(Z) \geq 1$.

COROLLARY 5.33. If Z is a proper closed subscheme of X and c is a positive integer, the $\mathrm{lcd}(X,Z) \leq c$ if and only if

$$R^{j+i}\pi_*\Omega_Y^{n-i}(\log E) = 0 \quad \text{for all } i \geq 0, j \geq c.$$

PROOF. We have $\mathrm{lcd}(X,Z) \leq c$ if and only if $\mathcal{H}^q_Z(\mathcal{O}_X) = 0$ for all $q \geq c+1$, condition which can be rewritten as follows: for every $q \geq c+1$, we have $\mathcal{H}^\ell_Z(\mathcal{O}_X) = 0$ for $\ell > q$ and the filtration on $\mathcal{H}^q_Z(\mathcal{O}_X)$ is generated at level -1. Therefore we obtain the assertion in the corollary by applying Theorem 5.30 with $p = -1$ for all $q \geq c+1$ (note that in this case $q \geq 2$). □

5.3. Basic properties of Hodge ideals

We return to the setting in Section 5.1: we fix an n-dimensional smooth, irreducible, complex algebraic variety X and a reduced hypersurface Z in X.

We begin with the following easy property concerning the behavior of Hodge ideals under pull-back by smooth morphisms:

PROPOSITION 5.34. If $p\colon X' \to X$ is a smooth morphism, then

$$I_k\big(p^*(Z)\big) = I_k(Z) \cdot \mathcal{O}_{X'} \quad \text{for all} \quad k \geq 0.$$

PROOF. Working locally, we may and will assume that Z is defined by $f \in \mathcal{O}_X(X)$, hence $p^*(Z)$ is defined by $f' = f \circ p$. Let ι and ι' be the graph embeddings corresponding to f and f', respectively. It is clear that if $q = (p, \mathrm{Id})\colon X' \times \mathbf{A}^1 \to X \times \mathbf{A}^1$, then we have an isomorphism

$$q^*\big(\iota_+(\mathcal{O}_X)\big) \simeq \iota'_+(\mathcal{O}_{X'})$$

such that $u \in F_p\iota_+(\mathcal{O}_X)$ if and only if $q^*(u) \in F_p\iota'_+(\mathcal{O}_{X'})$. It is also easy to check, using the definition, that if $V^\bullet\iota_+(\mathcal{O}_X)$ is the V-filtration on $\iota_+(\mathcal{O}_X)$ with respect to t, then $q^*V^\bullet\iota_+(\mathcal{O}_X)$ corresponds to the V-filtration on $\iota'_+(\mathcal{O}_X)$ with respect to t. The assertion in the proposition then follows from Proposition 5.10. □

The following result from [**MP18**] extends to Hodge ideals a fundamental result on multiplier ideals, the Restriction Theorem (see [**Laz04**, Theorem 9.5.1]).

THEOREM 5.35. *If Y is a smooth, irreducible hypersurface in X, not contained in Z and such that $Z_Y := Z \cap Y$ is reduced, then*

$$I_k(Z_Y) \subseteq I_k(Z) \cdot \mathcal{O}_Y \quad \textit{for all} \quad k \geq 0.$$

PROOF. We give a simplified argument following [**Sai16**]. We may and will assume that Z and Y are defined by the global regular functions f, respectively y, and let $g = f|_Y$.

Let $j\colon U = X \smallsetminus Z \hookrightarrow X$ and $j_Y\colon U_Y = Y \smallsetminus H_Y \hookrightarrow Y$ be the inclusion maps. We apply the base-change formula (4.18) for the Cartesian diagram

$$\begin{array}{ccc} U_Y & \xrightarrow{\beta} & U \\ \downarrow{\scriptstyle j_Y} & & \downarrow{\scriptstyle j} \\ Y & \xrightarrow{\alpha} & X, \end{array}$$

in which α and β are the defining closed immersions. This gives an isomorphism

$$\alpha^! j_* \big(\mathbf{Q}_U^H[n]\big) \simeq (j_Y)_* \beta^! \big(\mathbf{Q}_U^H[n]\big). \tag{5.8}$$

Note that since U_Y is a smooth hypersurface in U, we have

$$\beta^! \big(\mathbf{Q}_U^H[n]\big) \simeq \big(\mathbf{Q}_{U_Y}^H[n-1]\big)(-1)[-1]$$

(this follows, for example, via the definition and Example 4.9). Since Y is a smooth hypersurface in X, it follows from the isomorphism (5.8) and Remark 4.17 that we have an exact sequence of filtered $\mathcal{D}_H$-modules

$$0 \to \mathrm{Gr}_V^0\big(\mathcal{O}_X(*Z)\big) \xrightarrow{t} \mathrm{Gr}_V^1\big(\mathcal{O}_X(*Z)\big) \to \mathcal{O}_Y(*Z_Y) \to 0,$$

where the V-filtration is the one with respect to t. In particular, we see that for every $k \geq 0$, we have a surjective $\mathcal{O}_X$-linear map

$$\gamma \colon F_k\mathcal{O}_X(*Z) \cap V^1\mathcal{O}_X(*Z) \to F_k\mathcal{O}_Y(*Z_Y).$$

While we don't have a priori information regarding this homomorphism, this is not an issue. Indeed, let $W = X \smallsetminus (Z_Y)_{\mathrm{sing}}$. After possibly replacing X by a suitable neighborhood of Y, we may assume that $W \cap Z$ is smooth, hence we can apply Example 3.29 on W to conclude that we have

$$\gamma|_W \colon F_k\mathcal{O}_X(*Z)|_W \cap V^1\mathcal{O}_X(*Z)|_W = \tfrac{1}{f^{k+1}|_W}\mathcal{O}_W \to F_k\mathcal{O}_Y(*Z_Y)|_{W\cap Y} = \tfrac{1}{g^{k+1}|_W}\mathcal{O}_{W\cap Y}.$$

This implies that $\gamma|_W$ is given by restriction to $Y \cap W$ and multiplication with some (invertible) $h \in \mathcal{O}_Y(W \cap Y)$. Since $\frac{1}{g^{k+1}}\mathcal{O}_Y$ is a line bundle on Y and we have $\mathrm{codim}_Y(Y \smallsetminus W) \geq 2$, it follows that $h \in \mathcal{O}_Y(Y)$ is invertible and the composition

$$F_k\mathcal{O}_X(*Z) \cap V^1\mathcal{O}_X(*Z) \xrightarrow{\gamma} F_k\mathcal{O}_Y(*Z_Y) \hookrightarrow \tfrac{1}{g^{k+1}}\mathcal{O}_Y$$

is again given by restriction to Y and multiplication with h. Then it is clear that $I_k(Z_Y) \subseteq I_k(Z) \cdot \mathcal{O}_Y$. Note that we only use the condition that Z_Y is reduced here, to translate the inclusion between the pieces of the Hodge filtration to an inclusion at the level of Hodge ideals (the rest of the argument would work, with $Z_Y = (Z \cap Y)_{\mathrm{red}}$). □

Remark 5.36. Given any morphism of smooth varieties $f \colon Y \to X$ such that $f^{-1}(Z)$ is a proper reduced subscheme of Y, by factoring f as $Y \xhookrightarrow{\alpha} Y \times X \xrightarrow{p} X$, where α is the graph of f and p is the projection, then by applying Proposition 5.34, as well as Theorem 5.35 $\dim(Y)$ times, we deduce that

$$I_k\big(f^{-1}(Z)\big) \subseteq I_k(Z) \cdot \mathcal{O}_Y \quad \text{for all} \quad k \geq 1.$$

The next result treats the case of restriction to a general smooth subvariety, where by "general smooth subvariety" we mean the general fiber of a smooth surjective morphism. For a morphism $\pi \colon X \to T$, we put $X_t := \pi^{-1}(t)$ and $Z_t := Z \cap X_t$.

