Lecture 2: Further general notions about graphs

September 3, 2020

The adjacency matrix

Let G be a finite graph and let's label the vertices $x_1, \ldots, x_n$. The adjacency matrix $A_G = (a_{i,j})_{1 \leq i,j \leq n}$ is given by

$$a_{i,j} = \#\{\text{edges with ends } x_i \text{ and } x_j\}.$$

Remarks. 1) This completely determines G, up to isomorphism. We will discuss later information that we get on G from the eigenvalues of A_G. This is what *spectral graph theory* is about.
2) A_G is a symmetric matrix.
3) If G is a simple graph, then $a_{ii} = 0$ for all i and $a_{ij} \in \{0, 1\}$ for all i, j.

Remark. For every $m \geq 1$ and every i and j, with $1 \leq i, j \leq n$, the ij entry of the matrix A_G^m is equal to the number of walks $x_i - x_j$ of length m (HW problem).

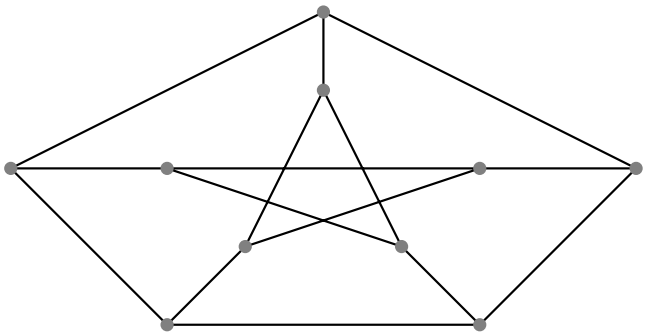
Regular graphs

Definition. A graph G is regular, of degree d (or d-regular), if every vertex x of G has degree d.

Example. The complete graph K_n is regular, of degree $n-1$.

Example. A polygon is a finite, connected graph, which is regular of degree 2. Note that a polygon with n vertices is isomorphic to P_n.

Example. The following graph, known as the Petersen graph



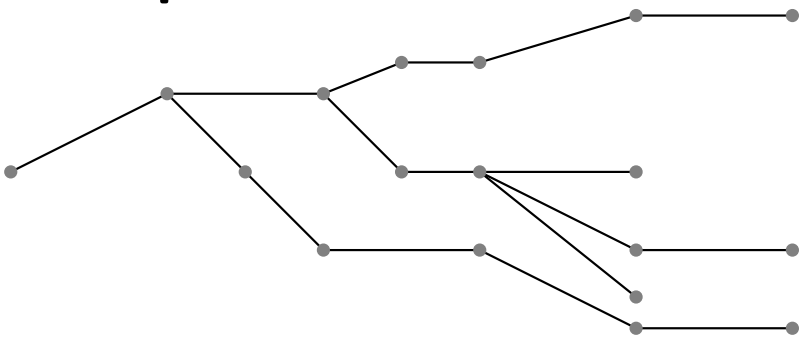
is 3-regular. It is an important example since it provides counterexamples to many questions in graph theory.

Trees

Definition. A tree is a connected graph that contains no polygons as subgraphs.

Remark. Since a tree contains no subgraphs isomorphic to P_1 or P_2, it follows that a tree is a simple graph.

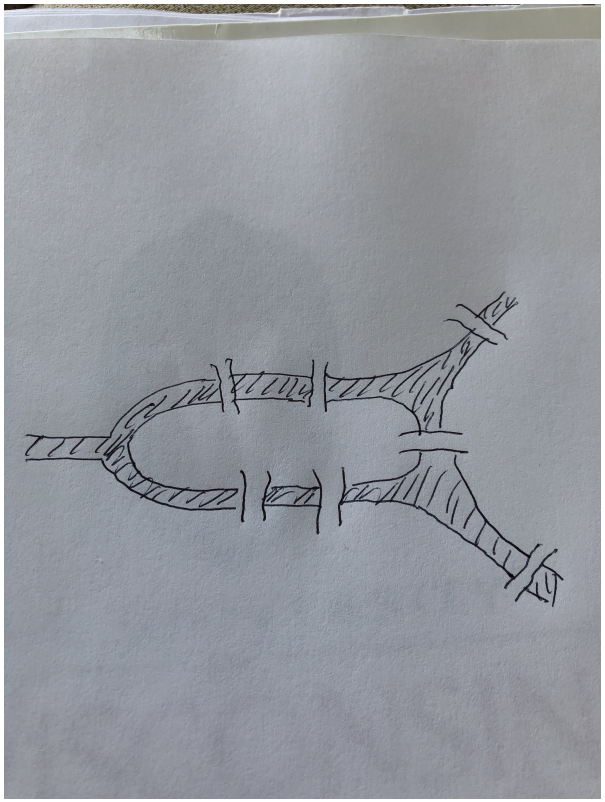
Example of a tree.



