Lecture 13. Connections between eigenvalues and properties of graphs

October 22, 2020

Review: the eigenvalues associated to a graph

Review from the last time: if G is a finite simple graph, with vertices labeled $1, \ldots, n$, we have two important matrices associated to G:

- The adjacency matrix $A_G = (a_{ij}) \in M_n(\mathbf{Z})$, with $a_{ij} = 1$ if $i \neq j$ and $\{i, j\} \in E(G)$, and $a_{ij} = 0$, otherwise.
- The Laplacian $L_G = (L_{ij}) \in M_n(\mathbf{Z})$, with $L_{ii} = \deg(i)$ and $L_{ij} = -a_{ij}$ for $i \neq j$.

Both A_G and L_G are symmetric matrices, hence all their eigenvalues are real.

We order them as $\mu_1 \geq \ldots \geq \mu_n$ for A_G and $\lambda_1 \leq \ldots \leq \lambda_n$ for L_G.

Review: the eigenvalues associated to a graph

Here are some easy properties that we have already seen:

1) We have $\mu_1 + \ldots + \mu_n = \mathrm{trace}(A_G) = 0$.

2) We have $\mu_1 = 0$ if and only if $E(G) = \emptyset$.

3) If G is connected, then the following hold: $\mu_1 \geq |\mu_i|$ for all i and μ_1 has multiplicity 1 as a root of the characteristic polynomial. Moreover, we have an eigenvector for μ_1 with positive entries.

The largest eigenvalue of A_G

Let $\deg_{\text{ave}}(G)$ be the average degree of the vertices of G, that is

$$d_{\text{ave}}(G) = \frac{\sum_{i=1}^{n} \deg(i)}{n} = \frac{2 \cdot \#E(G)}{\#V(G)}.$$

We also consider the maximal degree $\Delta(G) = \max_i \deg(i)$.

Proposition 1. For every finite simple graph G, we have $d_{\text{ave}}(G) \leq \mu_1 \leq \Delta(G)$. In particular, if G is d-regular, then $\mu_1 = d$.

Proof. We have seen last time that $\mu_1 = \max_{x \neq 0} \frac{\langle A_G x, x \rangle}{\langle x, x \rangle}$. If $\mathbf{1} = (1, \ldots, 1)$, then

$$\mu_1 \geq \frac{\langle A_G \mathbf{1}, \mathbf{1} \rangle}{\langle \mathbf{1}, \mathbf{1} \rangle} = \frac{\sum_{i=1}^{n} \deg(i)}{n} = d_{\text{ave}}(G).$$

This proves the first inequality.

The largest eigenvalue of A_G, cont'd

For the second inequality, choose an eigenvector $x = (x_1, \ldots, x_n)$ for μ_1. Say $x_i = \max_j x_j$. After possibly replacing x by $-x$, we may assume that $x_i > 0$. We thus have

$$\mu_1 = \frac{\sum_{j=1}^{n} a_{ij} x_j}{x_i} = \frac{\sum_{\{i,j\} \in E(G)} x_j}{x_i} \leq \deg(i) \leq \Delta(G).$$

This completes the proof of the proposition.

Remark. Given a subset S of $V(G)$, we denote by $d_{\mathrm{ave}}(S)$ the average degree of the subgraph of G spanned by S. In this case, we also have $\mu_1 \geq d_{\mathrm{ave}}(S)$.
Indeed, the incidence matrix of the subgraph H spanned by S is a symmetric submatrix of A_G. Interlacing implies that μ_1 is at least as large as the largest eigenvalue of A_H and we obtain the assertion applying the proposition to H.

The largest eigenvalue of A_G, cont'd

Proposition 2. If G is connected and $\mu_1 = \Delta(G)$, then G is a regular graph.

Proof. As in the proof of the second assertion in Proposition 1, consider an eigenvector $x = (x_1, \ldots, x_n)$ for μ_1, with

$$x_i = \max_j x_j > 0.$$

Since $\mu_1 = \Delta(G)$, the inequalities in

$$\mu_1 = \frac{\sum_{j=1}^n a_{ij} x_j}{x_i} = \frac{\sum_{\{i,j\} \in E(G)} x_j}{x_i} \leq \deg(i) \leq \Delta(G)$$

become equalities. This implies $\deg(i) = \Delta(G)$ and $x_j = x_i$ if $\{i,j\} \in E(G)$. We can then run the same argument with i replaced by j. Repeating, since G is connected, we obtain that $\deg(\ell) = \Delta(G)$ for all ℓ.

