HIGHER ELASTICA: GEODESICS IN JET SPACE.

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Abstract. Carnot groups are subRiemannian manifolds. As such, they admit geodesic flows, which are left-invariant Hamiltonian flows on their cotangent bundles. Some of these flows are integrable; some are not. The k-jets space of for real-valued functions on the real line forms a Carnot group of dimension $k+2$. In this study, it is shown that its geodesic flow is integrable and that its geodesics generalize Euler's elastica, with the case $k=2$ corresponding to the elastica, as shown in [1, 2, 3].

1. Introduction

The k-jets space of real functions of a single real variable, denoted here by $\mathcal{J}^k$, is a $k+2$-dimensional manifold endowed with a canonical rank 2 distribution, i.e., a linear sub-bundle of its tangent bundle. This distribution is framed by two vector fields, below denoted X_1, X_2 , whose iterated Lie brackets give $\mathcal{J}^k$ the structure of a stratified group. Declaring X_1 and X_2 to be orthonormal endows $\mathcal{J}^k$ with the structure of a subRiemannian manifold, which is (left-) invariant under the group multiplication. Like any subRiemannian structure, the cotangent bundle $T^*\mathcal{J}^k$ is endowed with a Hamiltonian system whose underlying Hamiltonian H is that whose solution curves project to the subRiemannian geodesics on $\mathcal{J}^k$. We call this Hamiltonian system the geodesic flow on $\mathcal{J}^k$.

This paper has two main goals, the following theorem is the first.

Theorem 1.1. *The geodesic flow for the subRiemannian structure on $\mathcal{J}^k$ is integrable.*

$\mathcal{J}^1$ is isometric to the Heisenberg group where this theorem is well-known, see [4] and [5]. $\mathcal{J}^2$ is isometric to the Engel's group, and Ardentov and Sachkov showed that its subRiemannian geodesics correspond to Euler elastica. Their result inspired Theorem 1.2, see below.

$\mathcal{J}^k$ comes with a projection $\Pi : \mathcal{J}^k \to \mathbb{R}^2 = \mathbb{R}^2_{x,u_k}$ onto the Euclidean plane which projects the frame X_1, X_2 projects onto the standard coordinate frame $\frac{\partial}{\partial x}, \frac{\partial}{\partial u_k}$ of $\mathbb{R}^2$. (See 2 below for the meaning of the coordinates). As a consequence, a horizontal curve γ in $\mathcal{J}^k$ is parameterized by (subRiemannian) arc-length if and only if its planar projection $\Pi \circ \gamma$ to $\mathbb{R}^2$ is parameterized by arc-length. We will characterize geodesics on $\mathcal{J}^k$ in terms of their planar

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projections. As alluded to already, Ardentov and Sachkov [2], proved that when $k = 2$, the planar projections of geodesics are Euler elastica. These elastica have "'directrix" the u_2-line, the line orthogonal to the x-axis. There are several ways to characterize Euler's elastica, see, e.g., [6, 7, 8, 9]. The one we will use is as follows. Take a planar curve $c(s) = (x(s), y(s))$ and consider its curvature $\kappa = \kappa(s)$, where s is arc-length. Then the curve c is an Euler elastica with a line directrix parallel to the y-axis, if and only if, $\kappa(s) = P(x(s))$ for $P(x)$ some **linear** polynomial in x, that is, $P(x) = ax + b$ for some constants a and b (See FIGURE 1.1).

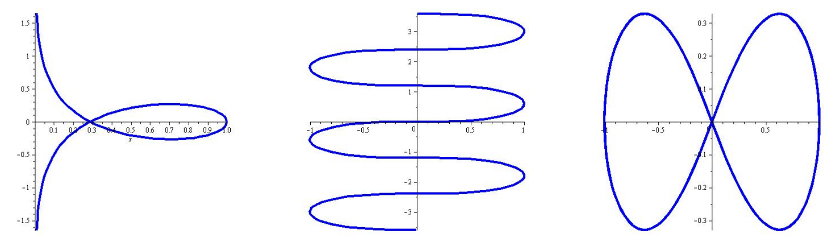

FIGURE 1.1. Some classic solutions of the Elastica equation generated by $F_2(x) = ax^2 + \alpha$, on the left the Euler Soliton with $\alpha = 1$ and $a = -2$, in the center the pseudo-sinusoid $\alpha = 0$ and $a = -1$, on the right the pseudo-lemniscate with $\alpha = .65222...$ and $a = 1.65222....$

The following theorem is the second main goal.