Theorem 5.37. *If $\pi \colon X \to T$ is a smooth morphism such that for every $t \in T$, the fiber Z_t is a proper reduced subscheme of X_t, then there is a nonempty open subset $U \subseteq T$ such that for every $t \in U$, we have*

$$I_k(Z_t) = I_k(Z) \cdot \mathcal{O}_{X_t} \quad \textit{for all} \quad k \geq 0.$$

Proof. It is shown in [**MP19**, Theorem 16.1], using the description via log resolutions in Proposition 5.20, that $I_k(Z \cap H) = I_k(Z) \cdot \mathcal{O}_H$ for all $k \geq 0$ whenever H is a smooth irreducible hypersurface in X such that $Z \cap H$ is a proper reduced subscheme of H and the following holds: there is a log resolution $\mu \colon Y \to X$ of

(X, Z) that is an isomorphism over $X \smallsetminus Z$ such that μ is also a log resolution for $(X, H + Z)$ and $\mu^*(H)$ is the strict transform of H, and it is a smooth divisor. In our setting, working locally over T, we may assume that T is affine. In this case, if $H_1, \ldots, H_d$ are general hyperplane sections of T, where $d = \dim(T)$, then by Generic Smoothness we may apply successively the above-mentioned result for each connected component of $\pi^{-1}(H_1), \pi^{-1}(H_1 \cap H_2), \ldots, \pi^{-1}(H_1 \cap \ldots \cap H_d)$. The assertion in the theorem then follows easily.

Alternatively, we can deduce the assertion in the theorem using the description of Hodge ideals via the V-filtration in Proposition 5.10 and the following result from [**DMST09**]: suppose that Z is defined by $f \in \mathcal{O}_X(X)$. If $\alpha\colon Y \hookrightarrow X$ is a general smooth subvariety[2] of X, and $\iota\colon X \hookrightarrow X \times \mathbf{A}^1$ and $\iota_Y\colon Y \hookrightarrow Y \times \mathbf{A}^1$ are the graph embeddings corresponding to f and $f|_Y$, respectively, then the canonical isomorphism

$$(\alpha \times \mathrm{Id})^*\big(\iota_+(\mathcal{O}_X)\big) \simeq (\iota_Y)_+(\mathcal{O}_Y)$$

induces isomorphisms

$$(\alpha \times \mathrm{Id})^*\big(F_pV^\alpha\iota_+(\mathcal{O}_X)\big) \simeq F_pV^\alpha(\iota_Y)_+(\mathcal{O}_Y) \quad \text{for all} \quad k \in \mathbf{Z}, \alpha \in \mathbf{Q}$$

(see [**DMST09**, Theorem 1.1 and Lemma 4.1]. □

Given Theorems 5.35 and 5.37, we can get the following semicontinuity statement as in the case of multiplier ideals (see [**Laz04**, Chapter 9.5.D]). For details, we refer to [**MP18**, Theorem E].

THEOREM 5.38. *Let $\pi\colon X \to T$ be a smooth morphism and let $s\colon T \to X$ be a section of π with $\pi(T) \subseteq Z$. If Z_t is a proper reduced closed subscheme of X_t for all $t \in T$, then for every $k \geq 0$, the subset*

$$\big\{t \in T \mid I_k(Z_t)_{s(t)} = \mathcal{O}_{X_t, s(t)}\big\} \subseteq T$$

is open.

We also obtain the following version of the Subadditivity Theorem for multiplier ideals (cf. [**Laz04**, Chapter 9.5.B]):

THEOREM 5.39. *If Z_1 and Z_2 are reduced hypersurfaces in X, without common components, then for every $k \geq 0$, we have*

$$I_k(Z_1 + Z_2) \subseteq \sum_{p+q=k} I_p(Z_1) \cdot I_q(Z_2) \cdot \mathcal{O}_X(-qZ_1 - pZ_2) \subseteq I_k(Z_1) \cdot I_k(Z_2).$$

PROOF. We use the fact that if Z_1 and Z_2 are reduced hypersurfaces in X_1 and X_2, respectively, and if $D = \pi_1^*(Z_1) + \pi_2^*(Z_2)$, where $\pi_i\colon X_1 \times X_2 \to X_i$, for $i = 1, 2$ are the projections, then there is an isomorphism of filtered $\mathcal{D}_{X_1 \times X_2}$-modules

$$\mathcal{O}_{X_1 \times X_2}(*D) \simeq \mathcal{O}_{X_1}(*Z_1) \boxtimes \mathcal{O}_{X_2}(*Z_2)$$

(see [**MP18**, Proposition 4.1]). This implies

$$I_k(D) = \sum_{p+q=k} \big(I_p(Z_1)\mathcal{O}_{X_1}(-qZ_1) \cdot \mathcal{O}_{X_1 \times X_2}\big) \cdot \big(I_q(Z_2)\mathcal{O}_{X_1}(-pZ_2) \cdot \mathcal{O}_{X_1 \times X_2}\big).$$

[2]The condition in *loc. cit.* is that Y is non-characteristic with respect to $\mathcal{O}_X(*Z)$ in the sense of $\mathcal{D}$-module theory, see [**HTT08**]. This condition is satisfied if Y is the general fiber of a smooth morphism.

The first inclusion in the theorem thus follows by applying Remark 5.36 for the diagonal inclusion $X \hookrightarrow X \times X$, and the second inclusion is a consequence of Remark 5.9. □

5.4. Basic properties of the minimal exponent

The properties of Hodge ideals that we discussed in the previous section can be used to deduce properties of the minimal exponent, much in the same way that corresponding properties of multiplier ideals can be used to deduce properties of the log canonical threshold as in [**Laz04**, Chapter 9.5]). One issue in our setting is that Hodge ideals do not allow us to detect the minimal exponent, but only its integral part. In order to go around this issue, we will make use of the following Thom-Sebastiani type result for minimal exponents:

THEOREM 5.40. *Let X_1 and X_2 be smooth, irreducible complex algebraic varieties, $X = X_1 \times X_2$, and f_1 and f_2 nonzero regular functions on X_1 and X_2, respectively. If $p_i \colon X_1 \times X_2 \to X_i$, for $i = 1, 2$ are the two projections, and $f = f_1 \circ p_1 + f_2 \circ p_2$, then for every $P_1 \in X_1$ and $P_2 \in X_2$, with $f_1(P_1) = 0 = f_2(P_2)$, we have*

$$\widetilde{\alpha}_{(P_1,P_2)}(f) = \widetilde{\alpha}_{P_1}(f_1) + \widetilde{\alpha}_{P_2}(f_2).$$

For a proof of this theorem, we refer to [**MSS20**]. For almost all our results on minimal exponents, it is enough to use the theorem when $X_2 = \operatorname{Spec} \mathbf{C}[x]$ and $f_2 = x^m$, in which case a stronger result, describing all the roots of $\widetilde{b}_f(s)$ can be found in [**Sai94**, Theorem 0.8].