Eigenvalues and the chromatic number: Wilf's theorem

Recall that the chromatic number $\chi(G)$ is the smallest number of colors we can use to color the vertices of G such that no two vertices of the same color are joined by an edge. For a real number α, we denote by $\lfloor \alpha \rfloor$ the largest integer $\leq \alpha$.

Theorem 1 (Wilf). For a finite simple graph G, we have

$$\chi(G) \leq \lfloor \mu_1 \rfloor + 1.$$

Proof. It is enough to show that we can order the vertices of G such that every vertex has $\leq \mu_1$ edges joining it with vertices that come before him. Indeed, in this case we can color the vertices of G in this order with $\lfloor \mu_1 \rfloor + 1$ colors, making sure that when we color the mth vertex, we don't use the same color as for its neighbors that come before him.

Wilf's theorem, cont'd

In fact, we proceed backwards. Let $n = \#V(G)$. Since we know that $d_{\text{ave}}(G) \leq \mu_1$ by Prop. 1, it follows that there is a vertex v_n of G with $\deg_G(v_n) \leq \mu_1$. This will be our last vertex.

We now consider the subgraph G_{n-1} spanned by the other $n-1$ vertices. By the remark following Prop. 1, we know that $d_{\text{ave}}(G_{n-1}) \leq \mu_1$, hence there is a vertex v_{n-1} of G_{n-1} such that $\deg_{G_{n-1}}(v_{n-1}) \leq \mu_1$.
We continue in this way to choose the vertices $v_{n-2}, \ldots, v_1$. It is clear from the construction that each v_i has $\leq \mu_1$ neighbors among $v_1, \ldots, v_{i-1}$. This completes the proof of Wilf's theorem.

Eigenvalues and the chromatic number: Hoffman's theorem

A more interesting result is the following one, given a lower bound for the chromatic number in terms of spectral information.

Theorem 2 (Hoffman). If G is a finite simple graph on n vertices, with $E(G) \neq \emptyset$, then

$$\chi(G) \geq 1 + \frac{\mu_1}{-\mu_n}.$$

Note that since $\mu_1 + \ldots + \mu_n = 0$, we always have $\mu_n \leq 0$. As we discussed last time, the condition $E(G) \neq \emptyset$ implies that $\mu_n < 0$.

Remark. The bound in Hoffman's theorem is sharp. Recall that for the complete graph K_n, we computed the characteristic polynomial of A_{K_n}: it is $(x+1)^{n-1}\big(x-(n-1)\big)$. Therefore $\mu_1 = n-1$ and $\mu_n = -1$. We thus have

$$\chi(K_n) = n = 1 + \frac{\mu_1}{-\mu_n}.$$

Proof of Hoffman's theorem

Proof of Theorem 2. Consider a coloring of $V(G)$ with $k = \chi(G)$ colors $1, \ldots, k$ and let m_i be the number of vertices colored with color i. After suitably ordering the vertices, we can write the adjacency matrix as

$$A_G = \begin{bmatrix} 0 & A_{12} & \ldots & A_{1k} \\ A_{21} & 0 & \ldots & A_{2k} \\ \ldots & \ldots & \ldots & \ldots \\ A_{k1} & A_{k2} & \ldots & 0 \end{bmatrix}$$

where each A_{ij} is an $m_i \times m_j$ matrix. Let v be an eigenvector corresponding to μ_1, written as $v = (v_1, \ldots, v_k)$, with $v_i \in \mathbf{R}^{m_i}$. Let $w_i = (\| v_i \|, 0, \ldots, 0) \in \mathbf{R}^{m_i}$ and $w = (w_1, \ldots, w_k) \in \mathbf{R}^n$.

Proof of Hoffman's theorem, cont'd

Let B_i be an orthogonal $m_i \times m_i$ matrix such that $B_i w_i = v_i$ and let

$$B = \begin{bmatrix} B_1 & 0 & \dots & 0 \\ 0 & B_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & B_k \end{bmatrix}$$

so that $Bw = v$. We thus have

$$B^{-1} A_G B w = B^{-1} A_G v = \mu_1 B^{-1} v = \mu_1 w,$$

hence w is an eigenvector of $B^{-1} A_G B$ with eigenvalue μ_1.