Theorem 1.2. *Let $\gamma : I \to \mathcal{J}^k$ be a subRiemannian geodesic parameterized by arc-length s, and $\pi \circ \gamma = c(s) = (x(s), u_k(s))$ its planar projection. Let κ be the curvature of c. Then $\kappa(s) = p(x(s))$ for some degree $k-1$-polynomial $p(x)$ in the coordinate x. Conversely, any plane curve $c(s)$ in the (x, u_k) plane, which is parameterized by arc-length s and whose curvature $\kappa(s)$ equals $p(x(s))$ for some polynomial $p(x)$ of degree at most $k-1$ in x, is the projection of such a subRiemannian geodesic.*

Example For the case $k = 1$ of the Heisenberg group, the theorem asserts that $\kappa = P(x)$, where P is a degree 0 polynomial ,i.e., a constant function. The only curves having constant curvature are lines and circles, and these are well-known to be the projections of the Heisenberg geodesics.

A geodesic is called globally minimizing if each of its compact sub-arcs realizes the distance between its endpoints. The geodesic on $\mathcal{J}^k$ will be classified and used to present the Conjecture 6.2, which attempts to make a complete classification of global minimizing geodesics on $\mathcal{J}^k$.

The outline of the paper is as follows. In section 2 presents $\mathcal{J}^k$, the k-th jet space as a subRiemannian manifold, and the notation that will be followed throughout the work. Also, the Hamiltonian subRiemannian geodesic flow is defined and give some details are given about the Carnot structure of $\mathcal{J}^k$. In section 3, the Poisson-Lie reduction is used to prove Theorem 1.1 and an explicit expression of the Casimir functions are presented. In section 4 we use the Lie-Poisson bracket and the geodesic equations to prove Theorem 1.2. In section 5, it is shown that the geodesics are generically periodic on x-coordinates and the geodesics on the $\mathcal{J}^k$ are classified. Finally, in section 6, a Conjecture 6.2 is introduced about a complete classification of globally minimizing geodesics on $\mathcal{J}^k$.

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2. SET-UP

The k-jet of a smooth function $f : \mathbb{R} \to \mathbb{R}$ at a point $x_0 \in \mathbb{R}$ is its kth order Taylor expansion at x_0. We will encode this k-jet as a $k+2$-tuple of real numbers as follows:

$$(j^k f)(x_0) = (x_0, f^k(x_0), f^{k-1}(x_0), \dots, f'(x), f(x_0)) \in \mathbb{R}^{k+2} \tag{2.1}$$

As f varies over smooth functions and x_0 varies over $\mathbb{R}$, these k-jets sweep out the k-jet space, denoted by $\mathcal{J}^k$. One can see that $\mathcal{J}^k$ is diffeomorphic to $\mathbb{R}^{k+2}$ and its points are coordinatized according to

$$(x, u_k, u_{k-1}, \dots, u_1, y) \in \mathbb{R}^{k+2} := \mathcal{J}^k.$$

Recall that if $y = f(x)$, then $u_1 = dy/dx$, while $u_{j+1} = du_j/dx$, $j \geq 1$. Rearranging these equations into $dy = u_1 dx, du_j = u_{j+1}dx$, we see that $\mathcal{J}^k$ is endowed with a natural rank 2 distribution $\mathcal{D} \subset T\mathcal{J}^k$ characterized by the k Pfaffian equations

$$\begin{aligned} u_1 dx - dy &= 0 \\ u_2 dx - du_1 &= 0 \\ \vdots &= \vdots \\ u_k dx - du_{k-1} &= 0 \end{aligned}$$

The typical integral curves of $\mathcal{D}$ are the k-jet curves $x \mapsto (j^k f)(x)$. In addition to these integral curves we have a distinguished family of curves which arise by varying only the highest derivative u_k, and which are the integral curves of the vector field X_2 below (eq (2.2). These latter curves are C^1-rigid in the sense of Bryant-Hsu,[10], and they exhaust the supply of C^1-rigid curves.

A subRiemannian structure on a manifold consists of a non-integrable distribution together with a smoothly varying family of inner products on the distribution. We have our distribution $\mathcal{D}$ on $\mathcal{J}^k$. We arrive at our subRiemannian structure by observing that $\mathcal{D}$ is globally framed by the two vector fields

$$X_1 = \frac{\partial}{\partial x} + u_1 \frac{\partial}{\partial y} + \sum_{i=2}^{k} u_i \frac{\partial}{\partial u_{i-1}} \quad \text{and} \quad X_2 = \frac{\partial}{\partial u_k} \tag{2.2}$$

and then declaring these two vector fields to be orthonormal . Now the restrictions of the one-forms dx, du_k to $\mathcal{D}$ form a global co-frame for $\mathcal{D}^*$ which is dual to our frame (equation (2.2). Therefore an equivalent way to describe our subRiemannian structure is to say that its metric is $dx^2 + du_k^2$ **restricted to** $\mathcal{D}$.

