

# METRIC LINES IN THE SPACE OF CURVES.

DANIELLA CATALÁ, MIRIAM VOLLMAYR-LEE,  
AND ALEJANDRO BRAVO-DODDOLI

ABSTRACT. This paper investigates sub-Riemannian geodesics within the jet space of curves. We establish the existence of two distinct families of metric lines, that is, globally minimizing geodesics, in the 2-jet space of plane curves. This result provides an initial contribution toward the broader classification of metric lines in jet spaces. Additionally, we present precise criteria, which characterize when a sub-Riemannian geodesic in the 2-jet space of plane curves can be identified as a metric line.

MSC CODE

53C17, 53C22, 49K15, 20F65, 22E25.

## 1. INTRODUCTION

In this work, we employ the sequence method to demonstrate the existence of two distinct families of metric lines in the 2-jet space of plane curves, denoted  $\mathcal{J}^2(\mathbb{R}, \mathbb{R}^2)$ . This result marks an initial advance toward resolving the conjecture regarding the classification of metric lines in  $\mathcal{J}^2(\mathbb{R}, \mathbb{R}^2)$ . More broadly, in the general setting, the  $k$ -jet space of curves  $\mathcal{J}^k(\mathbb{R}, \mathbb{R}^n)$  admits the structure of a Carnot group [16]. Every Carnot group admits a sub-Riemannian structure, and so does  $\mathcal{J}^k(\mathbb{R}, \mathbb{R}^n)$ . To place our study in context, consider a sub-Riemannian geodesic on a sub-Riemannian manifold  $M$ , meaning a curve that locally minimizes arc length. A central question then arises: under what conditions does a sub-Riemannian geodesic serve as a global minimizer? We define a metric line as a geodesic  $\gamma : \mathbb{R} \rightarrow M$  such that  $\gamma(t)$  is globally minimizing.

The sequence method, introduced by the third author in [2, 3], offers a systematic framework for determining whether a given candidate geodesic is a metric line. The process begins by selecting a candidate geodesic, obtained by classifying geodesics in  $\mathcal{J}^k(\mathbb{R}, \mathbb{R}^n)$  into five distinct types: line, periodic, homoclinic, turn-back, and direct-type (see Section 2.1.4). Based on this classification, the candidates of interest are those of homoclinic or direct-type. With this context, we now state our conjecture regarding metric lines in  $\mathcal{J}^k(\mathbb{R}, \mathbb{R}^n)$ .

---

*Key words and phrases.* Sub-Riemannian Geometry, Carnot Groups, Jet-space, Metric lines.

**Conjecture 1.** *The metric lines in  $\mathcal{J}^k(\mathbb{R}, \mathbb{R}^n)$  are precisely the line, homoclinic, and direct-type geodesics.*

It is important to note that every Carnot group admits a family of sub-Riemannian geodesics of the line type, all of which are metric lines. In contrast, periodic and turn-back geodesics in  $\mathcal{J}^k(\mathbb{R}, \mathbb{R}^n)$  do not correspond to metric lines [3].

The first main contribution of this work is as follows.

**Theorem A.** *Besides line geodesics,  $\mathcal{J}^2(\mathbb{R}, \mathbb{R}^2)$  admits two families of metric lines of homoclinic type. Modulo Carnot dilatations and translations, these families depend on a single parameter and are defined in Section 2.2.1.*

We have two remarks about Theorem A. First,  $\mathcal{J}^2(\mathbb{R}, \mathbb{R}^2)$  is a 3-step Carnot group, which implies that it does not admit geodesics of the direct-type (a higher step is required). Second, modulo Carnot translations and dilatations on  $\mathcal{J}^2(\mathbb{R}, \mathbb{R}^2)$ , the sub-Riemannian geodesics of homoclinic type in  $\mathcal{J}^2(\mathbb{R}, \mathbb{R}^2)$  are a two-parameter family (see Section 2.2). Finally, in this paper, we consider the general case of homoclinic geodesics, setting the foundation for the general proof of Conjecture 1, which will be completed by proving two additional results concerning an auxiliary function for the sequence method known as the period map (see Section Future Work and details below).

The second main result of this paper is Theorem B, which establishes the condition under which a sub-Riemannian geodesic in  $\mathcal{J}^2(\mathbb{R}, \mathbb{R}^2)$  is a metric line. We prove Theorem B using the sequence method. The ultimate goal of the sequence method is to tackle a general conjecture that classifies metric lines in Carnot groups (see Conjecture 5). It is important to note that Conjecture 5 states that the metric lines are precisely the geodesics exhibiting asymptotic velocity. Moreover, Proposition 6 gives a necessary condition for a sub-Riemannian geodesic to be classified as a metric line. This condition states that the average velocity of the geodesic is parallel to a direction. Metric lines do not include periodic and turn-back geodesics, as these do not exhibit the required asymptotic behavior.

Another key contribution of this paper is to clarify the significance of the asymptotic velocity introduced above. Given a candidate geodesic, we will construct an auxiliary sub-Riemannian manifold  $\mathbb{R}_{(a,b)}^5$ . The sequence method relies on the period map, which encodes the asymptotic behavior of the sub-Riemannian geodesics. A crucial requirement of the sequence method is that the period map be one-to-one. In previous works [5, 2, 3], we considered Carnot groups whose period map had a 1-dimensional domain, which enabled us to establish this requirement using tools from calculus. In the case of  $\mathbb{R}_{(a,b)}^5$ , the period map has a 2-dimensional domain. In this paper, we use the Hadamard Global Diffeomorphism Theorem to show that the period map is one-to-one [10].

The problem of classifying metric lines in  $\mathcal{J}^2(\mathbb{R}, \mathbb{R}^2)$  remains open for the general three-parameter family of homoclinic geodesics. However, thanks to

our work, this problem has been reduced to proving that the image of the period map is simply connected.

## STRUCTURE OF THE PAPER

Section 2 introduces the foundational concepts and tools employed throughout this work. Section 2.1 provides a concise overview of sub-Riemannian geometry and constructs the sub-Riemannian structure on  $\mathcal{J}^k(\mathbb{R}, \mathbb{R}^n)$ . Section 2.2 focuses on the specific case of  $\mathcal{J}^2(\mathbb{R}, \mathbb{R}^2)$ , where we parametrize the family of homoclinic geodesics relevant to our study in terms of the parameters  $(a, b)$ . Section 2.3 constructs the auxiliary sub-Riemannian space  $\mathbb{R}_{(a,b)}^5$ , which serves as the framework for applying the sequence method. Section 3 presents the sequence method, states and proves Theorem B, and concludes with the proof of Theorem A as a direct consequence of Theorem B. Appendix A presents the Carnot structure of  $\mathcal{J}^k(\mathbb{R}, \mathbb{R}^n)$  along with its symmetries. Appendix B contains the proofs of the propositions characterizing abnormal geodesics in both  $\mathcal{J}^k(\mathbb{R}, \mathbb{R}^n)$ , and  $\mathbb{R}_{(a,b)}^5$ .

## 2. PRELIMINARY

This section introduces the fundamental concepts and constructions needed to prove our main results. Section 2.1 presents  $\mathcal{J}^k(\mathbb{R}, \mathbb{R}^n)$  as a sub-Riemannian manifold. Section 2.1.1 reviews the notions of Carnot groups and formally states Conjecture 5, which proposes a classification of metric lines in Carnot groups. This is accompanied by Proposition 6, which characterizes them. Section 2.1.2 develops the sub-Riemannian structure of the jet space  $\mathcal{J}^k(\mathbb{R}, \mathbb{R}^n)$ , Section 2.1.3 outlines the construction of its geodesics, and Section 2.1.4 provides their classification. Section 2.2 presents the essential properties of  $\mathcal{J}^2(\mathbb{R}, \mathbb{R}^2)$  relevant to the proof of Theorem A. Section 2.3 describes the sub-Riemannian structure of  $\mathbb{R}_{(a,b)}^5$ , Section 2.3.1 constructs  $\mathbb{R}_{(a,b)}^5$ , Section 2.3.2 describes the sub-Riemannian geodesics in  $\mathbb{R}_{(a,b)}^5$ , Section 2.3.3 parametrizes the set of homoclinic geodesics, Section 2.3.4 introduces the cost function, an auxiliary tool used to define the period map, Section 2.3.5 defines the period map and examines its key properties, and Section 2.3.6 presents essential results on sequences of geodesics.