We will return to trees in the next lecture and discuss them in more detail.

The Bridges of Königsberg

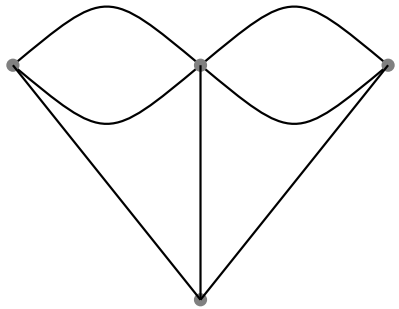
The following is known as the *Bridges of Königsberg* problem: the river Pregel, passing through Königsberg, splits in 2 parts, having 7 bridges, as follows:



Is it possible to walk from one point in the city, returning to the same point, and passing each bridge exactly once?

The Bridges of Königsberg: reformulation

Let's reformulate the problem graph-theoretically. Consider a graph with 4 vertices, corresponding to the 4 regions of the city. The graph has 7 edges, each of them corresponding to a bridge.



General setup. Given a graph G, is there a closed path in G that passes exactly once through each edge? Such a path is an Eulerian circuit. A graph that has an Eulerian circuit is an Eulerian graph.

Therefore the Bridges of Königsberg problem asks whether the above graph is Eulerian.

Characterization of Eulerian graphs

Theorem 1. Let G be a finite graph with no isolated vertices. Then G is Eulerian if and only if G is connected and $\deg(x)$ is even for every $x \in V(G)$.

Example. Note that in the Bridges of Königsberg problem, we have three vertices of degree 3, hence the answer is negative (the graph is not Eulerian).

Proof of the theorem. Suppose first that G is Eulerian and consider an Eulerian circuit. Since there are no isolated vertices, every vertex appears on this path, hence G is connected.

Given any $x \in V(G)$, in the Eulerian circuit, every edge incident to x appears exactly once. Moreover, it is clear that

$$\#\{\text{edges "arriving" at } x\} = \#\{\text{edges "departing" from } x\}.$$

This implies that $\deg(x)$ is even.

Characterization of Eulerian graphs, cont'd

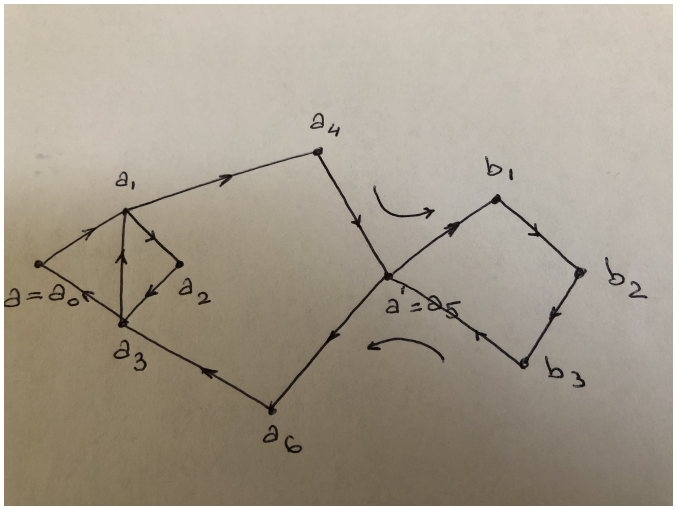
Conversely, let us assume that G is connected and that $\deg(x)$ is even for all $x \in V(G)$. We want to construct an Eulerian circuit.

We start from a vertex $a \in V(G)$ and construct a path, as long as we can, without using any edge more than once. Since all vertices have even degree, it follows that the path has to end at a. If we have used all edges, then we are done: we have an Eulerian circuit.