For the purposes of Theorem 1.2 the following alternative characterization of the subRiemannian metric is crucial. Consider the projection

$$\Pi : \mathcal{J}^k \to \mathbb{R}^2_{x,u_k}; \Pi(x, u_k, u_{k-1}, \ldots, u_1, y) = (x, u_k).$$

Its fibers are transverse to $\mathcal{D}$, and since $\Pi_* X_1 = \frac{\partial}{\partial x}, \Pi_* X_2 = \frac{\partial}{\partial u_k}$, the frame pushes down to the standard frame for $\mathbb{R}^2$. The metric on each two-plane $\mathcal{D}_p$, $p \in \mathcal{J}^k$ is characterized by the condition that $d\Pi_p$ (which is just Π since Π is linear), restricted to $\mathcal{D}_p$ is a linear isometry onto $\mathbb{R}^2$, where $\mathbb{R}^2$ is endowed with the standard metric $dx^2 + du_k^2$. It follows immediately that the length of any horizontal path equals the length of its planar projection, that Π is a "submetry", i.e., $\Pi(B_r(p)) = B_r(\Pi(p))$, where $B_r(p)$ denotes the metric ball of radius r about q, and the horizontal lift of a Euclidean line in $\mathbb{R}^2$ is a geodesic in $\mathcal{J}^k$.

2.1. **Hamiltonian.** Let $P_1, P_2 : T^*\mathcal{J}^k \to \mathbb{R}$ be the 'power functions' of the vector fields X_1, X_2 above (see [5], page 8) . In terms of traditional cotangent coordinates $(x, u_k, u_{k-1}, \ldots, u_1, y, p_x, p_k, p_{k-1}, \ldots, p_1, p_y)$ for $T^*\mathcal{J}^k$, with p_i short for p_{u_i} we have

$$P_1 = p_x + u_1 p_y + u_2 p_1 + \ldots + u_k p_{k-1}; \qquad P_2 = p_k.$$

Then the Hamiltonian governing the subRiemannian geodesic flow on $\mathcal{J}^k$ is

$$H = \frac{1}{2}(P_1^2 + P_2^2) \tag{2.3}$$

(see [5], 8 page). Where the condition $H = 1/2$ implies that the geodesics are parameterized by arc-length, and this we will do in what follows.

Remark. [C^1-rigidity]. A curve tangent to $\mathcal{D}$ is called C^1-rigid if it is a critical point of the endpoint map (see [5] chapter 3). The u_k curves are C^1-rigid for $\mathcal{D}$, and form what Liu-Sussmann christened as the "regular-singular" curves for $\mathcal{D}$. As such, they are geodesics for **any** subRiemannian metric $Edx^2 + 2Fdxdu_k + Gdu_k^2$, restricted to $\mathcal{D}$ such that ds^2 is positive definite and for E, F, G functions of the jet coordinates $(x, u_k, u_{k-1}, \ldots, y)$, regardless on whether or not they satisfy the corresponding (normal) geodesic equations. For the present metric each u_k-curve is indeed the projection to $\mathcal{J}^k$ of a solution to our H, so we do not go to extra effort to account for these abnormal geodesics (REF [5], chapter 3).

2.2. **Carnot Group structure.** Under iterated bracket, the frame $\{X_1, X_2\}$ generates a $k+2$-dimensional nilpotent Lie algebra which can be identified pointwise with the tangent space to $\mathcal{J}^k$. Specifically, if we write

$$X_3 = [X_2, X_1], \;\; X_4 = [X_3, X_1], \;\ldots, \;\; X_{k+2} = [X_{k+1}, X_1], \;\; 0 = [X_{k+2}, X_1],$$

then we compute that

$$X_{k+2} = \frac{\partial}{\partial y}, \;\; X_{k+1} = \frac{\partial}{\partial u_1}, \;\; X_k = \frac{\partial}{\partial u_2}, \ldots, \;\; X_3 = \frac{\partial}{\partial u_{k-1}}$$

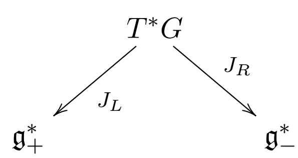
while all other Lie brackets $[X_i, X_j]$ are zero. The span of the X_i thus forms a $k+2$-dimensional graded nilpotent Lie algebra

$$\mathfrak{g}_k = V_1 \oplus V_2 \oplus \ldots V_{k+1}, \quad V_1 = span\{X_1, X_2\}, V_i = span\{X_{i+1}\}, 1 < i \leq k+1.$$

Like any graded nilpotent Lie algebra, this algebra has an associated Lie group which is a Carnot group G w.r.t. the subRiemannian structure, and by using the flows of the X_i, we can identify G with $\mathcal{J}^k$, and the X^i with left-invariant vector fields on $G \cong \mathcal{J}^k$.