We collect in the following result some important properties of the minimal exponent:

THEOREM 5.41. *Let X be a smooth, irreducible, n-dimensional complex algebraic variety and Z a hypersurface in X.*

i) *If Y is a smooth, irreducible subvariety of X that doesn't contain any irreducible component of Z, then for every $P \in Y$, we have*

$$\widetilde{\alpha}_P(Z) \geq \widetilde{\alpha}_P(Z \cap Y).$$

ii) *If $\pi \colon X \to T$ is a smooth morphism and $Z = \pi^{-1}(W)$, for some hypersurface W in T, then for every $P \in Z$, we have*

$$\widetilde{\alpha}_P(Z) = \widetilde{\alpha}_{\pi(P)}(W).$$

iii) *If $\pi \colon X \to T$ is a smooth morphism such that no fiber of π is contained in Z and $s \colon T \to X$ is a section of π whose image is contained in Z, then for all $\alpha \in \mathbf{Q}_{>0}$, the set*

$$\left\{ t \in T \mid \widetilde{\alpha}_{s(t)}(Z_t) \geq \alpha \right\}$$

is open in T.

iv) *If $\pi \colon X \to T$ is a smooth morphism such that no fiber of π is contained in Z, and $\alpha \in \mathbf{Q}_{>0}$ is such that for a general $t \in T$, we have $\widetilde{\alpha}(Z_t) \leq \alpha$, then the set $W_\alpha := \{P \in Z \mid \widetilde{\alpha}_P(Z) \leq \alpha\}$ dominates T.*

v) *If $f, g \in \mathcal{O}_X(X)$ are nonzero and P in X is such that $f(P) = 0 = g(P)$, and $f + g \neq 0$, then*

$$\widetilde{\alpha}_P(f + g) \leq \widetilde{\alpha}_P(f) + \widetilde{\alpha}(g).$$

PROOF. In order to prove the inequality in i), note first that if $r = \operatorname{codim}_X(Y)$, then in a neighborhood of P we can write $Y = H_1 \cap \ldots \cap H_r$, where the H_i are smooth hypersurfaces in X intersecting transversely. By successively cutting with $H_1, \ldots, H_r$, we see that it is enough to prove the assertion when $r = 1$, that is, Y is a hypersurface in X. Next, note that we may assume that $Z \cap Y$ is reduced in a neighborhood of P: otherwise, we have $\widetilde{\alpha}_P(Z \cap Y) \leq \frac{1}{2}$, and thus

$$\widetilde{\alpha}_P(Z \cap Y) = \operatorname{lct}_P(Z \cap Y) \leq \operatorname{lct}_P(Z) \leq \widetilde{\alpha}_P(Z),$$

where the middle inequality is a consequence of the corresponding property of log canonical thresholds (which follows, for example, using the characterization of log canonical thresholds in terms of multiplier ideals and the Restriction Theorem for multiplier ideals, see [**Laz04**, Theorem 9.5.1]).

Since $Z \cap Y$ is reduced in a neighborhood of P, it follows that Z is also reduced in a neighborhood of P. In this case, Theorem 5.35 gives $I_k(Z \cap Y) \subseteq I_k(Z) \cdot \mathcal{O}_Y$ for all $k \geq 0$, and we conclude via Corollary 5.13 that if $\widetilde{\alpha}_P(Z \cap Y) \geq k+1$, then $\widetilde{\alpha}_P(Z) \geq k+1$. We are thus done if $\widetilde{\alpha}_P(Z \cap Y)$ is an integer (or is infinite). Suppose now that $\alpha = \widetilde{\alpha}_P(Z \cap Y) \in \mathbf{Q}_{>0} \smallsetminus \mathbf{Z}$ and let us write $\alpha = \frac{p}{q}$, for positive integers p and q. Let m be a positive integer such that $p + m$ is divisible by q. After possibly replaxing X by a suitable open neighborhood of P, we may assume that Z is defined by $f \in \mathcal{O}_X(X)$, let Z' be the hypersurface defined in $X' = X \times \operatorname{Spec} \mathbf{C}[y_1, \ldots, y_m]$ by $f + y_1^q + \ldots + y_m^q$, and let $Y' = Y \times \operatorname{Spec} \mathbf{C}[y_1, \ldots, y_m]$. Note that if H is the hypersurface defined by (y^q) in $\operatorname{Spec} \mathbf{C}[y]$, then $\widetilde{\alpha}_0(H) = \operatorname{lct}(H) = \frac{1}{q}$, hence a repeated application of Theorem 5.40 gives

$$\widetilde{\alpha}_{(P,0)}(Z' \cap Y') = \widetilde{\alpha}_P(Z \cap Y) + \tfrac{m}{q} \in \mathbf{Z} \quad \text{and} \quad \widetilde{\alpha}_{(P,0)}(Z') = \widetilde{\alpha}_P(Z) + \tfrac{m}{q}.$$

We can now apply the assertion in the case of integral minimal exponent of $Z \cap Y$ to get the inequality in i) in general.

For the assertion in ii), we argue similarly. First, we may assume that W is reduced in a neighborhood of $\pi(P)$ (equivalently, Z is reduced in a neighborhood of P) since otheriwse the formula follows from the corresponding property of log canonical thresholds. In this case, Proposition 5.34 together with the connection between the minimal exponent and Hodge ideals in Corollary 5.13 imply that

$$\widetilde{\alpha}_P(Z) \geq k+1 \quad \text{if and only if} \quad \widetilde{\alpha}_{\pi(P)}(W) \geq k+1 \tag{5.9}$$

for all $k \in \mathbf{Z}_{\geq 0}$. Note that we need to prove the two inequalities

$$\widetilde{\alpha}_P(Z) \geq \widetilde{\alpha}_{\pi(P)}(W) \quad \text{and} \quad \widetilde{\alpha}_{\pi(P)}(W) \geq \widetilde{\alpha}_P(Z).$$

Using Theorem 5.40, we may assume for the first inequality that $\widetilde{\alpha}_{\pi(P)}(W) \in \mathbf{Z}$, while for the second inequality, we may assume that $\widetilde{\alpha}_P(Z) \in \mathbf{Z}$. We thus conclude using (5.9).

The proof of iii) follows the same pattern. We need to show that if $t_0 \in T$ is such that $\widetilde{\alpha}_{s(t_0)}(Z_{t_0}) = \beta$, then there is an open neighborhood U of t_0 such that $\widetilde{\alpha}_{s(t)}(Z_t) \geq \beta$ for all $t \in U$. If Z_{t_0} is not reduced in a neighborhood of $s(t_0)$, then the assertion follows from the semicontinuity property of log canonical thresholds (see [**Laz04**, Example 9.5.41]). Hence, after replacing X and T by suitable open neighborhoods of $s(t_0)$ and t_0, respectively, we may assume that Z_t is reduced for all $t \in T$. Using Theorem 5.40, we reduce to the case when $\widetilde{\alpha}_{s(t_0)}(Z_{t_0}) \in \mathbf{Z}$, in which case we deduce the existence of the open subset U from Theorem 5.38.

We argue similarly for iv). Arguing by contradiction, suppose that W_α does not dominate T. After replacing T by an open subset, we may assume that $W_\alpha = \emptyset$, hence $\beta := \widetilde{\alpha}(Z) > \alpha$. Arguing locally, we may and will assume that Z is defined by $f \in \mathcal{O}_X(X)$. After possibly replacing X and f by $X \times \operatorname{Spec} \mathbf{C}[y_1, \ldots, y_m]$ and $f + y_1^q + \ldots + y_m^q$, respectively, for suitable m and q, and using Theorem 5.40, we reduce to the case when $\beta = k + 1$ is a positive integer. In particular, this implies that Z is reduced, and thus, after possibly replacing T by an open subset, we may assume that Z_t is reduced for all $t \in T$. In this case, we apply Theorem 5.37 to get an open subset U of T such that

$$I_k(Z_t) = I_k(Z) \cdot \mathcal{O}_{X_t} \quad \text{for all} \quad t \in U.$$

The hypothesis and Corollary 5.13 thus give a contradiction.

The assertion in v) is the only one for which we need the general version of Theorem 5.40. Let $p_1, p_2 \colon X \times X \to X$ be the two projections and let $h = f \circ p_1 + g \circ p_2$. By Theorem 5.40, we have

$$\widetilde{\alpha}_{(P,P)}(h) = \widetilde{\alpha}_P(f) + \widetilde{\alpha}_P(g)$$

and the inequality in v) follows by restricting h to the diagonal embedding $X \hookrightarrow X \times X$ and using i). □

Remark 5.42. By using the characterization of the minimal exponent in terms of Hodge ideals of $\mathbf{Q}$-divisors, as in [**MP20**], one can get a stronger assertion than the one in Theorem 5.41iv): one can show that given a smooth morphism $\pi \colon X \to T$ and a hypersurface Z in X that does not contain any fiber of π, then there is a nonempty open subset U of T such that

$$\widetilde{\alpha}_x(Z_t) = \widetilde{\alpha}_x(Z)$$

for all $t \in U$ and all $x \in Z_t$.