Otherwise, since G is connected, there is a vertex a' on the path we constructed that is incident to an edge $\mathbf{e}$ not used for the path. We start constructing a path from a', beginning with $\mathbf{e}$, not using any edge that has already been used, and continuing for as long as possible. Again, since all vertices have even degree, we must end up back at a'.

Characterization of Eulerian graphs, cont'd

We now combine the two paths as in the picture to get a new path, involving a strictly larger number of edges.



More precisely, we replace the path $a_0, a_1, a_2, a_3, a_1, a_4, a_5, a_6, a_3, a_0$ by

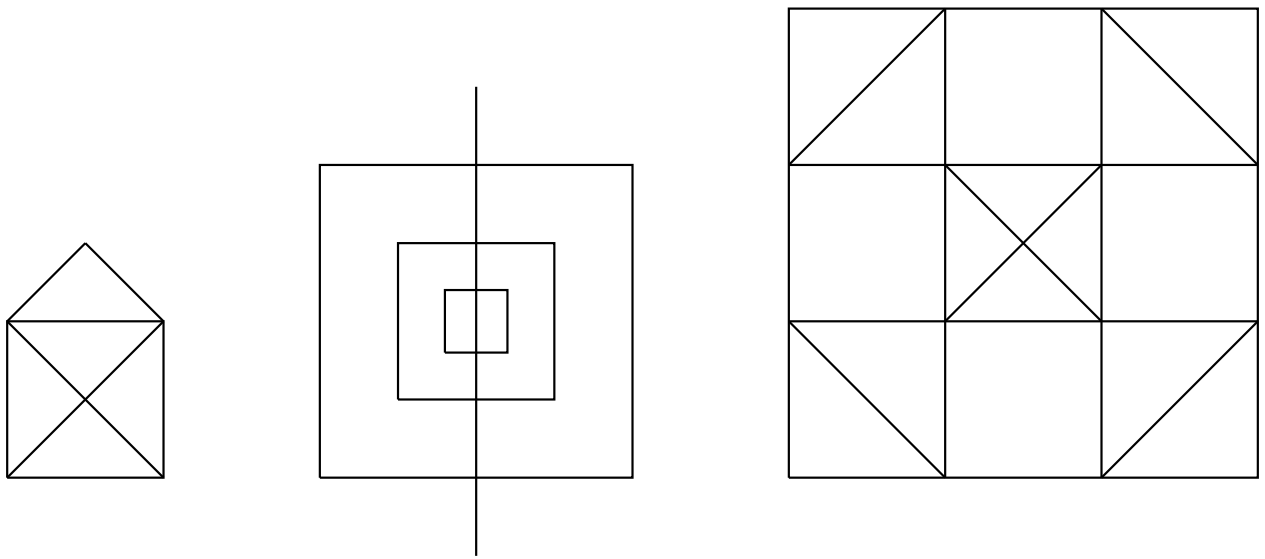
$$a_0, a_1, a_2, a_3, a_1, a_4, a_5, b_1, b_2, b_3, a_5, a_6, a_3, a_0.$$

After finitely many steps, we end up with an Eulerian path.

An exercise

The following problem is from Bondy and Murty, *Graph Theory*:

Which of the following pictures can be drawn without lifting one's pen from the paper and without tracing a line more than once?



Oriented graphs

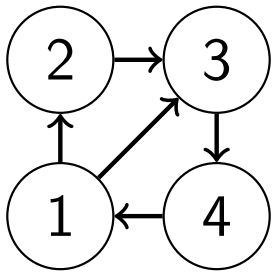
We next want to give a version of the previous theorem for oriented graphs. First, let's introduce this notion.

Definition. A directed graph (or simply, a digraph) consists of

- A set $V(G)$ of vertices
- a set $E(G)$ of edges, and
- a map $E(G) \to V(G) \times V(G)$, that is, to each edge we associate an ordered pair of vertices.

When representing a digraph, we use an arrow pointing from the first vertex (the tail) to the second vertex (the head).

Example.



Oriented graphs, cont'd

Clear: to each digraph we can associate a graph.

Giving a digraph $\leftrightarrow$ giving a graph + choice of orientation for each edge.