**2.1. Jet Space as a Sub-Riemannian Manifold.** A sub-Riemannian manifold is a triple  $(M, \mathcal{D}, \langle \cdot, \cdot \rangle)$ , where  $M$  is a smooth manifold,  $\mathcal{D}$  is a bracket generating distribution, and  $\langle \cdot, \cdot \rangle$  is an inner product on  $\mathcal{D}$  [14, 1]. A curve  $\gamma(t)$  is called horizontal if  $\dot{\gamma}(t) \in \mathcal{D}$  whenever  $\dot{\gamma}$  is defined. Chow's Theorem states that if a distribution  $\mathcal{D}$  is bracket generating, then any two points in  $M$  are connected by a horizontal curve [14]. The sub-Riemannian arc length of a horizontal curve is defined in the usual way, as in Riemannian geometry [1, 14]. A curve is called a geodesic if it is locally minimizing. The Pontryagin maximum principle implies the existence of two families of sub-Riemannian geodesics: normal and abnormal. The sub-Riemannian kinetic

energy is a Hamiltonian function  $H_{sR} : T^*M \rightarrow \mathbb{R}$  whose solutions, when projected to  $M$ , are locally minimizing geodesics; these geodesics are called normal. There is no analogous notion for abnormal geodesics in Riemannian geometry. The primary goal of this paper is to classify metric lines in the space of curves  $\mathcal{J}^2(\mathbb{R}, \mathbb{R}^2)$ , and the principal difficulty in this study is the presence of abnormal geodesics.

**Definition 2.** *Let  $M$  be a sub-Riemannian manifold, let  $dist_M(\cdot, \cdot)$  be the sub-Riemannian distance on  $M$ . A curve  $\gamma : \mathbb{R} \rightarrow M$  is a metric line if it is a globally minimizing geodesic, i.e.,*

$$|a - b| = dist_M(\gamma(a), \gamma(b)) \text{ for all compact intervals } [a, b] \subset \mathbb{R}.$$

Alternative names for the term “metric line” are “globally minimizing geodesic”, “isometric embedding of the real line”, and “infinite geodesic” [1, 7].

The following concept is fundamental to this work.

**Definition 3.** *Let  $(M, \mathcal{D}_M, \langle \cdot, \cdot \rangle_M)$  and  $(N, \mathcal{D}_N, \langle \cdot, \cdot \rangle_N)$  be two sub-Riemannian manifolds, and  $\phi : M \rightarrow N$  be a submersion. The submersion  $\phi$  is called sub-Riemannian if it respects the sub-Riemannian structures, i.e.,  $\phi_*\mathcal{D}_M = \mathcal{D}_N$  and  $\phi^*\langle \cdot, \cdot \rangle_N = \langle \cdot, \cdot \rangle_M$  where  $\phi_*$  and  $\phi^*$  are the push-forward and pull-back of  $\phi$ , respectively.*

Sub-Riemannian submersions are a special case of a more general class of maps, called submetry maps, in the framework of metric geometry (see Definition 3.1.23 [13]). Every sub-Riemannian submersion (or submetry) comes with an inverse map at the level of horizontal curves, called the horizontal lift (see Proposition 3.1.24 from [13]). A classical result in sub-Riemannian geometry is the following.

**Lemma 4** (Proposition 1, [5]). *Let  $\phi : M \rightarrow N$  be a sub-Riemannian submersion (or submetry). If  $\gamma(t)$  is the horizontal lift of a metric line in  $N$ , then  $\gamma(t)$  is a metric line in  $M$ .*

2.1.1. *Carnot Groups and the General Conjecture.* A Carnot group  $\mathbb{G}$  is a simply connected Lie group whose Lie algebra  $\mathfrak{g}$  is graded and nilpotent, i.e., the Lie algebra satisfies:

$$\mathfrak{g} = V_1 \oplus \cdots \oplus V_s, \quad [V_1, V_i] \subset \mathfrak{g}_{i+1} \quad \text{and} \quad \mathfrak{g}_{s+1} = \{0\},$$

where  $[\cdot, \cdot]$  is the Lie bracket and the integer  $s$  is called the step of  $\mathbb{G}$  [14, 1, 13].

Every Carnot group  $\mathbb{G}$  possesses a canonical projection  $\pi : \mathbb{G} \rightarrow V$ , where we can identify  $V \simeq \mathbb{G}/[\mathbb{G}, \mathbb{G}] \simeq \mathfrak{g}_1$ . In addition, we equip  $\mathbb{G}$  with a left-invariant distribution  $\mathcal{D} := (L_g)_*V_1$ , where  $L_g$  is the left translation by the element  $g \in \mathbb{G}$ , and  $(L_g)_*$  is the push-forward of  $L_g$ . In this paper, we will consider a left-invariant sub-Riemannian metric in  $\mathbb{G}$  [1]. We will equip  $V$  with the Euclidean inner product, which makes  $\pi$  a sub-Riemannian submersion. With this construction, if  $\gamma(t)$  is the horizontal lift of a straight

line in  $V$ , then Lemma 4 implies  $\gamma(t)$  is a metric line in  $\mathbb{G}$ . This family of sub-Riemannian geodesics is called line geodesics (see Section 2.1.4 for more details).

With the above discussion in mind, a natural question arises: beyond geodesic lines, what constitutes a metric line in a Carnot group  $\mathbb{G}$ ? The conjecture proposed by the third author of this paper seeks to address this question as follows:

**Conjecture 5.** *Consider a left-invariant sub-Riemannian structure in a Carnot group  $\mathbb{G}$ . The metric lines in  $\mathbb{G}$  are precisely the sub-Riemannian geodesics  $\gamma(t)$  which are parameterized by arc length and for which there exists a unit  $v \in \mathfrak{g}$  such that:*

$$v = \lim_{t \rightarrow -\infty} (L_{\gamma^{-1}(t)})_* \dot{\gamma}(t) = \lim_{t \rightarrow \infty} (L_{\gamma^{-1}(t)})_* \dot{\gamma}(t).$$

The following result gives a necessary condition for a geodesic to be a metric line.

**Proposition 6** (Proposition 2.6, [3]). *Consider the same hypotheses as in Conjecture 5. A necessary condition for  $\gamma(t)$  to be a metric line is that there exists a unit vector  $v \in V_1$  such that:*

$$(2.1) \quad 1 = \lim_{t \rightarrow \infty} \frac{1}{2t} \int_{-t}^t \langle v, (L_{\gamma^{-1}(s)})_* \dot{\gamma}(s) \rangle ds.$$

Conjecture 5 implies the necessary condition from Proposition 6.

2.1.2. *The Jet Space.* We will construct  $\mathcal{J}^k(\mathbb{R}, \mathbb{R}^n)$  and present its sub-Riemannian structure here, consult [16] for more details. Let  $f(x)$  and  $g(x)$  be two smooth curves. We define an equivalence relation on the set of germs of smooth curves at  $x_0 \in \mathbb{R}$  by the following relation:

$$f(x) \sim_{x_0} g(x) \quad \text{if and only if} \quad \|f(x) - g(x)\| = O(|x - x_0|^{k+1}).$$

The  $k$ -jet space is the set of equivalence classes given by:

$$\mathcal{J}^k(\mathbb{R}, \mathbb{R}^n) = \bigcup_{x \in \mathbb{R}} C^k(\mathbb{R}, \mathbb{R}^n) / \sim_x.$$

Elements in  $\mathcal{J}^k(\mathbb{R}, \mathbb{R}^n)$  will be denoted by  $j_x^k(f)$ .

The space  $\mathcal{J}^k(\mathbb{R}, \mathbb{R}^n)$  admits global coordinates; we determine an element  $j_{x_0}^k(f)$  for the point  $x_0$ , a list of  $n$  functions, and their  $k$ -derivatives. Then, the point  $j_{x_0}^k(f)$  will be denoted by:

$$j_{x_0}^k(f) = (x_0, \mathbf{u}^k, \dots, \mathbf{u}^0), \quad \text{where } \mathbf{u}^i = (u_1^i, \dots, u_n^i) \text{ for all } i = 0, \dots, k,$$

$$\text{and } u_\ell^i = \frac{d^i f_\ell}{dx^i}(x_0) \text{ for all } \ell = 1, \dots, n \text{ and } i = 0, \dots, k.$$

The  $k$ -jets space  $\mathcal{J}^k(\mathbb{R}, \mathbb{R}^n)$  has a natural  $(n+1)$ -rank distribution  $\mathcal{D}$  defined by the following set of Pfaffian equations:

$$0 = du_\ell^i - u_\ell^{i+1} dx, \quad \text{for all } i = 0, \dots, k-1, \text{ and } \ell = 1, \dots, n.$$

The following vector fields provide a global frame for the distribution  $\mathcal{D}$ :

$$(2.2) \quad X := \frac{\partial}{\partial x} + \sum_{\ell=1}^n \sum_{i=0}^{k-1} u_{\ell}^{i+1} \frac{\partial}{\partial u_{\ell}^i}, \quad Y_{\ell} := \frac{\partial}{\partial u_{\ell}^k}, \quad \text{where } \ell = 1, \dots, n.$$

To endow  $\mathcal{J}^k(\mathbb{R}, \mathbb{R}^n)$  with a sub-Riemannian metric, we consider the quadratic form:

$$ds^2 = (dx^2 + (du_1^k)^2 + \dots + (du_n^k)^2) \Big|_{\mathcal{D}}.$$

The canonical projection  $\pi : \mathcal{J}^k(\mathbb{R}, \mathbb{R}^n) \rightarrow \mathbb{R}^{n+1}$  is given in coordinates by  $\pi(j_x^k(f)) = (x, \mathbf{u}^k)$ . In the appendix, we discuss the Lie algebra and Carnot structure of  $\mathcal{J}^k(\mathbb{R}, \mathbb{R}^n)$ , as well as the symmetries on  $\mathcal{J}^k(\mathbb{R}, \mathbb{R}^n)$  arising from the automorphisms of the Lie algebra.

