Lecture 6: Introduction to Ramsey theory, part I

September 22, 2020

Idea of Ramsey theory

Ramsey theory deals with questions of the type: how many elements should a certain combinatorial structure have such that certain special substructures appear?

Idea: in sufficiently large systems, however disordered, there must be some order.



British mathematician, logician, and philosopher 1903-1930

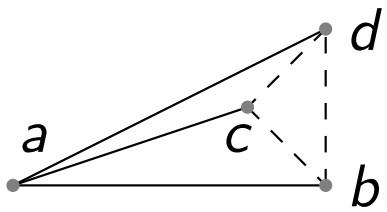
Best known for what's known these days as Ramsey's theorem (a lemma in a paper on logic)

A first example

Example. Any party with at least 6 people will contain a group of three mutual friends or a group of three mutual non-friends.

Graph-theoretical reformulation: If the edges of the complete graph K_6 are colored red or blue, then there is at least one monochromatic triangle.

Proof. Fix a vertex a of K_6. Since there are 5 edges starting from a, at least 3 of them have the same color. Suppose that these edges join a with b, c, and d. If none of abc, abd, and acd is monochromatic, it follows that bc, bd, and cd all have the same color, hence bcd is monochromatic.



The number of monochromatic triangles

In fact, for any K_n we can compute the number of monochromatic triangles, as follows:

Theorem 1. Suppose that the edges of K_n are colored red or blue and $r_1, \ldots, r_n$ are such that

$$r_i = \text{number of red edges with vertex } i \text{ as endpoint.}$$

If Δ is the number of monochromatic triangles, then

$$\Delta = \binom{n}{3} - \frac{1}{2} \cdot \sum_{i=1}^{n} r_i(n-1-r_i).$$

Proof. A triangle which is not monochromatic has precisely two vertices where a blue and a red edges meet.

For the ith vertex, we have precisely $r_i(n-1-r_i)$ choices of pairs formed by red and blue edges that meet in that vertex.

The number of monochromatic triangles, cont'd

If we sum over the non-monochromatic triangles the number of points where a red and blue edge meet, we thus get

$$\#\{\text{non-monochromatic triangles}\} = \frac{1}{2} \cdot \sum_{i=1}^{n} r_i(n-1-r_i), \quad \text{hence}$$

$$\Delta = \binom{n}{3} - \frac{1}{2} \cdot \sum_{i=1}^{n} r_i(n-1-r_i).$$

Corollary. With the notation in the theorem, we have

$$\Delta \geq \binom{n}{3} - \lfloor (n/2) \lfloor (n-1)^2/4 \rfloor \rfloor.$$

Proof. By arithmetic vs. geometric mean: $r_i(n-1-r_i) \leq \frac{(n-1)^2}{4}$. The left-hand side is an integer, hence $r_i(n-1-r_i) \leq \lfloor (n-1)^2/4 \rfloor$. Summing over all i and using the fact that Δ is an integer, we get the bound.
Example. For $n = 6$, we get $\Delta \geq 20 - 3\lfloor 25/4 \rfloor = 2$.

Ramsey's theorem

We now consider the following generalization of the example we started with:

Theorem 2. Given $s, t \geq 2$, there is a number $R(s, t)$ such that for every graph on $n \geq R(s, t)$ vertices, there is either a set of s vertices, no 2 of them adjacent, or there is a set of r vertices, any two of them adjacent.

Example. We saw that we can take $R(3, 3) = 6$.

Remark. Think about this: "there can't be complete chaos". If a structure is large enough, no matter how complicated, there is a smaller subsystem that exhibits a special structure.

We have the following more general version:

Ramsey's theorem, cont'd

Theorem 3 (Ramsey). Given $r \geq 1$ and $q_1, \ldots, q_s \geq r$, there is a number N such that the following holds: given a set S with $n \geq N$ elements such that all $\binom{n}{r}$ r-subsets of S are divided into s mutually exclusive families $T_1, \ldots, T_s$ ("colors"), there is i with $1 \leq i \leq s$ and some q_i-subset of S for which every r-subset is in T_i.

Remark. For $r = s = 2$, we recover the assertion in Theorem 2.

Notation. We write $N(q_1, \ldots, q_s; r)$ for the smallest N as in the theorem.

Proof of Ramsey's theorem. For simplicity, we restrict to the case $s = 2$. The proof of the general case is similar.

Case $r = 1$. It is clear that in this case we have $N(p, q; 1) = p + q - 1$.

Ramsey's theorem, cont'd

Case $q_1 = r$ **or** $q_2 = r$. Suppose for example that $q_1 = r$. In this case we may take $N = q_2$. Indeed, if there is an r-subset in T_1, then we can take $i = 1$. Otherwise, every r-subset is in T_2 and in this case we can take $i = 2$ and pick any q_2-subset of S

General case. We do induction on r, assuming that the assertion holds for $r - 1$ (and $s = 2$). Note that the case $r = 1$ has already been handled.