Definition. A walk in a digraph is strong if each edge is traversed according to its orientation (from tail to head). A digraph is *strongly connected* when for every two vertices x and y there is a strong walk from x to y.

If G a digraph, we can ask whether G has a directed Eulerian circuit (this is an Eulerian circuit that is at the same time a strong walk). The proof of the theorem we have discussed extends verbatim to directed graphs to give:
Theorem 2. A directed graph G with no isolated vertices has a strong Eulerian circuit if and only if G is strongly connected and for every vertex x of G, the in-degree of x is equal to the out-degree of x, where

$$\text{in-degree}(x) = \{e \in E(G) \mid \text{head}(e) = x\} \quad \text{and}$$

$$\text{out-degree}(x) = \{e \in E(G) \mid \text{tail}(e) = x\}.$$

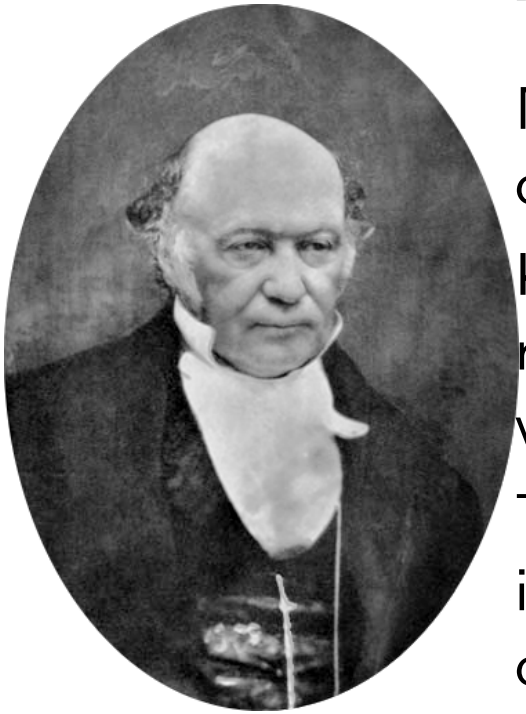
Hamiltonian graphs

A related notion to that of Eulerian circuit is that of Hamiltonian circuit: this is a simple closed path that passes through every point in the graph. A Hamiltonian graph is a graph that has a Hamiltonian circuit.

Remark. Giving a Hamiltonian circuit in G is the same as giving a spanning polygon for G.

Fact. While we have seen that it is pretty easy to decide whether a given graph is Eulerian, it turns out that to decide whether a graph is Hamiltonian is pretty hard.

Some information from Wikipedia



William Rowan Hamilton, 1805-1865, was an Irish mathematician, who worked at Trinity College, Dublin.

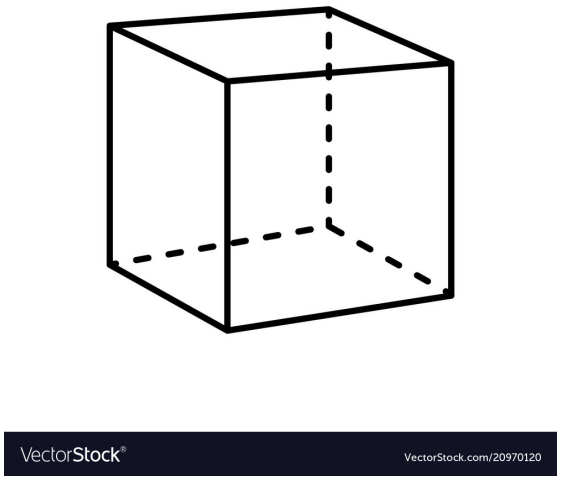
Made contributions to algebra, optics, and classical mechanics.

Known as the inventor of quaternions. His reformulation of Newtonian mechanics was very influential in Physics.

To study the symmetries of the icosahedron, he considered what are now called Hamilton circuits on various polyhedra.

Question 1: the cube

Is the graph with vertices given by the vertices of a cube and edges given by the edges of the cube a Hamiltonian graph?