3. Integrability: Proof of Theorem 1.1

Our Hamiltonian H is a left-invariant Hamiltonian on the cotangent bundle of a Lie group G. Let us recall the general 'Lie-Poisson' structure for such Hamiltonian flows, see Appendix [11] or ch 4 [12].



The arrows J_R, J_L are the momentum maps for the actions of G on itself by right and left translation, lifted to T^*G. The subscripts $\pm$ are for a plus or minus sign in front of the Lie-Poisson (also known as Kostant-Kirrilov-Souriau) bracket on $\mathfrak{g}^*$. J_R corresponds to *left* translation back to the identity and realizes the quotient of T^*G by the *left* action. J_L corresponds to *right* translation of a covector back to the identity and forms the components of the momentum map for *left* translation, lifted to the cotangent bundle. In this case, $\mathfrak{g}^* = \mathbb{R}^{k+2}$ and

$$J_R = (P_1, P_2, P_3, \ldots, P_{k+2})$$

with P_i as the power function associated to X_i, so that

$$P_3 = p_{k-1}, P_4 = p_{k-2}, \ldots, P_{k+2} = p_y$$

in terms of standard canonical coordinates as above.

When the Hamiltonian $H : T^*G \to \mathbb{R}$ is left-invariant, it can be expressed as a function of the components of J_R, that is, $H = h \circ J_R$ for some $h : \mathfrak{g}^* \to \mathbb{R}$, and H Poisson commutes with every component of the *left* momentum map J_L, so that these left-components are invariants. J_L and J_R are related by $J_L(g,p) = (Ad_g)^* J_R(g,p)$, where we have written $p \in T^*_g G$, and where Ad_g is the adjoint action of g.

The reason underlying the integrability of this system is a simple dimension count.

Proposition 3.1. *If the generic co-adjoint orbit of $\mathfrak{g}^*$ is 2-dimensional, then the left-invariant Hamiltonian flow on T^*G is integrable.*

Let us recall that the symplectic reduced spaces for the *left translation* action are the co-adjoint orbits, for $\mathfrak{g}^*_+$, and that J_R realizes this symplectic reduction procedure, mapping each $J_L^{-1}(\mu)$ onto the co-adjoint orbit through μ. The hypothesis of the Proposition asserts that the symplectic reduced spaces associated to the G-action are zero or two dimensional, so, morally speaking, the system is automatically integrable by reasons of dimension count.

Proof of Proposition. We must produce n commuting integrals in involution, where $n = dim(G)$. The hypothesis asserts that there are $n-2$ Casimirs $C_1, \ldots, C_{n-2}$ for $\mathfrak{g}^*$, these being the functions whose common level sets at a generic value define a generic co-adjoint orbit. These Casimirs are a functional basis for the Ad^*_G invariant polynomials on $\mathfrak{g}^*$. When viewed as functions on T^*G via $C_i \circ J_R$, the Casimirs Poisson commute with *any* left-invariant function on T^*G, and in particular with H and with each other. Thus, $H, C_1, C_2, \ldots, C_{n-2}$ yield $n-1$ integrals. To get the last commuting integral, take any component of J_L.

Q.E.D.

Proof of Theorem 1.1. In order to use the proposition, we need to verify that the co-adjoint orbits are generically 2-dimensional. We have the Poisson brackets

$$\{P_1, P_i\} = P_{i+1}, 1 < i < k+1, \qquad \text{and} \qquad \{P_1, P_{k+2}\} = 0, \tag{3.1}$$

with all other Poisson brackets $\{P_i, P_j\}, 1 < i < j \leq k+2$ being zero. Thus the Poisson tensor B at a point $Z = (P_1, P_2, P_3, \ldots, P_{k+2}) \in \mathfrak{g}^*_+$ is :

$$B := \begin{pmatrix} 0 & Z_k & 0 \\ Z_k^t & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \text{ where } Z_k = (P_3, \ldots, P_{k+2}), \tag{3.2}$$

which generically has rank 2, and rank 0, if and only if, $Z_k = 0$, i.e., if and only if, $P_i = 0$ for $2 < i$.

Q.E.D.

Note: After writing this paper, we learned about closely related work by Anzaldo-Meneses and Monroy-Perez, see [13, 14], who obtained many of the same results in a different way.

Thanks to Theorem 1.1, we know that the system has k Casimir functions.

Theorem 3.2. *Let C_i be the functions defined by $C_1 = P_{k+2}$ and*

$$C_i = P_{k+2}^{i-1} P_{k+2-i} + \sum_{j=1}^{i-2} (-1)^j P_{k+2}^{i-(j+1)} P_{k+2-j} \frac{P_{k+1}^j}{(j)!} + (-1)^{i-1} \frac{P_{k+1}^i}{(i-2)!\, i}, \tag{3.3}$$

for $i > 1$. If $P_{k+2} \neq 0$, then C_i's are constants of motion for the geodesic equations on $\mathcal{J}^k$, in others words they are Casimir functions[1].