Inside this induction on r, we do a second induction on $q_1 + q_2$. We have shown that we may assume $q_1, q_2 \geq r + 1$. By the inductive assumption, we may consider $q_1' = N(q_1 - 1, q_2; r)$ and $q_2' = N(q_1, q_2 - 1; r)$.

Suppose now that S is a set with n elements, where

$$n \geq N(q_1', q_2'; r - 1) + 1$$

and that the r-subsets of S are colored either red (corresponding to q_1) or blue (corresponding to q_2).

Ramsey's theorem, cont'd

Pick $a \in S$ and let $S' = S \setminus \{a\}$. Define a coloring on the $(r-1)$-subsets of S' by

$$(X \subseteq S',\ \#X = r-1) \to \text{color of } X \cup \{a\}.$$

By the condition on n, it follows that either there is $A \subseteq S'$, with $\#A = q_1'$, and with all $(r-1)$-subsetes of A red, or there is $B \subseteq S'$, with $\#B = q_2'$, with all $(r-1)$-subsets of B blue. We assume, for example, that we are in the former case.

Since $\#A = N(q_1 - 1, q_2; r)$, it follows that either A has a subset with q_2 elements, all of whose r-subsets are blue (in which case we are done), or A has a subset R with $q_1 - 1$ elements, all of whose r-subsets are red.

Suppose we are in the latter case.

Ramsey's theorem, cont'd

Conclusion so far:

- every $(r-1)$-subset of A is red and
- every r subset of $R \subseteq A$ is red.

We are done if we can show that every r-subset M of $R \cup \{a\}$ is red.

Case 1. If $a \in M$, then $M = (M \setminus \{a\}) \cup \{a\}$. Since $M \setminus \{a\}$ is a subset of A with $r-1$ elements, it follows that it is red; in other words, M is red.

Case 2. If $a \notin M$, then M is a subset of R, and thus it is red.

This completes the proof of the theorem.

An example

Let's consider another special case:

Exercise. Show that if the edges of K_{17} are colored with three colors, there must be a monochromatic triangle (that is, we have $N(3,3,3;2) \leq 17$).

An application of Ramsey's theorem

Theorem 4 (Erdös-Szekeres). For every $n \geq 1$, there is $N(n)$ such that any collection of $N \geq N(n)$ points in the plane, with no 3 of them on a line, has a subset of n points forming a convex polygon.

Proof. We will use the following fact: given n points in the plane, with no 3 of them on a line, they form a convex polygon if and only if any 4 of them form a polygon.

The goal will be to show that we can take $N = N(n, n; 3)$. Suppose that we have $N \geq N(n, n; 3)$ points in the plane, no 3 of them on a line. We begin by labeling the points $1, 2, \ldots, N$.

An application of Ramsey's theorem, cont'd

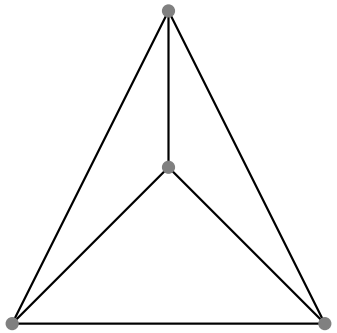
Let's color each of the 3-subsets of the set of points as follows:

• If the path from the smallest to the largest via the middle one is clockwise, then the set is colored blue.

• Otherwise, the set is colored red.

Since $N = N(n, n; 3)$, it follows that there is a subset A consisting of n of the points, such that any 3-subset of A is colored with the same color (say, blue).

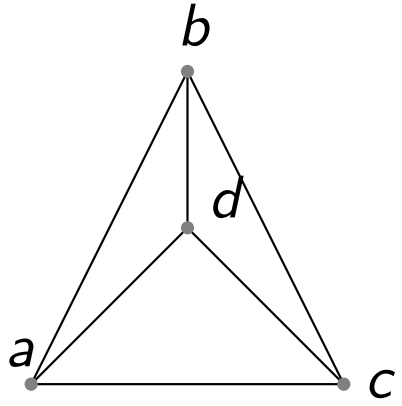
Claim: This implies that no 4 of the points in A form a configuration like this:



We saw: if this is the case, then the points of A form a convex polygon.

An application of Ramsey's theorem, cont'd

Proof of claim: suppose we have such a configuration



We also know that all 4 triangles in the picture are oriented clockwise. Suppose a is the smallest outside vertex.

Since $a < c$, the triangle acd gives $a < d < c$.
Since $a < b$ and we have seen $a < d$, the triangle abd gives $a < b < d$.
Hence $a < b < d < c$. This contradicts the fact that the triangle bcd is oriented clockwise.