Question 1: the cube

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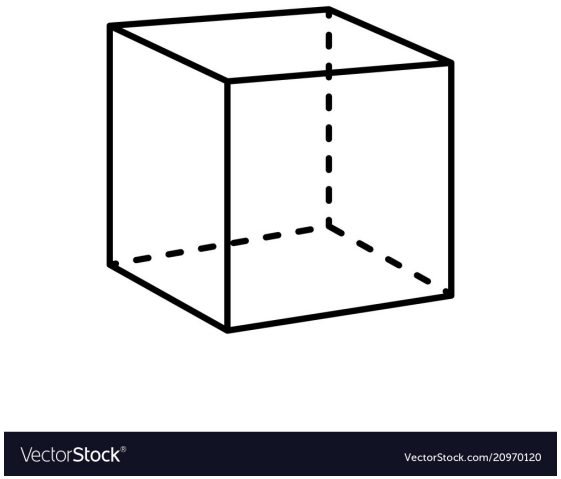
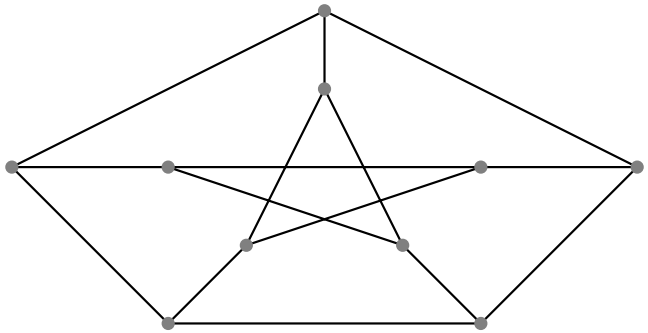


Answer: Yes

Question 2: the Petersen graph

Is the Pertersen graph Hamiltonian?



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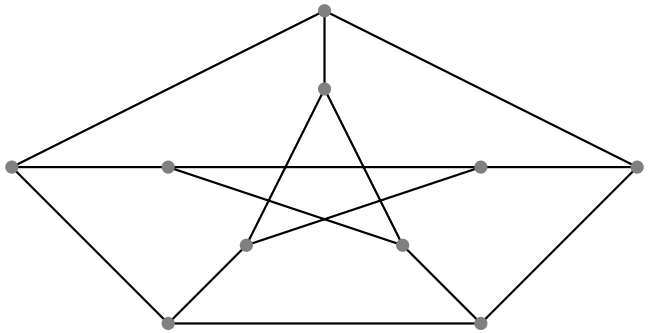
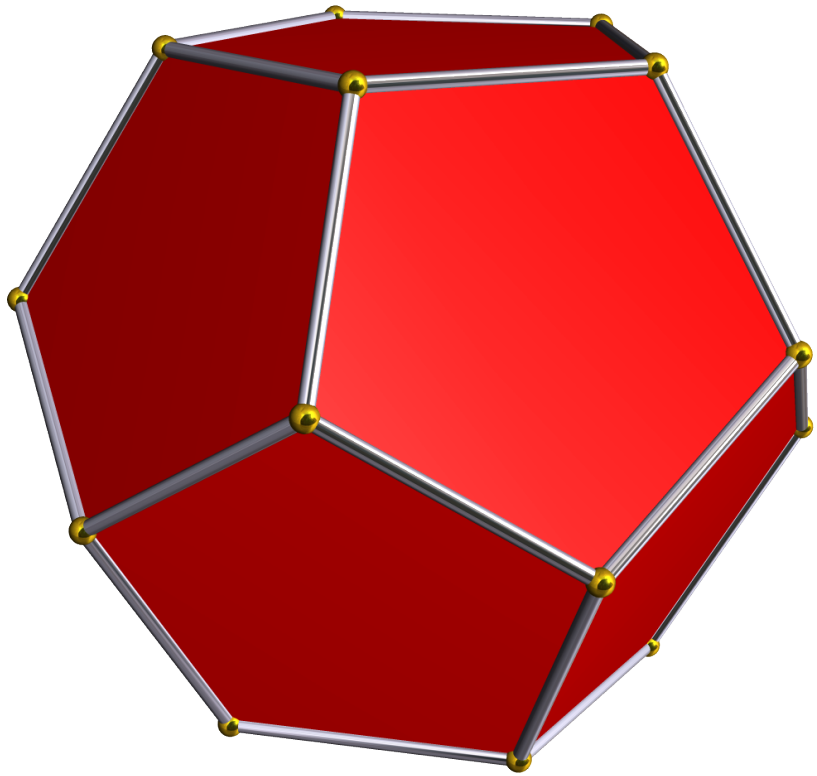