Note that

$$B^{-1} A_G B = \begin{bmatrix} 0 & B_1^{-1} A_{12} B_2 & \dots & B_1^{-1} A_{1k} B_k \\ B_2^{-1} A_{21} B_1 & 0 & \dots & B_2^{-1} A_{2k} B_k \\ \dots & \dots & \dots & \dots \\ B_k^{-1} A_{k1} B_1 & B_k^{-1} A_{k2} B_2 & \dots & 0 \end{bmatrix}$$

Proof of Hoffman's theorem, cont'd

Since each B_i is orthogonal, B_i^{-1} is the transpose of B_i, and we see that $B^{-1}A_GB$ is a symmetric matrix. For each of the k^2 matrices $B_i^{-1}A_{ij}B_j$ (we here take $A_{ii} = 0$), we consider the top-left entry. In this way we get a symmetric $k \times k$ submatrix D.

Let $u = (\| v_1 \|, \ldots, \| v_k \|)$. Note that u is an eigenvector of D, with eigenvalue μ_1. Indeed, w is an eigenvector of $B^{-1}A_GB$, with eigenvalue μ_1, and w has entries 0 on the places corresponding to the rows and columns of $B^{-1}A_GB$ that are deleted to get D.

Let $\alpha_1 \geq \ldots \geq \alpha_k$ be the eigenvalues of D. We have seen that μ_1 is an eigenvalue of D, hence $\alpha_1 \geq \mu_1$. On the other hand, interlacing implies $\mu_1 \geq \alpha_1$, hence $\mu_1 = \alpha_1$.

Proof of Hoffman's theorem, cont'd

On the other hand, D has 0 on the main diagonal, hence $\alpha_1 + \ldots + \alpha_k = \mathrm{trace}(D) = 0$.

Using interlacing one more time, we obtain

$$\mu_n \leq \alpha_k, \quad \mu_{n-1} \leq \alpha_{k-1}, \ldots, \mu_{n-k+2} \leq \alpha_2.$$

We thus conclude that

$$(k-1)\mu_n \leq \mu_n + \ldots + \mu_{n-k+2} \leq \alpha_k + \ldots + \alpha_2 = -\alpha_1 = -\mu_1.$$

Since $\mu_n < 0$, we deduce $k - 1 \geq \frac{\mu_1}{-\mu_n}$, completing the proof.

Bipartite graphs and eigenvalues

Remark. Recall that a graph G with $E(G) \neq \emptyset$ is bipartite if and only if $\chi(G) = 2$. In this case the theorem implies $-\mu_n \geq \mu_1$. On the other hand, we have seen that if G is connected, then $\mu_1 \geq |\mu_n|$. We thus conclude that if G is bipartite and connected, then $\mu_1 = -\mu_n$.

Example. Recall that we saw in the previous lecture that for the complete bipartite graph $K_{m,n}$ the characteristic polynomial of the adjacency matrix is $x^{m+n-2}(x - \sqrt{mn})(x + \sqrt{mn})$.

We next give a different proof of the assertion in the above remark that in fact shows more and also proves the converse statement. In particular, we obtain a necessary and sufficient criterion for a connected graph to be bipartite in terms of its eigenvalues.

Bipartite graphs and eigenvalues, cont'd

Proposition 3. If G is a connected graph with n vertices, then the following are equivalent:

i) G is bipartite.

ii) We have $\mu_i = -\mu_{n+1-i}$ for $1 \leq i \leq n$.

iii) We have $\mu_1 = -\mu_n$.

Remark. Note that if $P(x) = \prod_{i=1}^{n}(x - \mu_i)$ is the characteristic polynomial of A_G, then

$$P(-x) = (-1)^n(x + \mu_i),$$

hence the assertion in ii) is equivalent with the fact that $P(-x) = (-1)^n P(x)$.

Proof of the proposition. We may assume $n \geq 2$. To prove i)⇒ii), after reordering the vertices, we may assume that we have $1 \leq m \leq n-1$ such that if $A_G = (a_{ij})$, then $a_{ij} = 0$ if $i, j \leq m$ or if $i, j > m$.

Bipartite graphs and eigenvalues, cont'd

Consider the linear transformation $\varphi\colon \mathbf{R}^n \to \mathbf{R}^n$ given by

$$\varphi(u_1, \ldots, u_n) = (u_1, \ldots, u_m, -u_{m+1}, \ldots, -u_n).$$

For every $u = (u_1, \ldots, u_n) \in \mathbf{R}^n$, we have

$$A_G \circ \varphi(u) = \left(\sum_{j \le m} a_{ij} u_j - \sum_{j > m} a_{ij} u_j \right)_i =$$

$$\left(-\sum_{j>m} a_{1j} u_j, \ldots, -\sum_{j>m} a_{mj} u_j, \sum_{j \le m} a_{m+1,j} u_j, \ldots, \sum_{j \le m} a_{nj} u_j \right) = -\varphi \circ A_G(u).$$

Therefore we have $A_G = -\varphi^{-1} \circ A_G \circ \varphi$.