**2.1.3. Construction of sub-Riemannian geodesics.** We present a correspondence between vector-valued functions in  $\mathbb{R}^n$  and sub-Riemannian geodesics in  $\mathcal{J}^k(\mathbb{R}, \mathbb{R}^n)$ .

Let  $P_{\mu}(x)$  be a vector-valued function in  $\mathbb{R}^n$ , whose entries are polynomial functions of degree  $k$  depending on the parameter  $\mu \in \mathbb{R}^{kn}$ , i.e.,

$$P_{\mu}(x) = (P_1(x; \mu), \dots, P_n(x; \mu)),$$

where the entries have the form:

$$P_{\ell}(x; \mu) = a_{\ell}^0 + a_{\ell}^1 x + \dots + a_{\ell}^k x^k, \quad \text{for all } \ell = 1, \dots, n.$$

Here  $\mu = (\mathbf{a}^0, \dots, \mathbf{a}^k) \in \mathbb{R}^{k(n+1)}$ , and  $\mathbf{a}^i = (a_1^i, \dots, a_n^i)$ . Observe that the mapping  $\mu \rightarrow P_{\mu}(x)$  defines a linear bijection between  $\mathbb{R}^{kn}$  and the set of vector-valued functions  $P_{\mu}(x)$ . Thus, determining  $P_{\mu}(x)$  is equivalent to prescribing the value of  $\mu$ .

Let  $T^*\mathcal{H}$  be the cotangent bundle of  $\mathbb{R}$  with the canonical coordinates  $(p_x, x)$ . The vector-valued function  $P_{\mu}(x)$  defines a reduced Hamiltonian  $H_{\mu} : T^*\mathbb{R} \rightarrow \mathbb{R}$  as follows:

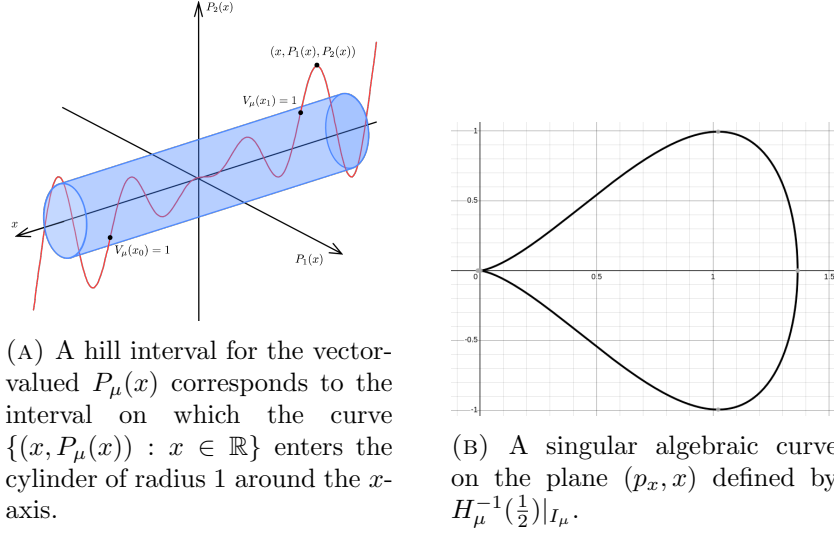
$$(2.3) \quad H_{\mu}(p_x, x) = \frac{1}{2} p_x^2 + \frac{1}{2} V_{\mu}(x), \quad \text{where } V_{\mu}(x) = \|P_{\mu}(x)\|^2.$$

Here  $\|\cdot\|$  is the Euclidean norm in  $\mathbb{R}^n$ . The dynamics of the  $x$ -component take place in a closed interval called the hill interval. We introduce its formal definition below.

**Definition 7.** *A hill interval for a polynomial potential  $V_{\mu}(x)$  is a closed interval  $I_{\mu}$  with the property that  $V_{\mu}(x) < 1$  if  $x \in \text{Int } I_{\mu}$  and  $V_{\mu}(x) = 1$  if  $x \in \partial I_{\mu}$ .*

We prescribe a two-step method to construct a normal sub-Riemannian geodesic: first, find a solution  $(p_x(t), x(t))$  to the reduced Hamiltonian system with the energy condition  $H(p_x(t), x(t)) = \frac{1}{2}$ . Second, define a horizontal curve  $\gamma(t) \in \mathcal{J}^k(\mathbb{R}, \mathbb{R}^n)$  by solving the differential equation:

$$(2.4) \quad \dot{\gamma}(t) = p_x(t)X + P_1(x(t); \mu)Y_1 + \dots + P_n(x(t); \mu)Y_n,$$



(A) A hill interval for the vector-valued  $P_\mu(x)$  corresponds to the interval on which the curve  $\{(x, P_\mu(x)) : x \in \mathbb{R}\}$  enters the cylinder of radius 1 around the  $x$ -axis.

(B) A singular algebraic curve on the plane  $(p_x, x)$  defined by  $H_\mu^{-1}(\frac{1}{2})|_{I_\mu}$ .

FIGURE 2.1. The panels illustrate the hill interval and the algebraic curves associated with momentum  $\mu$ .

where  $\{X, Y_1, \dots, Y_n\}$  is the frame of left-invariant vector fields as defined in Eq. (2.2). It is important to note that the condition  $H_\mu = \frac{1}{2}$  implies that  $\gamma(t)$  is parametrized by arc length. Moreover, the curve is defined for all  $t \in \mathbb{R}$  by virtue of the reduced Hamiltonian system being complete. The following result states that  $\gamma(t)$  is a sub-Riemannian geodesic.

**Theorem 8** (Theorem 1.2, [4]). *The above prescription yields a normal geodesic in  $\mathcal{J}^k(\mathbb{R}, \mathbb{R}^n)$ , parameterized by arc length and with momentum  $\mu$ . Conversely, every arc length parameterized normal geodesic in  $\mathcal{J}^k(\mathbb{R}, \mathbb{R})$  arises via this prescription by applying it to the vector-valued function  $P_\mu(x)$ .*

The observations made in [2, Remark 2.2] about an equivalent theorem in the framework of the jet space of functions  $\mathcal{J}^k(\mathbb{R}, \mathbb{R})$  are equally applicable in our framework.

The Hamiltonian  $H_\mu$  defines a Hamiltonian system obtained through the symplectic reduction of the geodesic flow (see Appendix A). Throughout this paper, we refer to this system as the reduced system or reduced dynamics. The classification of sub-Riemannian geodesics employed in the sequence method is based on the behavior of their reduced dynamics. To this end, we now characterize the system's equilibrium points.

**Proposition 9.** *Consider a point  $(p_x, x) \in H_\mu^{-1}(\frac{1}{2})$ . Then,  $(p_x, x)$  is an equilibrium point of the Hamiltonian system if and only if  $p_x = 0$  and  $x$  satisfies  $\|P_\mu(x)\| = 1$  and  $P_\mu(x) \perp P'_\mu(x)$ .*

*Proof.* A point  $(p_x, x) \in H_\mu^{-1}(\frac{1}{2})$  is an equilibrium point if and only if

$$(0, 0) = \nabla H_\mu = (p_x, P_\mu(x) \cdot P'_\mu(x)),$$

where  $P'_\mu(x) = \frac{d}{dx}P_\mu(x)$  and  $v_1 \cdot v_2$  is the Euclidean product in  $\mathbb{R}^n$ . The condition  $(0, 0) = \nabla H_\mu$  implies  $p_x = 0$ . Hence,  $(p_x, x) = (0, x) \in H_\mu^{-1}(\frac{1}{2})$  if and only if  $\|P_\mu(x)\| = 1$ .  $\square$

It is important to note that the equilibrium points of the reduced Hamiltonian system correspond to the singular points of the algebraic curve  $H_\mu^{-1}(\frac{1}{2})$ . This observation will play a fundamental role later (see Proposition 15, and figure 2.1).