Answer: No

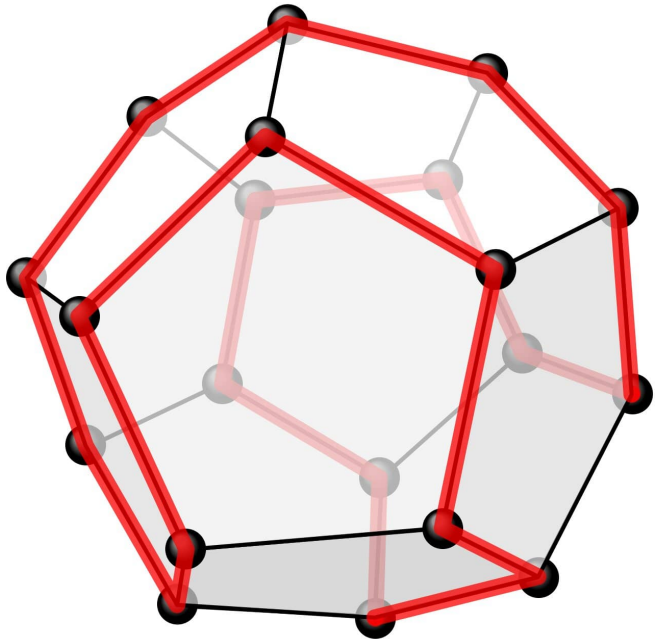
Question 3: the dodecahedron

Is the graph with vertices given by the vertices of a dodecahedron and edges given by the edges of the dodecahedron a Hamiltonian graph?



Question 3: the dodecahedron, cont'd

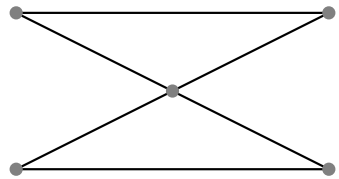
Answer: Yes



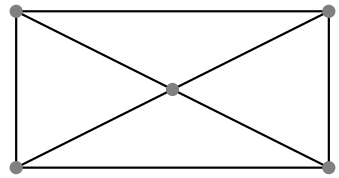
Some easy examples

While *Hamiltonian* and *Eulerian* graphs are similar notions, neither of them implies the other.

For example, the following graph is Eulerian, but not Hamiltonian:

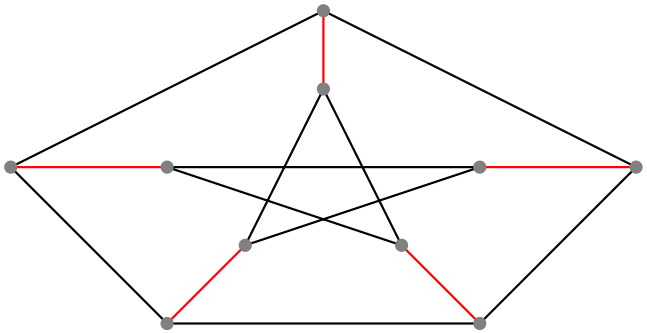


while the following graph is Hamiltonian, but not Eulerian:



The Petersen graph

Example. The Petersen graph $G = G(5, 2)$ is not Hamiltonian:

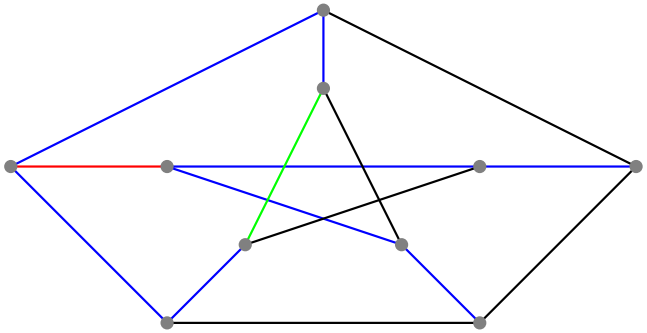


Proof (P. Cameron). Suppose that we have a Hamiltonian circuit Γ in G. Since this has to return to the starting point, it has to use an even number of "crossing edges" between the pentagon and the pentagram in the picture (the red edges); hence either 2 or 4 crossing edges.