4. Integration and curvature: proof of Theorem 1.2

Hamilton's equations read $\frac{df}{dt} = \{f, H\}$. With the Hamiltonian of this system, they expand to $\frac{df}{dt} = \{f, P_1\}P_1 + \{f, P_2\}P_2$. Returning to our coordinates x, u_k, we compute $\{x, P_1\} = 1, \{x, P_2\} = 0$, and $\{u_k, P_1\} = 0, \{u_k, P_2\} = 1$, so that

$$\frac{dx}{dt} = P_1 \qquad \text{and} \qquad \frac{du_k}{dt} = P_2. \tag{4.1}$$

Thus (P_1, P_2) are the components of the tangent vector to the plane curve $(x(s), u_k(s))$ obtained by projecting a geodesic to the plane. If $H = 1/2$, then this vector is a unit vector, the parameter t of the flow is the arc-length s, and we can write

$$(P_1, P_2, \dot{\theta}) = (\cos(\theta), \sin(\theta), \kappa).$$

Using $\{P_1, P_2\} = P_3$, we see that the P_1, evolves according to the equations

$$\begin{aligned} \dot{P}_1 &= P_3 P_2 \\ \dot{P}_2 &= -P_3 P_1. \end{aligned} \tag{4.2}$$

We also have that $\dot{P}_1 = -\dot{\theta}\sin(\theta) = -\dot{\theta}P_2$ and $\dot{P}_2 = +\dot{\theta}\cos(\theta) = +\dot{\theta}P_1$, from which we see that

$$-P_3 = \kappa.$$

Now, for $k + 2 > i > 2$ we have that $\{P_i, P_1\} = -P_{i+1}, \{P_i, P_2\} = 0$, and $\{P_j, P_{k+2}\} = 0$ for all j, so that

$$\begin{aligned} \dot{P}_3 &= -P_1 P_4, \\ \dot{P}_4 &= -P_1 P_5, \\ &\vdots \\ \dot{P}_{k+1} &= -P_1 P_{k+2}, \\ \dot{P}_{k+2} &= 0. \end{aligned} \tag{4.3}$$

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[1]In the case $i = 2$ the sum is empty

Proof of Theorem 1.2. Consider a geodesic γ and an arc of the geodesic for which $\dot{x} \neq 0$. Instead of arc-length $t = s$, we use x to parameterize this arc. From equation (4.1) along this arc

$$\frac{d}{dx} = \frac{1}{P_1}\frac{d}{ds},$$

so that the equations for the evolution of $P_3, P_4, \ldots, P_{k+2}$ along the curve become

$$\begin{aligned}\frac{dP_3}{dx} &= -P_4\\ \frac{dP_4}{dx} &= -P_5\\ \vdots &= \vdots\\ \frac{dP_{k+1}}{dx} &= -P_{k+2}\\ \frac{dP_{k+2}}{dx} &= 0,\end{aligned} \tag{4.4}$$

these equations can be summarized by

$$\frac{d^k P_3}{dx^k} = 0,$$

which asserts that the curvature P_3, of the projected curve $c = \pi \circ \gamma$, is a polynomial $p(x)$ of degree $k-1$ in x, at least along this arc-length. Finally, since γ is an analytic function of s, so are $c(s)$ and $\kappa(s)$, so that if $\kappa(s)$ enjoys a relation $\kappa(s) = p(x(s))$ along some subarc of $c(s)$, it enjoys this same relation everywhere along c.

To prove the converse, first consider a general smooth curve $c(s)$ in the (x,u) plane along which $dx/ds > 0$. We can parameterize the curve either by arc-length $c(s) = (x(s), u(s))$ or as a graph, $u = u(x)$. Define the function $F(x)$, with $-1 \leq F(x) \leq 1$, by way of relating the two parameterizations:

$$(\dot{x}, \dot{u}) := \left(\frac{dx}{ds}, \frac{du}{ds}\right) = (\sqrt{1 - F(x)^2}, F(x)) \tag{4.5}$$