**2.1.4. Classification of Sub-Riemannian Geodesics.** We now classify the sub-Riemannian geodesics by their reduced dynamics. According to the classical theory of one-degree-of-freedom systems, the sub-Riemannian geodesics fall into one of the following types:

**Line geodesics:** A geodesic  $\gamma(t)$  is of the line type if  $\pi(\gamma(t))$  is a straight line in  $\mathbb{R}^{n+1}$ . Line geodesics correspond to a constant polynomial  $P_\mu(x) = \mathbf{c}$ . The hill interval  $I_\mu = \mathbb{R}$  if  $\|\mathbf{c}\| < 1$  or  $I_\mu$  is a singleton if  $\|\mathbf{c}\| = 1$ .

**Periodic geodesics:** A geodesic  $\gamma(t)$  is of the periodic type with hill interval  $I_\mu$  if the reduced dynamics are periodic. The reduced dynamics are periodic if and only if  $H_\mu^{-1}(\frac{1}{2})|_{I_\mu}$  is smooth.

**Homoclinic geodesics:** A geodesic  $\gamma(t)$  is of the homoclinic type with hill interval  $I_\mu$  if the reduced dynamics have a homoclinic orbit. The reduced dynamics have a homoclinic orbit if and only if  $H_\mu^{-1}(\frac{1}{2})|_{I_\mu}$  only contains a singular point.

**Heteroclinic geodesics:** A geodesic  $\gamma(t)$  is of the heteroclinic type with hill interval  $I_\mu$  if the reduced dynamic has a heteroclinic orbit. The reduced dynamics have a heteroclinic orbit if and only if  $H_\mu^{-1}(\frac{1}{2})|_{I_\mu}$  contains two singular points.

Eq. (2.4) implies that every homoclinic geodesic satisfies the condition stated in Conjecture 5. However, this does not hold for every heteroclinic geodesic. The following definitions distinguish whether a heteroclinic geodesic satisfies the condition stated in Conjecture 5.

**Direct geodesics:** A heteroclinic geodesic  $\gamma(t)$  is of the direct-type with hill interval  $I_\mu = [x_0, x_1]$  if and only if  $P_\mu(x_0) = P_\mu(x_1)$ .

**Turn-back geodesics:** A heteroclinic geodesic  $\gamma(t)$  is of the Turn-back type with hill interval  $I_\mu = [x_0, x_1]$  if and only if  $P_\mu(x_0) \neq P_\mu(x_1)$ .

The following result characterizes the abnormal geodesics in  $\mathcal{J}^2(\mathbb{R}, \mathbb{R}^n)$ .

**Proposition 10.** *Let  $2 \leq k$ . The singular curves in  $\mathcal{J}^k(\mathbb{R}, \mathbb{R}^n)$  are the precisely the curves  $\gamma(t)$  with the property that*

$$\dot{\gamma}(t) \in \text{span}\{Y_1, \dots, Y_n\}, \text{ whenever } \dot{\gamma}(t) \text{ exist.}$$

*Consequently, the abnormal geodesics in  $\mathcal{J}^k(\mathbb{R}, \mathbb{R}^n)$  are geodesic lines whose  $x$ -component is constant.*

**2.2. The two-jet space of plane curves.** For the remainder of the paper, we focus on  $\mathcal{J}^2(\mathbb{R}, \mathbb{R}^2)$ . We begin by classifying its singular geodesics.

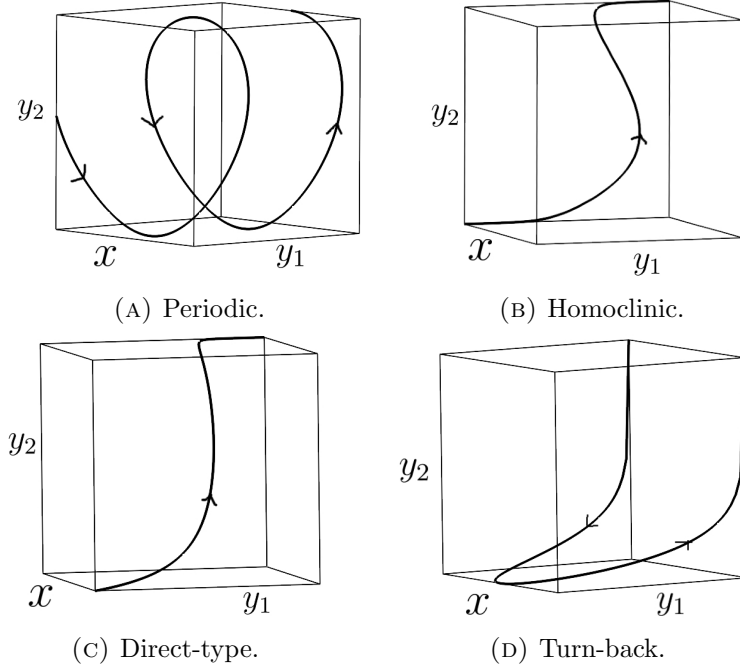


FIGURE 2.2. The panel displays the projection of the different types of geodesics in  $\mathcal{J}^2(\mathbb{R}, \mathbb{R}^2)$  to  $\mathbb{R}^3$ , with coordinates  $(x, y_1, y_2)$

**Proposition 11.** *The only singular geodesics in  $\mathcal{J}^2(\mathbb{R}, \mathbb{R}^2)$  are homoclinic.*

*Proof.* The potential function  $V_\mu(x)$  corresponding to geodesics in  $\mathcal{J}^2(\mathbb{R}, \mathbb{R}^2)$  has degree 4. To admit a geodesic of the type heteroclinic, the potential  $V_\mu(x)$  must have at least degree 5. Indeed, if  $V_\mu(x)$  is a potential for a heteroclinic geodesic and hill interval  $I_\mu = [x_0, x_1]$ , then  $V_\mu(x)$  must admit the factorization  $1 - V_\mu(x) = (x - x_0)^{r_0}(x - x_1)^{r_1}q(x)$  where  $2 \leq r_0$  and  $2 \leq r_1$ . If  $2 = r_0$  and  $2 = r_1$ , then  $V_\mu(x)$  must have local minimum inside  $(x_0, x_1)$ .  $\square$

Our goal is to classify metric lines up to isometries; that is, we will employ the symmetries described in Appendix A.1 to reduce the number of parameters. First, by applying Carnot translations and dilations, we restrict our attention to homoclinic geodesics with hill interval  $I_\mu = [0, x_1]$ . Second, using rotational symmetry, we restrict our attention to the asymptotic velocity  $v := (0, 1, 0)$ .

We denote by  $\Lambda \subset \mathbb{R}^6$  the subset of momenta corresponding to homoclinic geodesics with the properties mentioned above, i.e.,  $\mu \in \Lambda$  if and only if the associated vector-valued function  $P_\mu(x)$  satisfies the following properties:

- $\mu$  admits a hill interval  $I_\mu = [0, x(\mu)]$ , where  $0 < x(\mu)$ .
- $P_\mu(0) = (1, 0)$  and  $P_\mu(0) \cdot P'_\mu(0) = 0$ .

Let  $\mathcal{A}$  be the set whose interior and boundary are defined by

$$\begin{aligned} \text{Int } \mathcal{A} &:= \{(\lambda, a, b) \in (0, \infty) \times \mathbb{R}^2 : 0 < 2 - b^2, \ a \in \mathbb{R}\}, \\ \partial \mathcal{A} &:= \{(\lambda, a, b) \in (0, \infty) \times \mathbb{R}^2 : b = \pm\sqrt{2} \ ab < 0\}. \end{aligned}$$

Then, a bijection  $\mu : \mathcal{A} \rightarrow \Lambda$  is given by

$$(2.5) \quad \mu(\lambda, a, b) = \left(1, 0, 0, \frac{b}{\lambda}, \frac{-1}{\lambda^2}, \frac{a}{\lambda^2}\right) := (a_1^0, a_2^0, a_1^1, a_2^1, a_1^2, a_2^2) \in \Lambda.$$

The function  $\mu : \mathcal{A} \rightarrow \Lambda$  is clearly injective. To see that it is surjective, note that the condition  $P_\mu(0) = (1, 0)$  implies  $(a_1^0, a_2^0) = (1, 0)$ . The condition  $P_\mu(0) \cdot P'_\mu(0) = 0$  implies that  $P'_\mu(0)$  is parallel to  $(0, 1)$ , so  $a_1^1 = 0$ . The condition  $0 < x(\mu)$  implies  $P_1(x; \mu)$  is decreasing for  $0 < x$ , so  $a_1^2 < 0$ . By setting  $a_1^2 = \frac{-1}{\lambda^2}$ ,  $a_2^1 = \frac{b}{\lambda}$ , and  $a_2^2 = \frac{a}{\lambda^2}$ , we find that  $0 < x(\mu)$  if and only if  $(\lambda, a, b) \in \mathcal{A}$ .