Remark 5.43. In these notes, we have discussed the minimal exponent for arbitrary hypersurfaces, from a $\mathcal{D}$-module theoretic point of view. However, it is worth pointing out that for hypersurfaces with isolated singularities, the minimal exponent agrees with the *Arnold exponent* (or *complex singularity index*), an invariant that can be defined in terms of asymptotic expansions of integrals over vanishing cycles. In this setting, the invariant appears as the smallest number in the *spectrum of the singularity*, and was extensively studied in the 80s by Varchenko, Steenbrink, Malgrange, Loeser and others. For example, for isolated singularities, the assertion in Theorem 5.41ii) follows from [**Loe84**, Théorème 1] and the assertion in Theorem 5.41iii) is a consequence of [**Ste85b**, Theorem 2.11].

Remark 5.44. One important result concerning the minimal exponent in the setting of isolated singularities is its invariance under deformations with constsnt Milnor number. More precisely, suppose that X is smooth and irreducible, $\pi \colon X \to T$ is a smooth morphism, Z is a hypersurface in X that contains no fiber of π, and $s \colon T \to X$ is a section of π such that $s(T) \subseteq Z$. If Z_t has an isolated singularity at $s(t)$ for all $t \in T$ and the Milnor number[3] $\mu_{s(t)}(Z_t)$ is independent of t, then $\widetilde{\alpha}_{s(t)}(Z_t)$ is independent of$t$. This is a result due to Varchenko [**Var82**].

[3]Recall that if a hypersurface Z in X is defined is a neighborhood of P by $f \in \mathcal{O}_X(X)$ and $x_1, \ldots, x_n$ are local coordinates in a neighborhood of P, then the Milnor number $\mu_P(Z)$ is $\dim_{\mathbf{C}}(\mathcal{O}_{X,P}/J_f)$, where J_f is the *Jacobian ideal* $(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n})$.

The first part of the following result gives an upper bound for the minimal exponent in terms of multiplicity. As such, it makes more precise the implication ii)⇒i) in Proposition 5.14.

PROPOSITION 5.45. Let X be a smooth, irreducible, n-dimensional variety and Z a hypersurface in X.

i) If $P \in Z$ is such that $\mathrm{mult}_P(Z) = d \geq 2$, then
$$\widetilde{\alpha}_P(Z) \leq \frac{n}{d}.$$

ii) If, in addition, the dimension of the singular locus of the tangent cone of Z at P is r, then
$$\widetilde{\alpha}_P(Z) \geq \tfrac{n-r}{d}.$$

PROOF. The key point for both statements is that if $P \in Z$ is an ordinary singular point (that is, the projectivized tangent cone of Z at P is smooth), then $\widetilde{\alpha}_P(Z) = \frac{n}{d}$. This can be deduced using the result mentioned in Remark 5.44: if we consider the deformation of Z to the tangent cone at P, then all members of this deformation have an isolated singular point at P, with Milnor number equal to $(d-1)^n$. On the other hand, the tangent cone of Z at P has minimal exponent $\frac{n}{d}$ by Example 3.16.

Suppose now that $x_1, \ldots, x_n$ are local coordinates in a neighborhood U of P, with $x_i(P) = 0$ for $1 \leq i \leq n$. Let us consider the hypersurface $\mathcal{Z}$ in $X \times T$ defined by $f + y_1 x_1^d + \ldots + y_n x_n^d$, where $T = \mathbf{A}^n$, with coordinates $y_1, \ldots, y_n$. Since the zero-locus of $(x_1^d, \ldots, x_n^d)$ is $\{P\}$, it follows from the Kleinman-Bertini theorem that for general $t \in T$, the fiber $\mathcal{Z}_t$ has an ordinary singular point at P, and thus $\widetilde{\alpha}_P(\mathcal{Z}_t) = \frac{n}{d}$. Since $Z = \mathcal{Z}_0$, the bound in ii) follows from Theorem 5.41ii).

For the assertion in ii), if $x_1, \ldots, x_n$ are as above, let $H_1, \ldots, H_r$ be hypersurfaces in X defined by general linear combinations of $x_1, \ldots, x_n$, with coefficients in $\mathbf{C}$. If $Y = H_1 \cap \ldots \cap H_r$, then Y is a smooth subvariety of X of dimension $n-r$ and $Z \cap Y$ has an ordinary singular point at P, hence $\widetilde{\alpha}_P(Z \cap Y) = \frac{n-r}{d}$. The inequality in ii) again follows from Theorem 5.41ii). □

COROLLARY 5.46. Let Z be a hypersurface in the smooth, irreducible, n-dimensional complex algebraic variety X. If for $d \geq 2$, we have $\dim(W_d) = s$, where
$$W_d = \{P \in Z \mid \mathrm{mult}_P(Z) \geq d\},$$
then $\widetilde{\alpha}(Z) \leq \frac{n-s}{d}$. In particular, if $\dim(Z_{\mathrm{sing}}) = e$, then $\widetilde{\alpha}(Z) \leq \frac{n-e}{2}$.

PROOF. Working locally, we may assume that X is a closed subvariety in some $\mathbf{A}^N$. Let T be the variety parametrizing s-tuples $(H_1, \ldots, H_s)$ of hyperplanes in $\mathbf{A}^N$ such that $X \cap H_1 \cap \ldots \cap H_s$ is a smooth subvariety of X, of dimension $n-s$. Let $\mathcal{X} \hookrightarrow X \times T$ be the closed subvariety whose fiber over $(H_1, \ldots, H_s) \in T$ is $X \cap H_1 \cap \ldots \cap H_s$. The two projections induce morphisms $\pi\colon \mathcal{X} \to T$ and $q\colon \mathcal{X} \to X$ that are both smooth, and let $\mathcal{Z} = q^{-1}(Z)$. Note that for every general $t = (H_1, \ldots, H_s) \in T$, if $Y = X \cap H_1 \ldots \cap H_s$, then $Y \cap W_d \neq \emptyset$, hence the fiber $\mathcal{Z}_t = Z \cap Y$ is a hypersurface in a smooth $(n-s)$-dimensional variety, having a point of multiplicity d, and thus $\widetilde{\alpha}(\mathcal{Z}_t) \leq \frac{n-s}{d}$ by the proposition. Since this holds for all $t \in T$ general, we deduce from Theorem 5.41iv) that $\widetilde{\alpha}(\mathcal{Z}) \leq \frac{n-s}{d}$, and thus $\widetilde{\alpha}(Z) \leq \frac{n-s}{d}$ by Theorem 5.41i). The last assertion in the corollary follows from the fact that $Z_{\mathrm{sing}} = W_2$. □

CHAPTER 6

The Du Bois complex and the minimal exponent

In the first section, we give a brief introduction to the Du Bois complex $\underline{\Omega}^{\bullet}_Z$ of a complex algebraic variety Z, explaining Steenbrink's description of the graded pieces $\underline{\Omega}^p_Z$, and using it to relate these graded pieces to the graded de Rham complex of the local cohomology along Z for the structure sheaf of an ambient smooth variety X. In the remaining sections, we restrict ourselves to the case of hypersurfaces. In Section 6.2, we discuss higher Du Bois singularities for hypersurfaces and the connection to the minimal exponent following [**MOPW23**] and [**JKSY22**]. Roughly speaking, the main result here says that the minimal exponent is large if and only if the first graded pieces $\underline{\Omega}^p_Z$ of the Du Bois complex agree with the corresponding sheaves of differentials Ω^p_Z. Finally, in the last section we discuss another result from [**MOPW23**], which says that singular hypersurfaces with minimal exponent > 2 have nonvanishing $\mathcal{H}^i(\underline{\Omega}^p_Z)$, for suitable p and $i > 0$.