so that $dx = \sqrt{1 - F(x)^2}ds$ and

$$u'(x) := \frac{du}{dx} = \frac{F(x)}{\sqrt{1 - F(x)^2}}.$$

Therefore,

$$u'' = \frac{F'}{(1 - F(x)^2)^{3/2}}.$$

The curvature of $c(s)$, when viewed as a graph, is well-known to be

$$\kappa(x) = \frac{u''(x)}{(1 + u'(x)^2)^{3/2}}$$

and so

$$(1 + u'(x)^2) = \frac{1}{1 - F(x)^2},$$

from which we conclude that

$$\kappa = F'(x). \tag{4.6}$$

To end this proof, suppose that we are given a curve $c(s)$ in the plane (x, u), with $u = u_k$, whose curvature κ is a degree $k - 1$ polynomial in x. Define $F(x)$ by equation (4.5) along an arc of $c(s)$ for which $dx/ds > 0$. From equation (4.6), we know that F is an anti-derivative of κ and so a polynomial of degree k in x. The constant term in the integration $F(x) = \int^x \kappa dx$ is fixed by choosing any point $(x_*, u_*) = (x(s_*), u(s_*))$ along $c(s)$ for which $dx/ds > 0$, so that $-1 < du/ds|_{s=s_*} < 1$ and setting $F(x_*) = du/ds|_{s=s_*}$.) By the preceding analysis, $c(s)$ has curvature $\kappa(x(s))$ along the entire arc $dx/ds > 0$ of $c(s)$ which contains (x_*, u_*). Moreover, $u'(x) = F(x)/\sqrt{1 - F(x)^2}$. Set

$$(P_1(x), P_2(x)) := (\sqrt{1 - (F(x))^2}, F(x)), \tag{4.7}$$

and

$$P_3(x) := F'(x), \text{ and } P_{i+2} := (-1)^i \frac{d^i F}{d^i x}(x), i > 1. \tag{4.8}$$

We look at the P_i as momentum functions. Reparameterize the momentum functions by s using $dx/ds = P_1(x)$. Then we verify that the P_i satisfies 4.2 and 4.3, so that the horizontal curve $\gamma(x(s))$ in $\mathcal{J}^k$ with these momenta satisfies the geodesic equations and projects on our given curve $c(s)$.

Q.E.D.

Corollary 4.1. *Suppose that the momentum functions P_i are related to the degree k polynomial $F(x)$ as per equations (4.7, 4.8), and that $H = 1/2$, then a critical point x_0 of $F(x)$ corresponds to a relative equilibrium for the reduced equations 4.2 and 4.3 if and only if $F(x_0) = \pm 1$.*

Proof. The equilibria of equations (4.2) and (4.3) are the points with $P_1 = 0$ and $P_3 = 0$, as long as $H \neq 0$. If $H = \frac{1}{2}$, the condition $P_1 = 0$ forces $P_2 = \pm 1$, but $P_2 = F(x)$. Finally, $P_3 = F'(x)$.

5. Structure of the higher Elastica

From the proof of Theorem 1.2, we have some freedom selecting a primitive function for $p(x)$, then, given $F_k(x) = \int p(x)dx$, the dynamic is trivial if $F_k(x(s))$ is constant for all s, that is, when $F_k^{-1}([-1, 1])$ is empty or a set of isolated points. Then we take $F_k^{-1}([-1, 1])$ as follows

$$F_k^{-1}([-1, 1]) := \cup_I [x_0^i, x_1^i], \qquad \text{where} \qquad F_k^2(x_0^i) = 1, \quad F_k^2(x_1^i) = \pm 1,$$

$x_0^i < x_1^i \leq x_0^{i+1} < x_1^{i+1}$ and the condition that $|F_k(x)| < 1$ if $x \in (x_0^i, x_1^i)$; note that we allow $F_k(x_0^i)F_k(x_1^i) = 1$ or $F_k(x_0^i)F_k(x_1^i) = -1$, this dichotomy will help us to classify the geodesic in subsection 5.1. Choose $[x_0^i, x_1^i] = [x_0, x_1]$.

Theorem 5.1. *The curve $c(s)$ in (x,u) with curvature $k((x(s)) = p(x)$ is bounded in the x-direction; generically the curve is periodic in x, and the period L is given by*

$$L := \int_{x_0}^{x_1} \frac{2dx}{\sqrt{1-F_k^2(x)}}, \quad \textit{we also define}\ , \quad \Delta u := \int_{x_0}^{x_1} \frac{2F_k(x)dx}{\sqrt{1-F_k^2(x)}}.$$

Finally, we have that $u(s+L) = u(s) + \Delta u$.

Due corollary 4.1, we know, the points x_0 and x_1 are equilibrium points if and only if they are critical points of the function $F_k(x)$, so it takes infinite time to arrive to them.