**2.2.1. The Candidate Geodesics.** Here, we will introduce the two families of homoclinic geodesics mentioned in Theorem A. To define these families, we will specify their momentum  $\mu$ , in other words, we will define two subsets of  $\Lambda$  using Eq. (2.5).

**Family 1:** If  $\mathcal{A}_1 := \mathcal{A}|_{b=0}$  and  $\Lambda_1 := \mu(\mathcal{A}_1)$ , then the geodesics in the *Family 1* corresponding to a momentum  $\mu \in \Lambda_1$ .

**Family 2:** If  $\mathcal{A}_2 := \mathcal{A}|_{a=0}$  and  $\Lambda_2 := \mu(\mathcal{A}_2)$ , then the geodesics in the *Family 2* corresponding to a momentum  $\mu \in \Lambda_2$ .

We will devote the rest of the paper to proving Theorem 12, which states that the homoclinic geodesics with momentum  $\mu \in \Lambda_i$ , with  $i = 1, 2$ , are metric lines.

Before continuing our study, let us exclude the case  $(a, b) = (0, 0) \in \mathcal{A}_1 \cap \mathcal{A}_2$ . The Carnot group  $\mathcal{J}^2(\mathbb{R}, \mathbb{R}^2)$  contains a normal subgroup  $\mathcal{N}$  such that  $\mathcal{J}^2(\mathbb{R}, \mathbb{R}^2)/\mathcal{N} \simeq \mathcal{J}^2(\mathbb{R}, \mathbb{R})$  (for the explicit construction of this subgroup see Appendix A). It follows that if canonical projection  $\pi_{\mathcal{N}} : \mathcal{J}^2(\mathbb{R}, \mathbb{R}^2)/\mathcal{N} \rightarrow \mathcal{J}^2(\mathbb{R}, \mathbb{R})$  is a submetry map by Corollary 6.3.5 from [13]. Moreover, if  $\gamma(t)$  is geodesic with momentum  $\mu = \mu(\mathcal{A}_1 \cap \mathcal{A}_2)$ , then  $\pi_{\mathcal{N}}(\gamma(t))$  is a geodesic in  $\mathcal{J}^2(\mathbb{R}, \mathbb{R})$ . This geodesic was studied already in [2], where Theorem 1.4 implies that  $\pi_{\mathcal{N}}(\gamma(t))$  is a metric line in  $\mathcal{J}^2(\mathbb{R}, \mathbb{R})$ . Therefore, Lemma 4 implies  $\gamma(t)$  is a metric line in  $\mathcal{J}^2(\mathbb{R}, \mathbb{R}^2)$ . Having in mind this discussion, we define the set

$$\Lambda_0 := \{\mu = (1, 0, 0, 0, \frac{-1}{\lambda^2}, 0) : \lambda \in (0, \infty)\}.$$

For the rest of the paper, we restrict our attention to geodesics with momentum  $\mu \in \Lambda \setminus \Lambda_0$ .

**2.3. Magnetic Space.** Let  $\gamma_h(t)$  be a homoclinic geodesic with momentum  $\mu \in \Lambda \setminus \Lambda_0$ . Without loss of generality, by applying a Carnot dilatation, we may focus our attention on the case  $\lambda = 1$ . Thus, for the remainder of the

paper, we consider the vector-valued function

$$P_{(a,b)}(x) := (P_1(x), P_2(x)) = (1 - x^2, bx + ax^2), \quad \text{with } (a, b) \neq (0, 0).$$

We will avoid using the notation  $\mu(1, a, b)$  and  $P_i(x; \mu)$ , since  $\mu$  is uniquely determined by  $(a, b)$ , and we wish to simplify notation.

We will build a sub-Riemannian magnetic space  $\mathbb{R}_{(a,b)}^5$ , whose geometry depends on the vector-valued function  $P_{(a,b)}(x)$ . We also construct a sub-Riemannian submersion  $\pi_{(a,b)} : \mathcal{J}^k(\mathbb{R}, \mathbb{R}^n) \rightarrow \mathbb{R}_{(a,b)}^5$  factorizing the submersion  $\pi$ , i.e.  $\pi = pr \circ \pi_{(a,b)}$  where  $pr : \mathbb{R}_{(a,b)}^5 \rightarrow \mathbb{R}^3$  is a submersion. Therefore, by Lemma 4, the goal of the sequence method is to prove that the sub-Riemannian geodesic  $c_h(t) := \pi_{(a,b)}(\gamma_h(t))$  is a metric line in  $\mathbb{R}_{(a,b)}^5$ , we will call  $c_h(t)$  the candidate geodesic. Let us introduce the formal statement.

**Theorem 12.** *Consider  $\mu(1, a, b) \in \Lambda_i \setminus \Lambda_0$ , with  $i = 1, 2$ , and let  $\gamma_h(t)$  be a homoclinic geodesic with momentum  $\mu(1, a, b)$ . If  $c_h(t) := \pi_{(a,b)}(\gamma_h(t))$ , then  $c_h(t)$  is metric line in  $\mathbb{R}_{(a,b)}^5$ . As a consequence,  $\gamma_h(t)$  is a metric line in  $\mathcal{J}^2(\mathbb{R}, \mathbb{R}^2)$ .*

**2.3.1. The Magnetic Space.** The magnetic space  $\mathbb{R}_{(a,b)}^5$  associated to  $\mathcal{J}^2(\mathbb{R}, \mathbb{R}^2)$  is a sub-Riemannian structure whose construction is as follows.

Consider  $\mathbb{R}_{(a,b)}^5$  with coordinates  $(x, y^1, y^2, z^1, z^2)$ . We will adopt the short notation  $\mathbf{y} = (y^1, y^2)$  and  $\mathbf{z} = (z^1, z^2)$ . We define a non-integrable distribution  $\mathcal{D}_{(a,b)}$  on  $\mathbb{R}_{(a,b)}^5$ , which is defined by the following Pfaffian system:

$$dz^1 - P_1(x)dy^1 = 0, \quad \text{and} \quad dz^2 - P_2(x)dy^2 = 0.$$

Hence, the following vector fields provide a global frame for  $\mathcal{D}_{(a,b)}$

$$\tilde{X} = \frac{\partial}{\partial x}, \quad \tilde{Y}_1 = \frac{\partial}{\partial y^1} + P_1(x) \frac{\partial}{\partial z^1}, \quad \text{and} \quad \tilde{Y}_2 = \frac{\partial}{\partial y^2} + P_2(x) \frac{\partial}{\partial z^2}.$$

The quadratic form  $ds_M^2 = (dx^2 + (dy^1)^2 + (dy^2)^2)|_{\mathcal{D}_{(a,b)}}$  induces an sub-Riemannian inner product on  $\mathcal{D}_{(a,b)}$ . The sub-Riemannian submersion  $\pi_{(a,b)}$  is given by

$$\pi_{(a,b)}(j_x^k(f)) = (x, u_1^2, u_2^2, \sum_{i=0}^2 (-1)^i u_1^{2-i} \frac{d^i P_1}{dx^i}, \sum_{i=0}^2 (-1)^i u_2^{2-i} \frac{d^i P_2}{dx^i}).$$

**2.3.2. Sub-Riemannian Geodesic in the Magnetic Space.** Consider the cotangent bundle  $T^*\mathbb{R}_{(a,b)}^5$  with the canonical coordinates  $(p_x, \mathbf{p}_y, \mathbf{p}_z, x, \mathbf{y}, \mathbf{z})$ . The sub-Riemannian kinetic energy  $H_{(a,b)} : T^*\mathbb{R}_{(a,b)}^5 \rightarrow \mathbb{R}$  is given by

$$H_{(a,b)}(p_x, \mathbf{p}_y, \mathbf{p}_z, x, \mathbf{y}, \mathbf{z}) = \frac{1}{2}p_x^2 + \frac{1}{2}(p_{y^1} + p_{z^1}P_1(x))^2 + \frac{1}{2}(p_{y^2} + p_{z^2}P_2(x))^2.$$

The Hamiltonian function  $H_{(a,b)}$  does not depend on the coordinates  $(\mathbf{y}, \mathbf{z})$ , so the momentum  $\nu := (\mathbf{p}_y, \mathbf{p}_z)$  remains constant under the reduced Hamiltonian flow. Consequently, the translation  $\Phi_{(\mathbf{y}_0, \mathbf{z}_0)} : \mathbb{R}_{(a,b)}^5 \rightarrow \mathbb{R}_{(a,b)}^5$  given

by:

$$(2.6) \quad \Phi_{(\mathbf{y}_0, \mathbf{z}_0)}(x, \mathbf{y}, \mathbf{z}) = (x, \mathbf{y} + \mathbf{y}_0, \mathbf{z} + \mathbf{z}_0),$$

is an isometry.

Analogously to the case of  $\mathcal{J}^2(\mathbb{R}, \mathbb{R}^2)$ , the sub-Riemannian geodesics in  $\mathbb{R}_{(a,b)}^5$  are in bijection with a pencil of polynomials. The following definitions formalize this fact.