6.1. The Du Bois complex via Hodge filtrations

Let Z be a complex algebraic variety (assumed to be reduced and separated, but not necessarily irreducible). By building on Deligne's work related to the construction of a mixed Hodge structure on the cohomology of a singular variety, Du Bois introduced in [**DB81**] a filtered complex $\underline{\Omega}^{\bullet}_Z$ (more precisely, this is an object in a suitable filtered derived category). By taking the (suitably shifted) graded pieces, one gets

$$\underline{\Omega}^p_Z := \mathrm{Gr}^p_F(\underline{\Omega}^{\bullet}_Z)[p],$$

which are objects in $D^b_{\mathrm{coh}}(X)$. If Z is smooth, then $\underline{\Omega}^{\bullet}_Z$ is the de Rham complex of Z, with its "stupid" filtration, so $\underline{\Omega}^p_Z = \Omega^p_Z$. However, when Z is singular, it turns out that the Du Bois complex and its truncations are much better behaved from a Hodge-theoretic point of view than the de Rham complex and the sheaves of differential forms.

The definition of $\underline{\Omega}^{\bullet}_Z$ makes use of a simplicial resolution of Z, taking the de Rham complex of this resolution and pushing it forward to Z. However, we will not need the precise definition. We list a few properties that are useful to keep in mind:

i) For every Z and every $p \geq 0$, there is a canonical morphism $\Omega_Z \to \underline{\Omega}^p_Z$. This is an isomorphism if Z is smooth.
ii) For every morphism $f\colon W \to Z$, there is a canonical morphism $\underline{\Omega}^p_Z \to \mathbf{R}f_*\underline{\Omega}^p_W$ that is compatible with composition and with the push-forward of differential forms via the morphisms given by i).
iii) If $f\colon Y \to X$ is a proper morphism, Z is a closed subvariety of X and $W = f^{-1}(Z)$, and f is an isomorphism over $X \smallsetminus Z$, then for every p, we

have an exact triangle

$$\underline{\Omega}^p_X \to \underline{\Omega}^p_Z \oplus \mathbf{R}f_*\underline{\Omega}^p_Y \to \mathbf{R}f_*\underline{\Omega}^p_W \xrightarrow{+1}, \tag{6.1}$$

in which the first two maps are given (up to an obvious sign change) by the maps in ii).

While we will not need the following properties, they illustrate the fact that the Du Bois complex allows extending to the singular case some important properties of the de Rham complex of smooth proper varieties (for proofs, we refer to [**PS08**]):

1) The canonical morphism $\mathbf{C}_{Z^{\mathrm{an}}} \to \underline{\Omega}^\bullet_Z$ is an isomorphism (in the derived category). In particular, we have canonical isomorphisms
$$H^i(Z^{\mathrm{an}}, \mathbf{C}) \simeq H^i(Z, \underline{\Omega}^\bullet_Z) \quad \text{for all} \quad i \in \mathbf{Z}.$$
2) If Z is proper, then the spectral sequence
$$E_1^{p,q} = H^q(Z, \underline{\Omega}^p_Z) \Rightarrow H^{p+q}(Z, \underline{\Omega}^\bullet_Z)$$
associated to the canonical filtration on $\underline{\Omega}^\bullet_Z$ degenerates at E_1.
3) If Z is a projective variety and $\mathcal{L}$ is an ample line bundle on Z, then
$$H^q(Z, \underline{\Omega}^p_Z \otimes \mathcal{L}) = 0 \quad \text{if} \quad p+q > \dim(Z)$$
(note that this extends the Kodaira-Akizuki-Nakano Vanishing theorem to singular varieties).

6.1.1. Steenbrink's description of $\underline{\Omega}^p_Z$. Our main tool for analyzing the Du Bois complex is the following description due to Steenbrink [**Ste85a**]. Suppose that Z is a closed subvariety of the variety X such that $X \smallsetminus Z$ is smooth. We consider a proper morphism $\pi\colon Y \to X$ that is an isomorphism over $X \smallsetminus Z$ such that Y is smooth and $E = \pi^{-1}(Z)_{\mathrm{red}}$ is a simple normal crossing divisor (the existence of such a morphism is a consequence of Hironaka's theorem on embedded resolution of singularities).

PROPOSITION 6.1. With the above notation, for every $p \in \mathbf{Z}$, we have an exact triangle:
$$\mathbf{R}\pi_*\Omega^p_Y(\log E)(-E) \to \underline{\Omega}^p_X \to \underline{\Omega}^p_Z \xrightarrow{+1},$$
where the second morphism is the canonical one associated to the inclusion map $Z \hookrightarrow X$.

PROOF. We consider the exact triangle (6.1), in which $W = E$. Let us apply the octahedral axiom for the maps
$$\mathbf{R}\pi_*\underline{\Omega}^p_Y \xrightarrow{u} \mathbf{R}\pi_*\underline{\Omega}^p_Y \oplus \underline{\Omega}^p_Z \xrightarrow{v} \mathbf{R}\pi_*\underline{\Omega}^p_E,$$
where $u = (\mathrm{Id}, 0)$ and v is the sum of the obvious morphisms, to get an exact triangle
$$\underline{\Omega}^p_Z = \mathrm{cone}(u) \to \mathrm{cone}(v \circ u) \to \mathrm{cone}(v) \xrightarrow{+1}. \tag{6.2}$$
Since Y is smooth, we have $\underline{\Omega}^p_Y \simeq \Omega^p_Y$, while E being a simple normal crossing divisor implies that $\underline{\Omega}^p_E \simeq \Omega^p_Y/\Omega^p_Y(\log E)(-E)$ (see [**PS08**]), hence
$$\mathrm{cone}(v \circ u) \simeq \mathbf{R}\pi_*\Omega^p_Y(\log E)(-E)[1].$$
Since $\mathrm{cone}(v) \simeq \underline{\Omega}^p_X[1]$ by (6.1), we obtain the assertion in the proposition by suitably rotating the exact triangle (6.2). □

REMARK 6.2. We will apply Proposition 6.1 by choosing an embedding $Z \hookrightarrow X$, with X smooth, in which case we have $\underline{\Omega}^p_X \simeq \Omega^p_X$. Another way to apply it is by starting with a variety X and taking Z to be the singular locus of X. Arguing by induction on $\dim(X)$, we thus deduce from Proposition 6.1 that $\underline{\Omega}^p_X = 0$ unless $0 \leq p \leq \dim(X)$. Moreover, if X is irreducible and n-dimensional, then $\underline{\Omega}^n_X$ is isomorphic to the *Grauert-Riemenschneider canonical sheaf* $\omega^{\mathrm{GR}}_X = \pi_*(\omega_Y)$, where $\pi\colon Y \to X$ is a resolution of singularities (note that $R^i\pi_*(\omega_Y) = 0$ for $i > 0$ by Grauert-Riemenschneider vanishing).

6.1.2. The connection to the Hodge filtration on local cohomology. Let X be a smooth, irreducible, n-dimensional complex algebraic variety, Z a closed subvariety of X, and $i\colon Z \hookrightarrow X$ the corresponding closed immersion. We consider a log resolution $\pi\colon Y \to X$ of (X, Z) that is an isomorphism over $X \smallsetminus Z$ and let $E = \pi^{-1}(Z)_{\mathrm{red}}$. In this case, Steenbrink's description dualizes to the following

PROPOSITION 6.3. With the above notation, for every p, we have an exact triangle

$$\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(\underline{\Omega}^p_Z, \omega_X) \to \Omega^{n-p}_X \xrightarrow{u} \mathbf{R}\pi_*\Omega^{n-p}_Y(\log E) \xrightarrow{+1},$$

with u being the canonical morphism induced by $\pi^*(\Omega^{n-p}_X) \to \Omega^{n-p}_Y \hookrightarrow \Omega^{n-p}_Y(\log E)$.