Bipartite graphs and eigenvalues, cont'd

If $P(x)$ is the characteristic polynomial of A_G, we have

$$P(x) = \det(xI - A_G) = \det(xI + \varphi^{-1} A_G \varphi) = \det(xI + A_G) = (-1)^n P(-x).$$

This completes the proof of i)⇒ii).

Since the implication ii)⇒iii) is clear, to complete the proof it is enough to prove iii)⇒i). The argument is very close to an argument we discussed in the previous lecture. Suppose that $v = (v_1, \ldots, v_n) \in \mathbf{R}^n$ is an eigenvector corresponding to μ_n and let $u = (u_1, \ldots, u_n)$, with $u_i = |v_i|$. We thus have

$$|\mu_n| \langle v, v \rangle = |\langle A_G v, v \rangle| = \left| \sum_{i,j} a_{ij} v_i vj \right|$$

$$\leq \sum_{i,j} a_{ij} u_i u_j = \langle A_G u, u \rangle \leq \mu_1 \langle u, u \rangle,$$

where the last inequality follows from the description of μ_1 given by the Min-Max Theorem.

Bipartite graphs and eigenvalues, cont'd

The inequalities on the previous slide lead to

$$|\mu_n|\langle v, v\rangle \leq \mu_1 \cdot \langle u, u\rangle.$$

However, $\mu_1 = -\mu_n = |\mu_n|$ by hypothesis and $\langle v, v\rangle = \langle u, u\rangle$ by the definition of u. Hence the above inequality is an equality. First, this gives

$$\left|\sum_{i,j} a_{ij} v_i v_j\right| = \sum_{i,j} a_{ij} |v_i| \cdot |v_j|,$$

which is the case if and only if all $v_i v_j$ with $a_{ij} \neq 0$ have the same sign. Second, it gives $\langle A_G u, u\rangle = \mu_1 \langle u, u\rangle$, hence u is an eigenvector corresponding to μ_1 by an exercise given in the previous lecture. Since the entries of u are nonnegative, they are positive by a lemma from Lecture 12.

Bipartite graphs and eigenvalues, cont'd

We thus conclude that $v_i \neq 0$ for all i and all $v_i v_j$ with $a_{ij} \neq 0$ have the same sign.

Since $\sum_j a_{ij} v_j = \mu_n v_i$ for all i and $\mu_n < 0$, it follows that the above mentioned sign is negative.

We thus see that if $A = \{i \mid v_i < 0\}$ and $B = \{i \mid v_i > 0\}$, then $G = G(A, B)$ is a bipartite graph.

The second eigenvalue

So far, we have focussed on μ_1. The second eigenvalue μ_2 (more precisely, the difference between μ_1 and μ_2) turns out to be related with expansion properties of graphs. In what follows we only describe one such statement.

We will be mostly interested in the case of regular graphs. Recall that if G is d-regular, then $\mu_1 = d$, hence $\mu_1 - \mu_2 = d - \mu_2$.

The graphs one is interested in this setting are sparse (they are d-regular, for small d), but they have strong connectivity properties. The connectivity property is measured via the following notion.

Definition. A finite graph G is an ϵ-expander (for some $\epsilon > 0$) if for every $S \subseteq V(G)$, with $\#S \leq \frac{1}{2}\#V(G)$, we have

$$\#\{v \in V(G) \setminus S \mid v \text{ has a neighbor in } S\} \geq \epsilon \cdot \#S.$$

The second eigenvalue, cont'd

Example. If G is a complete graph, then G is a 1-expander.

However, in practice one is interested in d-regular ϵ-expanders, where $d \geq 3$ is a small integer. Constructing such examples is hard.

Theorem 3. Let G be a d-regular graph.

i) If $d - \mu_2 \geq 2\epsilon d$, then G is an ϵ-expander.

ii) If G is an ϵ-expander, then $d - \lambda_2 \geq \frac{\epsilon^2}{5}$.

Remark. The spectral theory of graphs that we briefly discussed can be considered as a discrete analogue of the spectral study of compact Riemannian manifolds. For example, the above theorem provides an analogue for a result of Cheeger in differential geometry.