Let x_0 be a regular point, we notice that if $c(s)$ is reparameterized as a graph $u = u(x)$, as in the proof of Theorem 1.2, $c(s)$ stop to be a graph at the point x_0, (See FIGURES 1.1). We will answer the question of how to extend the curve $c(s)$ to a "multiple value function" on x at $x = x_0$ such that its lift to $\mathcal{D}^k$ is a smooth solution to the geodesic equations. Set $(P_1, P_2) = (\cos\theta, \sin\theta)$ and $\dot{\theta} = p(x)$, since $F_k(x_0) = \pm 1$ define $\theta(x_0) = \pm\frac{\pi}{2}$ and $\dot{\theta}(x_0) \neq 0$, then P_1 changes sign, while P_2 does not change. Therefore, if $x(s_0) = x_0$ we define

$$(\dot{x}, \dot{u}) = \begin{cases} (\pm\sqrt{1-F_k^2(x)}, F_k(x)) & \text{if} \quad s_0 - \frac{L}{2} \leq s \leq s_0, \\ (\mp\sqrt{1-F_k^2(x)}, F_k(x)) & \text{if} \quad s_0 \leq s \leq s_0 + \frac{L}{2}. \end{cases} \tag{5.1}$$

Hence, the curve stays in the interval $[x_0, x_1]$, we extend the curve $c(s)$ to a "multiple value function" on x at $x = x_1$ in the same way. If both are regular points, we can read the equation $P_1 = \sqrt{1-F_k(x)}$ as the restriction $P_1(x)|_{\{H=\frac{1}{2}, C_1, \ldots, C_k\}}$ and consider action function I given by the area under the graph $\sqrt{1-F_k(x)}$ going from x_0 to x_1 and the area of $-\sqrt{1-F_k(x)}$ going from x_1 to x_0, i.e.,

$$\mathcal{I}(H = \frac{1}{2}, C_1, \ldots, C_k) := 2\int_{x_0}^{x_1} \sqrt{1-F_k(x)}dx.$$

Finally, the period is given by $\frac{\partial \mathcal{I}}{\partial H}|_{\{H=\frac{1}{2}, C_1, \ldots, C_k\}} = L$, (see [11] chapter 10). The period goes to infinity when x_0 or x_1 are critical points, much like the well-known homoclinic connection of a pendulum.

Let us consider (x_0, u_*) the initial point of the curve, $x(s) \in [x_0, x_1]$ and $2s \leq L$, then

$$u(s) + \tau = \int_{x_0}^{x(s)} \frac{F_k(x)dx}{\sqrt{1 - F_k^2(x)}} + \int_{x_0}^{x_1} \frac{2F_k(x)dx}{\sqrt{1 - F_k^2(x)}} + u_*,$$
$$= (\int_{x_0}^{x(s)} + \int_{x(s)}^{x_1} + \int_{x_0}^{x_1} + \int_{x_0}^{x(s+L)}) \frac{F_k(x)dx}{\sqrt{1 - F_k^2(x)}} + u_* = u(s + L).$$

Once again, we use the fact that $\int_{x_0}^{x_1} \frac{F_k(x)dx}{\sqrt{1-F_k^2(x)}} = \int_{x_1}^{x_0} \frac{F_k(x)dx}{-\sqrt{1-F_k^2(x)}}$.

Q.E.D.

5.1. **Geodesic classification.** There are three cases:

- x-periodic - geodesics whose x-coordinate is periodic, $p(x_0) \neq 0$ and $p(x_1) \neq 0$.
- Homoclinic - geodesics whose plane curve is asymptotic to a one line in both directions - $p(x_0) = 0$ and $p(x_1) \neq 0$, or $p(x_0) \neq 0$ and $p(x_1) = 0$.
- Heteroclinic- geodesics whose plane curve is asymptotic to two lines - $p(x_0) = 0$ and $p(x_1) = 0$.

In the heteroclinic case, we add one more dichotomy into the mix.

Definition 5.2. *A heteroclinic geodesic is said to be of turn-back type if $F_k(x_0)F_k(x_1) = -1$. Otherwise, it is said that the heteroclinic is of direct type, in which case $F_k(x_0)F_k(x_1) = -1$.*

To have a better understanding of definition 5.2 (See FIGURES 5.2 and 5.3).

5.2. **General Euler Soliton.** The elastica equation has a distinguished solution called the **Euler Kink**. Other names for it are the Euler soliton or Convict's curve (See FIGURES 1.1 and 5.2), see [2, 8, 9, 6]. We define the **Euler soliton** at the level k in the sense that the curvature of the curve $(x, u_k(x))$ is always proportional to x^{k-1}. See FIGURE 5.2.

Theorem 5.3. *If $1 < k$ then the level k has a Soliton curve.*

Consider the polynomial $F_k(x) = \frac{x^k}{a^k} - \alpha$. Set $x = a\sqrt[k]{\alpha + \cos(t)}$, we find the following expressions

$$u(t) = \int_{t_0}^{t_1} \frac{\cos(t)dt}{(\alpha + \cos(t))^{\frac{k-1}{k}}}, \quad t(x) = \frac{a}{k}\int_{t_0}^{t_1} \frac{dt}{(\alpha + \cos(t))^{\frac{k-1}{k}}}.$$

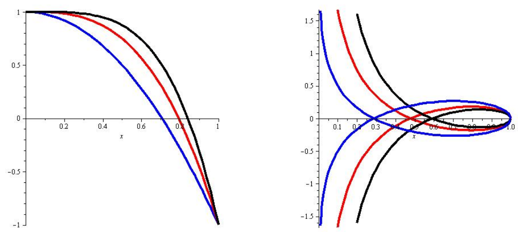

FIGURE 5.1. On the left is the graph of $F(x) = 1 - 2x^k$ and on the right the corresponding plane curves in (x, u) for $k = 2, 3, 4$.