PROOF. We apply $\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(-, \omega_X)$ to the exact triangle

$$\mathbf{R}\pi_*\Omega_Y(\log E)(-E) \to \Omega^p_X \to \underline{\Omega}^p_Z \xrightarrow{+1}$$

provided by Proposition 6.1. Note that we have

$$\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(\Omega^p_X, \omega_X) \simeq (\Omega^p_X)^\vee \otimes_{\mathcal{O}_X} \omega_X \simeq \Omega^{n-p}_X,$$

and using Grothendienck duality for π, we obtain

$$\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}\big(\mathbf{R}\pi_*\Omega^p_Y(\log E)(-E), \omega_X\big) \simeq \mathbf{R}\pi_*\mathbf{R}\mathcal{H}om_{\mathcal{O}_Y}\big(\Omega^p_Y(\log E)(-E), \omega_Y\big)$$

$$\simeq \mathbf{R}\pi_*\big(\Omega^p_Y(\log E)^\vee(E) \otimes_{\mathcal{O}_Y} \omega_Y\big) \simeq \mathbf{R}\pi_*\Omega^{n-p}_Y(\log E).$$

We thus obtain the exact triangle in the proposition (we leave the identification of u as an easy exercise). □

We can now interpret the graded pieces of the Du Bois complex in terms of the graded de Rham complex of $i_*i^!\mathbf{Q}^H_X[n]$:

COROLLARY 6.4. For every p, we have an isomorphism

$$\underline{\Omega}^p_Z \simeq \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}\big(\mathrm{Gr}^F_{p-n}\mathrm{DR}_X(i_*i^!\mathbf{Q}^H_X[n]), \omega_X\big)[p]$$

in $D^b_{\mathrm{coh}}(X)$.

PROOF. Note that if we define $\mathbf{D} = \mathbf{R}\mathcal{H}om(-\omega_X)\colon D^b_{\mathrm{coh}}(X) \to D^b_{\mathrm{coh}}(X)$, then $\mathbf{D} \circ \mathbf{D} \simeq \mathrm{Id}$. Therefore it is enough to show that

$$\mathbf{D}(\underline{\Omega}^p_Z)[p] \simeq \mathrm{Gr}^F_{p-n}\mathrm{DR}_X\big(i_*i^!\mathbf{Q}^H_X[n]\big). \tag{6.3}$$

The exact triangle

$$i_*i^!\mathbf{Q}^H_X[n] \to \mathbf{Q}^H_X[n] \to j_*\mathbf{Q}^H_U[n] \xrightarrow{+1}$$

in $D^b\big(\mathrm{MHM}(X)\big)$ induces an exact triangle

$$\mathrm{Gr}^F_{p-n}\mathrm{DR}_X\big(i_*i^!\mathbf{Q}^H_X[n]\big) \to \mathrm{Gr}^F_{p-n}\mathrm{DR}_X\big(\mathbf{Q}^H_X[n]\big) \xrightarrow{v} \mathrm{Gr}^F_{p-n}\mathrm{DR}_X\big(j_*\mathbf{Q}^H_U[n]\big) \xrightarrow{+1}$$

in $D^b_{\rm coh}(X)$. By Proposition 5.24, we have isomorphisms

$$\mathrm{Gr}^F_{p-n}\mathrm{DR}_X\big(\mathbf{Q}^H_X[n]\big) \simeq \Omega^{n-p}_X[p] \quad \text{and} \quad \mathrm{Gr}^F_{p-n}\mathrm{DR}_X\big(j_*\mathbf{Q}^H_U[n]\big) \simeq \mathbf{R}\pi_*\Omega^{n-p}_Y(\log E)[p].$$

We leave it as an exercise to check that $v = u[p]$, where u is the map in Proposition 6.3. The assertion in the corollary thus follows from the proposition, using the fact that two cones of a morphism in the derived category are isomorphic. □

Remark 6.5. The assertion in Corollary 6.4 admits a better formulation in terms of $\mathbf{Q}^H_Z$: using the compatibility of duality with the graded de Rham functor (see (4.2)) and the fact that

$$\mathbf{D}\big(i_*i^!\mathbf{Q}^H_X[n]\big) \simeq i_*i^*\mathbf{D}\big(\mathbf{Q}^H_X[n]\big) \simeq i_*\mathbf{Q}^H_Z[n](n),$$

we see that

$$\underline{\Omega}^p_Z \simeq \mathrm{Gr}^F_{n-p}\mathrm{DR}_X\big(\mathbf{D}(i_*i^!\mathbf{Q}^H_X[n])\big)[p-n] \simeq \mathrm{Gr}^F_{-p}(\mathbf{Q}^H_Z)[p]. \tag{6.4}$$

Remark 6.6. It is worth pointing out that while our approach to the isomorphism in Corollary 6.4 and its reformulation in (6.4) is rather elementary, it has two disadvantages. On one hand, the isomorphism is not clearly canonical, since it was obtained as an isomorphism between two cones of the same morphism in the derived category. On the other hand, the objects on both sides of (6.4) lie in $D^b_{\rm coh}(Z)$ and one could expect this isomorphism to already hold in this derived category. In fact, it is true that the isomorphism (6.4) is canonical and holds in $D^b_{\rm coh}(Z)$: this was proved, via more sophisticated methods, by Saito in [**Sai00**, Theorem 0.2].

6.2. Higher Du Bois singularities

Recall that a variety Z has *Du Bois singularities* if the canonical morphism $\mathcal{O}_Z \to \underline{\Omega}^0_Z$ is an isomorphism. By analogy with this, for any $p \in \mathbf{Z}_{\geq 0}$, we say that Z has *p-Du Bois singularities* if the canonical morphism

$$\Omega^q_Z \to \underline{\Omega}^q_Z \tag{6.5}$$

is an isomorphism for all $0 \leq q \leq p$. The following is the main result of this section.

Theorem 6.7. *Let X be a smooth, irreducible, n-dimensional complex algebraic variety and let Z be a reduced hypersurface in X. For every nonnegative ineteger $p \geq 0$, we have $\widetilde{\alpha}(Z) \geq p+1$ if and only if Z has p-Du Bois singularities.*

The "only if" part in the above statement was proved in [**MOPW23**] and the "if" part was proved in [**JKSY22**].

Proof of Theorem 6.7. Note that since Z is a hypersurface in X, we have $\mathcal{H}^i_Z(\mathcal{O}_X) = 0$ for all $i \neq 1$, hence the formula in Corollary 6.4 becomes

$$\underline{\Omega}^q_Z \simeq \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}\big(\mathrm{Gr}^F_{q-n}\mathrm{DR}_X(\mathcal{H}^1_Z(\mathcal{O}_X)), \omega_X\big)[q+1] \quad \text{in} \quad D^b_{\rm coh}(X). \tag{6.6}$$

We argue by induction on p, hence we may and will assume that $\widetilde{\alpha}(Z) \geq p$ and (6.5) is an isomorphism for $q \leq p-1$, and we need to show that $\widetilde{\alpha}(Z) \geq p+1$ if and only if (6.5) is an isomorphism also for $q = p$.

We may and will assume that Z is singular, since otherwise the assertion in the theorem is trivial. Note that in this case $\widetilde{\alpha}(Z) \leq \frac{n}{2}$ by Proposition 5.45, hence our induction assumption implies $p \leq \frac{n}{2}$.