If Γ uses 2 such edges, then in between these edges it contains a path of length 4 in the pentagon and a path of length 4 in the pentagram. The ends of such paths (either in the pentagon or the pentagram) are neighbors in the respective subgraphs. However, given any two neighbors in the pentagon, the corresponding vertices in the pentagram are not neighbors in the pentagram. Hence this case is impossible.

The Petersen graph, cont'd

Suppose now that Γ contains 4 crossing edges. Say the red edge in the picture is the one that is *not* used and the blue ones are the used ones.



It is then easy to see that Γ has to contain 3 edges of the pentagon and 3 edges of the pentagram. Moreover, these should contain the blue edges in the picture. The green edge then clearly can't be part of Γ, which forces the other two edges in the pentagram that have common ends with the green edge to be part of Γ. This gives 4 pentagram edges in Γ, a contradiction.

Another example

Example/Exercise. Consider the hypercube $C = [0,1]^n \subseteq \mathbf{R}^n$ and consider the graph G given by the *1-skeleton* of C (the vertices are the vertices of C and two vertices are connected by an edge precisely when they are joined by an edge of C).

Equivalently, we have $V(G) = \{0,1\}^n$ and two vertices $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ are joined by an edge if and only if $\#\{i \mid a_i \neq b_i\} = 1$.

Show that if $n \geq 2$, then G is a Hamiltonian graph.

Ore's theorem

Typically, results about Hamiltonian graphs say that if G has a lot of edges, then G is Hamiltonian. Here is one such result:

Theorem 3 (Ore). If G is a simple graph with $n \geq 3$ vertices such that for any two non-adjacent vertices x and y, we have

$$\deg(x) + \deg(y) \geq n,$$

then G is Hamiltonian.

Remark. The condition in the theorem is only a sufficient condition for being a Hamiltonian graph. For example, if $G = P_n$, then G is clearly Hamiltonian, while for $n \geq 5$ it does not satisfy the hypothesis of the theorem.

An immediate consequence of the theorem is the following

Corollary (Dirac). If G is a simple graph with $n \geq 3$ vertices such that $\deg(x) \geq \frac{n}{2}$ for every vertex x of G, then G is Hamiltonian.

Proof of Ore's theorem

Easy to see: if $n = 3$, then $G \simeq K_3$, hence it is Hamiltonian. We thus may assume $n \geq 4$.
Suppose that G has no Hamiltonian circuits. We construct a sequence of graphs $G_1 = G, G_2, \ldots, G_m \simeq K_n$ on $V(G)$ such that each G_i is obtained from G_{i-1} by adding one edge that is not in $E(G_{i-1})$. It is clear that each G_i still satisfies the hypothesis of the theorem (the degrees do not go down).

Since K_n certainly has Hamiltonian circuits (since $n \geq 3$), we see that after replacing G by some G_i, with $i \geq 1$, we may assume that G has no Hamiltonian circuits, but after adding one new edge $\{a, b\}$, the resulting graph G' has Hamiltonian circuits. Let Γ be such a circuit in G'.

Since G has no Hamiltonian circuits, it follows that the edge $\{a, b\}$ is part of Γ. After possibly choosing a different starting point on this circuit, we may assume that Γ is the simple closed path $x_1, \ldots, x_{n-2}, a, b, x_1$.

Proof of Ore's theorem, cont'd

Consider the following two sets:

$$A = \{i \mid 2 \leq i \leq n-2, x_i \text{ is a neighbor of } b \text{ in } G\} \quad \text{and}$$

$$B = \{i \mid 2 \leq i \leq n-2, x_{i-1} \text{ is a neighbor of } a \text{ in } G\}.$$

Since G is a simple graph, with $V(G) = \{x_1, \ldots, x_{n-2}, a, b\}$, we see that

$$\#A = \deg(b) - 1 \quad \text{and} \quad \#B = \deg(a) - 1.$$

Since $\{a, b\} \notin E(G)$, the hypothesis implies $\#A + \#B \geq n-2$. Since A, B are subsets of a set with $n-3$ elements, we see that there is $i \in A \cap B$. In this case, we get a Hamiltonian circuit in G given by

$$x_i, \ldots, x_{n-2}, a, x_{i-1}, \ldots, x_1, b, x_i,$$

a contradiction.