**Definition 13.** *The pencil  $\text{Pen}_{(a,b)}$  of  $P_{(a,b)}(x)$  is a 4-dimensional vector space of vector-valued functions with the form  $G_\nu(x) := (G_1(x; \nu), G_2(x; \nu))$  given by*

$$\{G_\nu(x) = (\alpha_1 + \beta_1 P_1(x), \alpha_2 + \beta_1 P_2(x)) : \nu = (\alpha_1, \alpha_2, \beta_1, \beta_2) \in \mathbb{R}^4\}.$$

Let us delineate the sub-Riemannian geodesics in  $\mathbb{R}_{(a,b)}^5$ , as well as their horizontal lift and connection with sub-Riemannian geodesics in  $\mathcal{J}^2(\mathbb{R}, \mathbb{R}^2)$ :

A sub-Riemannian geodesic  $c(t) \in \mathbb{R}_{(a,b)}^5$  corresponds to a vector-valued function  $G_\mu(x)$ , and its  $x$ -component satisfies a one-degree-of-freedom system given by the Hamiltonian function:

$$(2.7) \quad H_\nu(p_x, x) = \frac{1}{2}p_x^2 + \frac{1}{2}V_\nu(x), \quad \text{where } V_\nu(x) = \|G_\nu(x)\|^2.$$

Once  $x(t)$  is found as the solution to the above Hamiltonian system, the coordinates  $\mathbf{y}$  and  $\mathbf{z}$  satisfy the equations

$$(2.8) \quad \dot{y}_i = G_i(x(t); \nu), \quad \text{and } \dot{z}_i = G_i(x(t); \nu)P_i(x(t)) \text{ for all } i = 1, 2.$$

The horizontal lift of the geodesic  $c(t)$  is the solution to the differential equation:

$$\dot{\gamma}(t) = p_x(t)X(\gamma(t)) + G_1(x(t); \nu)Y_1(\gamma(t)) + G_2(x(t), \nu)Y_2(\gamma(t)).$$

With the above discussion, we have proven the following lemma.

**Lemma 14.** *Every sub-Riemannian geodesic in  $\mathbb{R}_{(a,b)}^5$  is the  $\pi_{(a,b)}$ -projection of a geodesic in  $\mathcal{J}^2(\mathbb{R}, \mathbb{R}^2)$  corresponding to a vector-valued function  $G_\mu(x) \in \text{Pen}_{(a,b)}$ . Conversely, the horizontal lift of a geodesic in  $\mathbb{R}_{(a,b)}^5$  is a geodesic corresponding to a vector-valued function in  $\text{Pen}_{(a,b)}$ .*

It is important to note that  $c_h(t) \in \mathbb{R}_{(a,b)}^5$  is a sub-Riemannian geodesic corresponding to the momentum  $\nu = (0, 0, 1, 1)$ .

The dynamics of the reduced system  $H_\nu$  take place on the hill interval  $I_\nu$  as described in Definition 7. A vector-valued function  $G_\mu(x) \in \text{Pen}_{(a,b)}$  defines an isoenergetic algebraic curve given by

$$(2.9) \quad \mathcal{C}(\nu, I_\nu) := \{(p_x, x) : 1 = p_x^2 + V_\nu(x) \text{ and } x \in I_\nu\}.$$

The algebraic curve  $\mathcal{C}(\nu, I_\nu)$  will take relevance later, when we analyze the smoothness of the period map and other functions. The relation between the singular points in  $\mathcal{C}(\mu, I_\nu)$  and the equilibrium point of the Hamiltonian system for  $H_\nu$  is given by the following result.

**Proposition 15.** *Consider  $(p_x, x) \in \mathcal{C}(\nu, I_\nu)$ . The point  $(p_x, x)$  is a singular point of  $\mathcal{C}(\nu, I_\nu)$  if and only if  $(p_x, x)$  is an equilibrium point of the Hamiltonian system for  $H_\mu$ .*

*Proof.* The equilibrium points of the Hamiltonian system for  $H_\mu$  are characterized as in Proposition 9. So  $(p_x, x)$  is an equilibrium point if and only if  $\nabla H_\nu = (0, 0)$ , which is the condition for a singular point in  $\mathcal{C}(\nu, I_\nu)$ .  $\square$

The classification of sub-Riemannian geodesics in  $\mathbb{R}_{(a,b)}^5$  is according to their reduced dynamics  $H_\mu$  as in Section 2.1.4. In particular, the candidate geodesic  $c_h(t)$  is homoclinic.

2.3.3. *The set of homoclinic geodesics.* We denote the subset of momenta corresponding to homoclinic geodesics with the same asymptotic velocity as the candidate geodesic  $c_h(t)$  by

$$\Lambda_{(a,b)} := \Lambda_{(a,b)}^+ \cup \Lambda_{(a,b)}^- \subset \mathbb{R}^4,$$

More precisely,  $\nu \in \Lambda_{(a,b)}$  if and only if the vector-valued function  $G_\nu(x)$  satisfies

- $G_\nu(0) = (1, 0)$  and  $G_\nu(0) \cdot G'_\nu(0) = 0$ .

In addition,

- If  $\nu \in \Lambda_{(a,b)}^+$ , then the hill interval has the form  $I_\nu^+ = [0, x^+(\nu)]$ , where  $0 < x^+(\nu)$ ;
- If  $\nu \in \Lambda_{(a,b)}^-$ , then the hill interval has the form  $I_\nu^- = [x^-(\nu), 0]$  where  $x^-(\nu) < 0$ .

(Case  $0 \leq ab$ ) We define  $\mathcal{A}_{(a,b)}^+$ :

$$(2.10) \quad \mathcal{A}_{(a,b)}^+ := \{(\tau, \eta) \in (0, \infty) \times \mathbb{R} : 0 < 2\tau - b^2\eta^2\}.$$

(Case  $0 < ab$ ) We define  $\mathcal{A}_{(a,b)}^-$ :

$$(2.11) \quad \mathcal{A}_{(a,b)}^- := \{(\tau, \eta) \in (0, \infty) \times \mathbb{R} : 0 \leq 2\tau - b^2\eta^2\}.$$

(Case  $ab < 0$ ) We define  $\mathcal{A}_{(a,b)}^+$ :

$$(2.12) \quad \mathcal{A}_{(a,b)}^+ := \{(\tau, \eta) \in (0, \infty) \times \mathbb{R} : 0 \leq 2\tau - b^2\eta^2\}.$$

(Case  $ab \leq 0$ ) We define  $\mathcal{A}_{(a,b)}^-$ :

$$(2.13) \quad \mathcal{A}_{(a,b)}^- := \{(\tau, \eta) \in (0, \infty) \times \mathbb{R} : 0 < 2\tau - b^2\eta^2\}.$$

Then a bijection  $\nu : \mathcal{A}_{(a,b)}^\pm \rightarrow \Lambda_{(a,b)}^\pm$  is given by

$$(2.14) \quad \nu(\tau, \eta) = (1 - \tau, 0, \tau, \eta).$$

Therefore, the vector-valued function  $G_\mu(x)$  has the form

$$(2.15) \quad G_\nu(x) = (G_1(x; \nu), G_2(x; \nu)) = (1 - \tau x^2, \eta(bx + ax^2)),$$

and the candidate geodesic  $c_h(t)$  has momentum  $\nu(1, 1)$ . Moreover,  $x^\pm(\nu)$  is given by:

$$(2.16) \quad x^\pm(\nu) = \frac{-ab\eta^2 \pm \sqrt{\tau^2(2\tau - b^2\eta^2) + 2a^2\eta^2\tau}}{\tau^2 + a^2\eta^2}.$$

The following result characterizes the abnormal geodesics in  $\mathbb{R}_{(a,b)}^5$ .

**Proposition 16.** *The magnetic space  $\mathbb{R}_{(a,b)}^5$  possesses a family of abnormal geodesics characterized by:*

$$\dot{c}(t) = v_1 \tilde{Y}_1 + v_2 \tilde{Y}_2,$$

where  $(v_1, v_2)$  is a constant vector such that  $0 = (v_1, v_2) \cdot (\frac{dP_1}{dx}, \frac{dP_2}{dx})$ .

The proof of Proposition 3.1.5 is in the Appendix B.

2.3.4. *The Cost Function.* The cost function is an auxiliary tool used to prove Theorem B and to define the period map.