Recall that by Proposition 5.6, after passing from $\mathcal{O}_X(*Z)$ to $\mathcal{H}^1_Z(\mathcal{O}_X)$, we have

$$F_\bullet\mathcal{H}^1_Z(\mathcal{O}_X) \subseteq P_\bullet\mathcal{H}^1_Z(\mathcal{O}_X), \tag{6.7}$$

where $P_k\mathcal{H}^1_Z(\mathcal{O}_X) = P_k\mathcal{O}_X(*Z)/\mathcal{O}_X$ for all k. In what follows, we also consider the graded de Rham complexes $\mathrm{Gr}^P_{k-n}\mathrm{DR}_X\big(\mathcal{H}^1_Z(\mathcal{O}_X)\big)$ with respect to the pole order filtration on $\mathcal{H}^1_Z(\mathcal{O}_X)$. The inclusion (6.7) implies that for every k, we have a morphism of complexes

$$\varphi_k\colon \mathrm{Gr}^F_{k-n}\mathrm{DR}_X\big(\mathcal{H}^1_Z(\mathcal{O}_X)\big) \to \mathrm{Gr}^P_{k-n}\mathrm{DR}_X\big(\mathcal{H}^1_Z(\mathcal{O}_X)\big).$$

By Corollary 5.13, our inductive assumption implies that φ_q is an isomorphism for all $q \leq p-1$ and φ_p is an injective morphism of complexes, whose cokernel only lives in cohomological degree 0. Moreover, $\widetilde{\alpha}(Z) \geq p+1$ if and only if φ_p is an isomorphism of complexes or, equivalently, gives an isomorphism in $D^b_{\mathrm{coh}}(X)$. Since $\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(-,\omega_X)$ is an antiequivalence, we see that φ_p is an isomorphism in $D^b_{\mathrm{coh}}(X)$ if and only if

$$\psi_p\colon \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}\big(\mathrm{Gr}^P_{p-n}\mathrm{DR}_X\big(\mathcal{H}^1_Z(\mathcal{O}_X)\big),\omega_X\big) \to \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}\big(\mathrm{Gr}^F_{p-n}\mathrm{DR}_X(\mathcal{H}^1_Z(\mathcal{O}_X)),\omega_X\big)$$

is an isomorphism. By (6.6), the target of $\psi_p[p+1]$ is isomorphic to $\underline{\Omega}^p_Z$.

We next describe the source of $\psi_p[p+1]$. It follows from the definition of the pole order filtration that $\mathrm{Gr}^P_{p-n}\mathrm{DR}_X\big(\mathcal{H}^1_Z(\mathcal{O}_X)\big)$ is the complex

$$C^\bullet:\ 0 \to \Omega_X^{n-p}\otimes_{\mathcal{O}_X}\mathcal{O}_Z(Z) \to \ldots \to \Omega_X^{n-1}\otimes_{\mathcal{O}_X}\mathcal{O}_Z(pZ) \to \Omega_X^n\otimes_{\mathcal{O}_X}\mathcal{O}_Z\big((p+1)Z\big) \to 0,$$

placed in cohomological degrees $-p,\ldots,0$. By relative duality for the inclusion $Z \hookrightarrow X$, we have

$$\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(C^\bullet,\omega_X) \simeq \mathbf{R}\mathcal{H}om_{\mathcal{O}_Z}(C^\bullet,\omega_Z)[-1].$$

Using the fact that all terms of $C^\bullet$ are locally free $\mathcal{O}_Z$-modules, and that

$$(\Omega_X^{n-i})^\vee \otimes_{\mathcal{O}_X} \mathcal{O}_Z\big(-(p+1-i)Z\big) \otimes_{\mathcal{O}_Z} \omega_Z \simeq \Omega^i_X|_Z \otimes_{\mathcal{O}_Z} \mathcal{O}_Z\big(-(p-i)Z\big)$$

for all i, we see that

$$\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}\big(\mathrm{Gr}^P_{p-n}\mathrm{DR}_X\big(\mathcal{H}^1_Z(\mathcal{O}_X)\big),\omega_X\big)[p+1]$$

is represented by the complex

$$B^\bullet:\ 0 \to \mathcal{O}_Z(-pZ) \to \Omega^1_X|_Z \otimes \mathcal{O}_Z\big(-(p-1)Z\big) \to \ldots \to \Omega^p_X|_Z \to 0, \tag{6.8}$$

placed in cohomological degrees $-p,\ldots,0$. Moreover, the maps in the complex are easy to identify: this complex is the truncation of the twist by $\mathcal{O}_Z(-pZ)$ of the Koszul complex associated to the canonical morphism

$$h\colon \mathcal{O}_Z(-Z) \to \Omega^1_X|_Z.$$

We note that since $\mathrm{Coker}(h) \simeq \Omega^1_Z$, it follows that $\mathcal{H}^0(B^\bullet) \simeq \Omega^p_Z$. On the other hand, the characterization of depth via Koszul homology (see [**Mat89**, Theorem 16.8]) implies that $\mathcal{H}^i(B^\bullet) = 0$ for all $i<0$ if and only if $\mathrm{depth}\big(I(h)\big) \geq p$, where $I(h)$ defines the zero-locus of h. Since Z is Cohen-Macaulay and the subset defined by $I(h)$ is the singular locus Z_{sing} of Z, it follows that this condition holds if and only if $\mathrm{codim}_Z(Z_{\mathrm{sing}}) \geq p$. However, this condition holds by our inductive assumption: if $s = \dim(Z_{\mathrm{sing}})$, then $s \leq n-2$ since Z is reduced, and the condition $\widetilde{\alpha}(Z) \geq p$ implies via Proposition 5.45 $p \leq \frac{n-s}{2}$ and thus

$$\mathrm{codim}_Z(Z_{\mathrm{sing}}) = n-s-1 \geq \tfrac{n-s}{2} \geq p.$$

Since $\mathcal{H}^i(\underline{\Omega}^p_Z) = 0$ for $i<0$, we thus conclude that ψ_p is an isomorphism if and only if $\mathcal{H}^i(\underline{\Omega}^p_Z) = 0$ for all $i>0$ and the induced morphism

$$\Omega^p_Z \simeq \mathcal{H}^0(B^\bullet) \to \mathcal{H}^0(\underline{\Omega}^p_Z)$$

is an isomorphism. However, note that this is morphism is the one associated to the canonical map (6.5) for $q = p$. Since $\mathcal{H}^0(\underline{\Omega}^p_Z)$ is always torsion-free (this follows, for example, from the description in [**HJ14**, Theorem 7.12]), we see that it is enough to show that this identification holds on the smooth locus of Z. We leave checking this as an exercise for the reader.

Since we have seen that, under our inductive assumption, $\widetilde{\alpha}(Z) \geq p+1$ if and only if ψ_p is an isomorphism, which in turn holds if and only if the morphism (6.5) is an isomorphism for $q = p$, this completes the proof of the theorem. □

6.3. A nonvanishing result for $\mathcal{H}^i(\underline{\Omega}^p_Z)$

Our goal in this section is to show, following [**MOPW23**], that for singular hypersurfaces Z with large minimal exponent (more precisely, with minimal exponent > 2), certain higher cohomology sheaves of suitable $\underline{\Omega}^p_Z$ are nonzero.

THEOREM 6.8. *Let X be a smooth, irreducible, n-dimensional complex algebraic variety and Z a reduced hypersurface in X, defined by $f \in \mathcal{O}_X(X)$. If $p \geq 2$ is an integer such that $\widetilde{\alpha}(Z) > p$, then for every singular point $x \in Z$, the following hold:*

i) *We have an isomorphism $\mathcal{H}^{p-1}(\underline{\Omega}^{n-p}_Z)_x \simeq \mathcal{O}_{Z,x}/J_f\mathcal{O}_{Z,x}$, where J_f is the Jacobian ideal of f. In particular, $\mathcal{H}^{p-1}(\underline{\Omega}^{n-p}_Z)_x \neq 0$.*

ii) *If x is an isolated singularity of Z and $p \geq 3$, then*

$$\mathcal{H}^{p-2}(\underline{\Omega}^{n-p}_Z)_x \simeq (J_f : f)/J_f$$

(hence, in particular, $\mathcal{H}^{p-2}(\underline{\Omega}^{n-p}_Z)_x \neq 0$), while $\mathcal{H}^i(\underline{\Omega}^{n-p}_Z)_x = 0$ for $0 < i < p-2$.

REMARK 6.9. We will see in the proof of the theorem that, without any assumptions on Z, we have $\mathcal{H}^i(\underline{\Omega}^{n-p}_Z) = 0$ for all $i \geq p$.