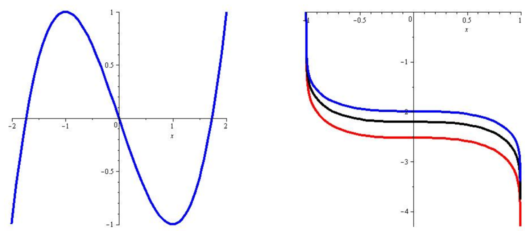

FIGURE 5.2. On the left is the graph of $F_{2k+1}(x)$ from proposition 5.4 and on the right the corresponding plane curves in (x, u) for $k = 3, 5, 7$.

The case $k = 2$ is the classic solution for the Elastica equation, (see [9] page 436). If $\alpha = 1$, then we have the explicit expression

$$u(x) = \int_{\sqrt[k]{2}}^{x} \frac{x^{\frac{k}{2}}\,dx}{\sqrt{2 - x^k}} - \frac{2}{k\sqrt{2}} \ln\left(\frac{\sqrt{2} - \sqrt{2 - x^k}}{x^{\frac{k}{2}}}\right),$$

$$t(x) = \frac{1}{\frac{3}{2}\sqrt{2}} \ln\left(\frac{\sqrt{2} - \sqrt{2 - x^k}}{x^{\frac{k}{2}}}\right).$$

We can find a explicit second order ODE for $\dot{\theta}$,

$$p_{u_{k-1}} = \frac{\partial P}{\partial x}(x) = kx^{k-1} \quad \text{and} \quad \frac{\dot{\theta}^2}{k^2 a^{2k}} = (\cos\theta + \alpha)^{\frac{2(k-1)}{k}}. \tag{5.2}$$

If $k > 1$ and $\alpha = 1$, the ODE defined by the right side from equation 5.2 has a unique equilibrium point $\theta = \pi$ and a homoclinic connection. In the case $k = 2$, this ODE is the same as the pendulum equation defined in [2].

5.3. **Heteroclinic geodesics.** $k = 1, 2$ does not have heteroclinic geodesics, since a polynomial $p(x)$ of degree 1 has at must one root. In the case $2 < k$ we have the opposite.

Proposition 5.4. *If $k > 2$, then $\mathcal{J}^k$ admits heteroclinic geodesics.*

We split in the even and odd case, in both case the Hill interval is $[-1, 1]$:

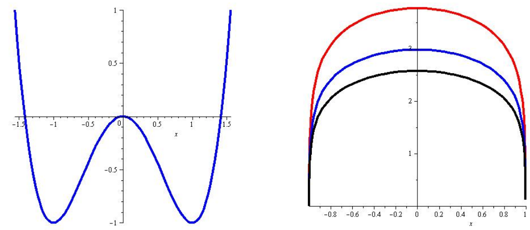

FIGURE 5.3. On the left is the graph for $F_{2k}(x)$ from proposition 5.4 and on the right the corresponding plane curves in (x, u) for $k = 4, 6, 8$.

- Turn-back type: Consider $F_{2k+1}(x) = -\frac{x^{2k+1}-(2k+1)x}{2k}$. The points $x = \pm 1$ are equilibrium points, since $F_{2k+1}(\pm 1) = \pm 1$ and $\frac{\partial F}{\partial x}(\pm 1) = 0$.
- Direct type: Consider $F_{2k}(x) = -\frac{x^{2k}-kx^2}{1-k}$. The points $x = \pm 1$ are equilibrium points, since $F_{2k}(\pm 1) = -1$ and $\frac{\partial F}{\partial x}(\pm 1) = 0$.

Q.E.D.

6. Future work

Definition 6.1. *A geodesic $\gamma(t)$ is globally minimizing if each of its compact sub-arcs realizes the distance between its endpoints, in other words, $\gamma(t)$ is an isometric embedding of the real line.*

In [1, 2, 3], Andertov and Sachkov proved that the Euler Soliton for $k = 2$ is globally minimizing geodesic and in, [4], Hakavuori and Donne showed that neither periodic nor turn back geodesics are globally minimizing geodesic. These results suggest the following conjecture.

Conjecture 6.2. *The global minimizers geodesics on J^k are homoclinic and heteroclinic of direct type.*

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