The following two sets play a fundamental role in this section.

$$(2.17) \quad \begin{aligned} \mathcal{C} &:= \{c(t) \in \mathbb{R}_F^{n+2} \mid c(t) \text{ is a geodesic parameterized by arc-length}\}, \\ \mathcal{I} &:= \{\mathcal{T} \subset \mathbb{R} \mid \mathcal{T} = [t_0, t_1] \text{ is a compact interval}\}. \end{aligned}$$

We denote by  $\Delta : \mathcal{C} \times \mathcal{I} \rightarrow [0, \infty) \times \mathbb{R}^4$  the map given by

$$(2.18) \quad \begin{aligned} \Delta(c, \mathcal{T}) &:= (\Delta t(c, \mathcal{T}), \Delta \mathbf{y}(c, \mathcal{T}), \Delta \mathbf{z}(c, \mathcal{T})) \\ &:= (t_1 - t_0, \mathbf{y}(t_1) - \mathbf{y}(t_0), \mathbf{z}(t_1) - \mathbf{z}(t_0)). \end{aligned}$$

The map  $\Delta(c, \mathcal{T})$  encodes the change in the time  $t$  and the coordinates  $\mathbf{y}$  and  $\mathbf{z}$  as the geodesic  $c(t)$  evolves during the interval  $\mathcal{T}$ . In addition, let us consider  $v := (0, \hat{v})$  where  $\hat{v} = (v_1, v_2) \in \mathbb{R}^2$  is a unit vector. We denote by  $\text{Cost}_t^v : \mathcal{C} \times \mathcal{I} \rightarrow [0, \infty) \times \mathbb{R}$  the map given by

$$\begin{aligned} \text{Cost}_t^v(c, \mathcal{T}) &= \Delta t(c, \mathcal{T}) - v_1 \cdot \Delta y(c, \mathcal{T}), \\ \text{Cost}_y^v(c, \mathcal{T}) &= v \cdot (\Delta y(c, \mathcal{T}) - \Delta z(c, \mathcal{T})). \end{aligned}$$

We call  $\text{Cost}_t^v(c, \mathcal{T})$  the cost function of  $c(t)$ .

For the remainder of this subsection, we will discuss the relevance and significance of the cost function. The value  $\text{Cost}_t^v(c, \mathcal{T})$  is the cost incurred by the geodesic  $c(t)$  when traveling in the direction  $v$ . The following result offers further insight into this interpretation.

**Proposition 17.** *Let  $q_1$  and  $q_2 \in \mathbb{R}_{(a,b)}^5$ . Let  $c(t)$  and  $\tilde{c}(t) \in \mathbb{R}_{(a,b)}^5$  be two sub-Riemannian geodesics with the property that they travel from  $q_1$  to  $q_2$  in a time interval  $\mathcal{T}$  and  $\tilde{\mathcal{T}}$ , respectively. If  $\text{Cost}_t^v(c, \mathcal{T}) < \text{Cost}_t^v(\tilde{c}, \tilde{\mathcal{T}})$ , then the arc length of  $c(t)$  is shorter than the arc length of  $\tilde{c}(t)$ .*

*Proof.* The property that  $c(t)$  and  $\tilde{c}(t)$  travel from  $\mathbf{A}$  to  $\mathbf{B}$  implies that  $\Delta \mathbf{y}(c, \mathcal{T}) = \Delta \mathbf{y}(\tilde{c}, \tilde{\mathcal{T}})$ . If  $\text{Cost}_t^v(\tilde{c}, \tilde{\mathcal{T}}) < \text{Cost}_t^v(c, \mathcal{T})$ , then:

$$\begin{aligned} \Delta t(c, \mathcal{T}) &= \text{Cost}_t^v(c, \mathcal{T}) + v \cdot \Delta \mathbf{y}(c, \mathcal{T}) \\ &< \text{Cost}_t^v(\tilde{c}, \tilde{\mathcal{T}}) + v \cdot \Delta \mathbf{y}(\tilde{c}, \tilde{\mathcal{T}}) = \Delta t(\tilde{c}, \tilde{\mathcal{T}}). \end{aligned}$$

□

Before continuing our analysis of the cost function, we introduce the following notation to simplify expressions: let  $d\phi_\nu$  be a closed but not exact one-form on the algebraic curve  $\mathcal{C}(G_\nu, I_\nu)$  given by:

$$d\phi_\nu := \frac{dx}{\sqrt{1 - V_\nu(x)}}.$$

The one-form  $d\phi_\nu$  is smooth if and only if  $\mathcal{C}(G_\nu, I_\nu)$  is smooth. Indeed, if  $I_\nu = [x_0, x_1]$ , then  $\mathcal{C}(G_\nu, I_\nu)$  has a natural parametrization in terms of  $x$  whenever  $x \in (x_0, x_1)$ , namely  $(p_x, x) = (\pm\sqrt{1 - V_\nu(x)}, x)$ . However, at the point  $x = x_0$ , the Implicit Function Theorem provides us a parametrization of  $\mathcal{C}(G_\nu, I_\nu)$  in terms of  $p_x$ . Indeed, the implicit derivative is given by:

$$\frac{\partial x}{\partial p_x} = -\frac{p_x}{V'_\nu(x)} \text{ which implies that } d\phi_\nu = \frac{dp_x}{V'_\nu(x)}.$$

If  $V'_\nu(x_0) = G_\nu(x) \cdot G'_\nu(x) \neq 0$ , then  $d\phi_\nu$  is smooth at  $(0, x_0)$ . Therefore,  $d\phi_\nu$  is a smooth if and only if  $\mathcal{C}(G_\nu, I_\nu)$  is smooth.

**Proposition 18.** *Let  $c(t) \in \mathbb{R}_{(a,b)}^5$  be a geodesic with momentum  $\nu$ . Let  $\mathcal{T} := [t_0, t_1]$  be a time interval. The map  $\Delta(c, \mathcal{T})$  can be expressed in terms of the vector-valued function  $G(x)$  as follows:*

$$\begin{aligned} \Delta t(c, \mathcal{T}) &= \int_{x(\mathcal{T})} d\phi_\nu, \\ \Delta y_i(c, \mathcal{T}) &= \int_{x(\mathcal{T})} G_i(x; \nu) d\phi_\nu, \text{ for } i = 1, 2, \\ \Delta z_i(c, \mathcal{T}) &= \int_{x(\mathcal{T})} G_i(x; \nu) P_i(x) d\phi_\nu, \text{ for } i = 1, 2. \end{aligned}$$

In the same way, the maps  $\text{Cost}_t^v(c, \mathcal{T})$  and  $\text{Cost}_y^v(c, \mathcal{T})$  are rewritten in term of  $G_\nu(x)$  as:

$$\begin{aligned} \text{Cost}_t^v(c, \mathcal{T}) &= \int_{x(\mathcal{T})} (1 - v \cdot G_\nu(x)) d\phi_\nu, \\ \text{Cost}_y^v(c, \mathcal{T}) &= \sum_{i=1}^2 v_i \int_{x(\mathcal{T})} G_i(x; \nu) (1 - P_i(x)) d\phi_\nu. \end{aligned}$$

*Proof.* Since  $H_\mu$  is a one-degree-of-freedom system, we reduced to quadrature using the standard method, i.e., we consider the energy condition  $H_\mu = \frac{1}{2}$ , and solve the algebraic equation for  $p_x$  to find the expression:

$$\dot{x} = p_x = \frac{1}{\sqrt{1 - V_\nu(x)}},$$

where we used Hamilton equation  $\dot{x} = \frac{\partial H_\nu}{\partial p_x}$ . By considering the change of variable  $t(x)$ , we find that  $dt = d\phi_\nu$ . Then the integration of  $dt$  yields the

formula from the proposition statement for  $\Delta t(c, \mathcal{T})$ . Using  $dt = d\phi_\nu$  and Eq. (2.8), we find the expressions:

$$dy^i = G_i(x; \nu) d\phi_\nu, \quad \text{and} \quad dz^i = P_i(x) G_i(x; \nu) d\phi_\nu.$$

Finally, using the definition of the cost function, we find the expressions from the proposition statement for  $\text{Cost}_t^v(c, \mathcal{T})$  and  $\text{Cost}_y^v(c, \mathcal{T})$ .  $\square$

**Remark 19.** *Remark about Proposition 18:*

(1) *There is no ambiguity in the sign of the integral expressions involving  $\dot{x} = \sqrt{1 - V_G(x)}$  [4, Remark 2.21].*

(2) *The function  $\Delta t(c, \mathcal{T})$  depends not only on the endpoint  $x(t_0)$  and  $x(t_1)$ , but also on the specific path  $x(\mathcal{T})$ . For instance, if  $c(t)$  is a periodic geodesic, then  $x(t + mL) = x(t)$  for  $m \in \mathbb{Z}$  and:*

$$\Delta t(c, [t, t + mL]) = mL(I_\nu), \quad \text{where} \quad L(I_\nu) := 2 \int_{I_\nu} d\phi_\nu.$$

*We concluded that  $m$  is the degree of map  $t \rightarrow (p_x(t), x(t))$ , where the solution is thought as a map from  $\mathbb{S} = \mathbb{R}/L\mathbb{Z}$  to  $\mathcal{C}(G, I_G)$ .*

(3) *The functions  $\Delta t(c, \mathcal{T})$ ,  $\Delta y^i(c, \mathcal{T})$ , and  $\Delta z^i(c, \mathcal{T})$  are finite whenever the one-forms  $dt$ ,  $dy^i$ , and  $dz^i$  are smooth over the curve  $\mathcal{C}(G, I_G)|_{x(\mathcal{T})}$ , respectively. The same statement holds for  $\text{Cost}_t^v(c, \mathcal{T})$  and  $\text{Cost}_y^v(c, \mathcal{T})$ .*

For the remainder of the paper, we restrict our attention to the homoclinic geodesics with momentum  $\nu \in \Lambda_{(a,b)}$ . Accordingly, we consider only the asymptotic velocity  $v = (0, 1, 0)$ , and omit the superscript  $v$  in the cost function for simplicity.