PROOF OF THEOREM 6.8. The main idea is to use duality and the computation via the V-filtration. Since Z is a hypersurface, hence $\mathcal{H}^j_Z(\mathcal{O}_X) = 0$ for all $j \neq 1$, Corollary 6.4 gves

$$\underline{\Omega}^{n-p}_Z \simeq \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}\big(\mathrm{Gr}^F_{-p}\mathrm{DR}_X(\mathcal{H}^1_Z(\mathcal{O}_X), \omega_X)[n-p+1] \quad \text{in} \quad D^b_{\mathrm{coh}}(X).$$

By the compatibility between duality and the graded de Rham complex (see (4.2), this becomes

$$\underline{\Omega}^{n-p}_Z \simeq \mathrm{Gr}^F_p\mathrm{DR}_X\mathbf{D}_X\big(\mathcal{H}^1_Z(\mathcal{O}_X)\big)[1-p] \quad \text{in} \quad D^b_{\mathrm{coh}}(X),$$

hence

$$\mathcal{H}^{p-j}(\underline{\Omega}^{n-p}_Z) \simeq \mathcal{H}^{1-j}\big(\mathrm{Gr}^F_p\mathrm{DR}_X\mathbf{D}_X(\mathcal{H}^1_Z(\mathcal{O}_X))\big) \quad \text{for all} \quad j \in \mathbf{Z}. \tag{6.9}$$

Note that since the de Rham complex of a filtered $\mathcal{D}_X$-module is concentrated in nonpositive degrees, it follows that $\mathcal{H}^i(\underline{\Omega}^{n-p}_Z) = 0$ for $i \geq p$.

Recall now that if $\iota\colon X \hookrightarrow X \times \mathbf{A}^1$ is the graph embedding corresponding to f, then we have a short exact sequence of filtered $\mathcal{D}_X$-modules:

$$0 \to \mathrm{Gr}^0_V\big(\iota_+(\mathcal{O}_X)\big)(-1) \xrightarrow{t\cdot} \mathrm{Gr}^1_V\big(\iota_+(\mathcal{O}_X)\big)(-1) \to \mathcal{H}^1_Z(\mathcal{O}_X) \to 0, \tag{6.10}$$

see (4.13). Since $\mathbf{Q}^H_X[n]$ is pure of weight n, the compatibility of vanishing and nearby cycles with duality in (4.6) gives isomorphisms

$$\mathbf{D}_X\big(\mathrm{Gr}^1_V(\iota_+(\mathcal{O}_X))\big) \simeq \mathrm{Gr}^1_V\big(\iota_+(\mathcal{O}_X)\big)(n-1) \quad \text{and}$$
$$\mathbf{D}_X\big(\mathrm{Gr}^0_V(\iota_+(\mathcal{O}_X))\big) \simeq \mathrm{Gr}^0_V\big(\iota_+(\mathcal{O}_X)\big)(n-2),$$

such that applying $\mathbf{D}_X$ to (6.11), we get a short exact sequence of filtered $\mathcal{D}_X$-modules:

$$0 \to \mathbf{D}_X\big(\mathcal{H}^1_Z(\mathcal{O}_X)\big) \to \mathrm{Gr}^1_V\big(\iota_+(\mathcal{O}_X)\big)(n) \xrightarrow{\partial_t \cdot} \mathrm{Gr}^0_V\big(\iota_+(\mathcal{O}_X)\big)(n-1) \to 0 \tag{6.11}$$

and corresponding short exact sequences of graded de Rham complexes.

We now use the hypothesis: the condition $\widetilde{\alpha}(Z) > p$ is equivalent, by Theorem 3.58, to $\partial_t^p \delta \in V^{>0}\iota_+(\mathcal{O}_X)$ (which implies $F_{p+1}\iota_+(\mathcal{O}_X) \subseteq V^{>0}\iota_+(\mathcal{O}_X)$ and $F_p\iota_+(\mathcal{O}_X) \subseteq V^1\iota_+(\mathcal{O}_X)$). First, this implies that $\mathrm{Gr}^F_i \mathrm{Gr}^0_V\big(\iota_+(\mathcal{O}_X)\big) = 0$ for $i \le p+1$. Second, for $1 \le i \le p$, we have $F_iV^1\iota_+(\mathcal{O}_X) = F_i\iota_+(\mathcal{O}_X)$ and

$$F_iV^{>1}\iota_+(\mathcal{O}_X) = t \cdot F_iV^{>0}\iota_+(\mathcal{O}_X) = t \cdot F_i\iota_+(\mathcal{O}_X),$$

hence

$$\begin{aligned}\mathrm{Gr}^F_i \mathrm{Gr}^1_V\big(\iota_+(\mathcal{O}_X)\big) &= F_iV^1\iota_+(\mathcal{O}_X)/\big(F_{i-1}V^1\iota_+(\mathcal{O}_X) + F_iV^{>1}\iota_+(\mathcal{O}_X)\big)\\ &= F_i\iota_+(\mathcal{O}_X)/\big(F_{i-1}\iota_+(\mathcal{O}_X) + t \cdot F_i\iota_+(\mathcal{O}_X) = \mathcal{O}_Z \cdot \overline{\partial_t^{i-1}\delta} \simeq \mathcal{O}_Z.\end{aligned}$$

The terms in the complex

$$\mathrm{Gr}^F_p \mathrm{DR}_X\big(\mathrm{Gr}^0_V(\iota_+(\mathcal{O}_X))(n-1)\big) = \mathrm{Gr}^F_{p-n+1}\mathrm{DR}_X\big(\mathrm{Gr}^0_V(\iota_+(\mathcal{O}_X))\big)$$

involve $\mathrm{Gr}^F_q \mathrm{Gr}^0_V\iota_+(\mathcal{O}_X)$ for $q \le p+1$, that are all 0, hence this is the 0 complex. We thus conclude that

$$\mathrm{Gr}^F_p \mathrm{DR}_X\big(\mathbf{D}_X(\mathcal{H}^1_Z(\mathcal{O}_X))\big) \simeq \mathrm{Gr}^F_{p-n}\mathrm{DR}_X\big(\mathrm{Gr}^1_V(\iota_+(\mathcal{O}_X))\big). \tag{6.12}$$

Moreover, the complex on the right-hand side of (6.12) is given by

$$0 \to \Omega_Z^{n-p+1} \otimes_{\mathcal{O}_X} \mathcal{O}_Z \to \ldots \to \Omega_X^{n-1} \otimes_{\mathcal{O}_X} \mathcal{O}_Z \to \omega_X \otimes \mathcal{O}_Z \to 0,$$

placed in cohomological degrees $-(p-1), \ldots, 0$. Furthermore, if we choose local coordinates $x_1, \ldots, x_n$ in a neighborhood of P, then we clearly have

$$\partial_{x_j} \cdot \overline{\partial_t^{i-1}\delta} = -\tfrac{\partial f}{\partial x_j}\overline{\partial_t^i \delta} \quad \text{for} \quad i \le p-1,$$

hence $\mathrm{Gr}^F_{p-n}\mathrm{DR}_X\big(\mathrm{Gr}^1_V(\iota_+(\mathcal{O}_X))\big)$ is isomorphic to the "stupid" truncation $\sigma^{\ge -p+1}$ of the Koszul complez on Z associated to the sequence $-\frac{\partial f}{\partial x_1}, \ldots, -\frac{\partial f}{\partial x_n}$. The assertion in part i) of the theorem follows from this and (6.9).

Suppose now that $p \ge 3$ and $x \in Z$ is an isolated singular point. In this case, after possibly replacing X with a neighborhood of P, we may assume that $-\frac{\partial f}{\partial x_1}, \ldots, -\frac{\partial f}{\partial x_n}$ form a regular sequence on X. In this case, the corresponding Koszul complex on X is acyclic, and thus (6.9) gives

$$\mathcal{H}^{p-j}(\underline{\Omega}^{n-p}_Z) \simeq \mathcal{T}or^{\mathcal{O}_X}_{j-1}(\mathcal{O}_Z, \mathcal{O}_X/J_f) \quad \text{for} \quad 2 \le j \le p-1,$$

and the assertions in ii) are immediate consequences. □

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