**Proposition 20.** *Let  $c(t) \in \mathbb{R}_{(a,b)}^5$  be a sub-Riemannian geodesic with hill interval  $I_\nu$ , and let  $\{\mathcal{T}_n\}_{n \in \mathbb{N}}$  denote a sequence of time interval such that  $\mathcal{T}_n \rightarrow [-\infty, \infty]$  when  $n \rightarrow \infty$ . Then,  $\text{Cost}_t(c, \mathcal{T}_n)$  and  $\text{Cost}_y(c, \mathcal{T}_n)$  converge when  $n \rightarrow \infty$  if and only if  $c(t)$  has momentum  $\nu \in \Lambda_{(a,b)}$  or  $G_\mu(x) = (1, 0)$ .*

*Proof.* Let  $c(t)$  be a sub-Riemannian geodesic in  $\mathbb{R}_{(a,b)}^5$  with momentum  $\nu$ . If  $c(t)$  is a periodic geodesic, then Remark 19 and the inequality  $0 \leq 1 - G_1(x)$  imply

$$0 < 2j \int_L (1 - G_1(x)) d\phi_\nu \leq \text{Cost}_t(c, \mathcal{T}_{n_\ell}).$$

Where  $\mathcal{T}_{n_\ell}$  is sub-sequence with the property that  $[-jL(I_\nu), jL(I_\nu)] \subset \mathcal{T}_{n_\ell}$ . The above inequality shows that  $\text{Cost}_t(c, \mathcal{T}_{n_\ell}) \rightarrow \infty$  when  $n_\ell \rightarrow \infty$ . Therefore, a necessary condition for the convergence of  $\text{Cost}_t(c, \mathcal{T}_n)$  is that  $c(t)$  is not a periodic geodesic, which is equivalent to the convergence of  $G(x(t))$  as  $t \rightarrow \pm\infty$ . A sufficient condition is that  $G(x(t)) = (1, 0)$  when  $t \rightarrow \pm\infty$ . Therefore,  $c(t)$  is a homoclinic geodesic with asymptotic velocity  $(0, 1, 0)$  because  $\mathbb{R}_{(a,b)}^5$  only admits singular geodesics of type homoclinic.

If  $c(t)$  is an homoclinic geodesic, then  $x(\mathcal{T}_n) \rightarrow I_\nu$  when  $n \rightarrow \infty$ . It follows that sequence  $\text{Cost}_y(c, \mathcal{T}_n)$  converge if and only if  $(1 - P_1(x))d\phi_\nu$  is smooth. Therefore, the singularity of  $d\phi_\nu$  at the endpoint of the hill interval must

be removed. Given the one-form  $(1 - P_1(x))d\phi_\nu$  with  $1 - P_1(x) = x^2$ , the singularity is removed if and only if  $x = 0$  is an endpoint of the hill interval.  $\square$

2.3.5. *Period Map.* The period map encodes the asymptotic behavior of the sub-Riemannian geodesic with momentum  $\nu \in \Lambda_{(a,b)}$ .

**Definition 21.** *The period map  $\Theta_{(a,b)}^\pm : \mathcal{A}_{(a,b)}^\pm \rightarrow \mathbb{R}^3$  is given by*

$$\Theta_{(a,b)}^\pm(\tau, \eta) := (\Theta_{(a,b)}^1(\tau, \eta, \pm), \Theta_{(a,b)}^2(\tau, \eta, \pm), \Theta_{(a,b)}^3(\tau, \eta, \pm)),$$

where

$$\Theta_{(a,b)}^1(\tau, \eta, \pm) := 2 \int_{I_\nu^\pm} G_2(x; \nu) d\phi_\nu = 2 \int_{I_\nu^\pm} \eta(bx + ax^2) d\phi_\nu,$$

$$\Theta_{(a,b)}^2(\tau, \eta, \pm) := 2 \int_{I_\nu^\pm} G_1(x; \nu)(1 - P_1(x)) d\phi_\nu = 2 \int_{I_\nu^\pm} x^2(1 - \tau x^2) d\phi_\nu,$$

$$\Theta_{(a,b)}^3(\tau, \eta, \pm) := 2 \int_{I_\nu^\pm} G_2(x; \nu) P_2(x) d\phi_\nu = 2 \int_{I_\nu^\pm} \eta(bx + ax^2)^2 d\phi_\nu.$$

We draw inspiration from the following limiting process to define the period map: Consider a geodesic  $c(t) \in \mathbb{R}_{(a,b)}^5$  with momentum  $\nu(\tau, \nu) \in \Lambda_{(a,b)}$  and hill interval  $I_\nu^+$ . Let  $\{\mathcal{T}_n\}_{n \in \mathbb{N}}$  be a sequence of time intervals such that  $\mathcal{T}_n \rightarrow \infty$  when  $n \rightarrow \infty$ . Then,

$$\lim_{n \rightarrow \infty} (\Delta y^2(c, \mathcal{T}_n), \text{Cost}_y(c, \mathcal{T}_n), \Delta z^2(c, \mathcal{T}_n)) = \Theta_{(a,b)}^+(\tau, \nu).$$

An analogous relation holds for the interval  $I_\nu^-$  and the period map  $\Theta_{(a,b)}^-(\tau, \nu)$ . Having established this interpretation of the period map, we now turn to its symmetries.

**Proposition 22.** *The period map has the following symmetries:*

- ( *$\eta$  reflection*)

$$\Theta_{(a,b)}^2(\tau, \eta, \pm) = \Theta_{(a,b)}^2(\tau, -\eta, \pm),$$

$$\Theta_{(a,b)}^i(\tau, \eta, \pm) = -\Theta_{(a,b)}^i(\tau, -\eta, \pm), \quad \text{for } i = 1, 3.$$

- (*Switch of hill interval*)

$$\Theta_{(a,b)}^-(\tau, \eta) = \Theta_{(a,-b)}^+(\tau, \eta).$$

A vector-valued function is called even if  $G(x) = G(-x)$ . We use  $\mathcal{A}_{(a,b)}^{\text{even}} \subset \mathcal{A}_{(a,b)}$  to denote the subset of parameters  $(\tau, \eta)$  with the property that the vector-valued function  $G_\nu(x)$  is even.

**Proposition 23.** *The subset  $\mathcal{A}_{(a,b)}^{\text{even}}$  is characterized as follows:*

**Case 1:** *If  $b = 0$ , then  $\mathcal{A}_{(a,0)}^{\text{even}} = \mathcal{A}_{(a,0)}$ .*

**Case 2:** *If  $b \neq 0$ , then  $(\tau, \eta) \in \mathcal{A}_{(a,b)}^{\text{even}}$  if and only if  $\eta = 0$ .*

Moreover, if  $G_\nu(x)$  is a even vector-valued function, then  $\Theta_{(a,b)}^+(\nu) = \Theta_{(a,b)}^-(\nu)$ .

*Proof.* The first part of the proposition is straightforward from the construction of  $G_\mu(x)$ . Note that if  $G_\nu(x)$  is even, then the roots from Eq. (2.16) satisfy  $x^+(\nu) = -x^-(\nu)$ . Setting  $u = -x$ , we obtain the desired result.  $\square$

Proposition 23 is consequence of a stronger result which states that if  $c(t) = (x(t), \mathbf{y}(t), \mathbf{z}(t))$  is a geodesic with momentum  $\nu \in \mathcal{A}_{(a,b)}^{\text{even}}$ , then  $\tilde{c}(t) = (-x(t), \mathbf{y}(t), \mathbf{z}(t))$  is also a geodesic with momentum  $\nu$ . As a consequence, every geodesic  $c(t)$  with momentum  $\nu \in \mathcal{A}_{(a,b)}^{\text{even}}$  which crosses  $x = 0$  twice fails to minimize (see [5, Lemma 3]).

**Theorem 24.** *Consider the period map  $\Theta_{(a,b)}^\pm : \mathcal{A}_{(a,b)} \rightarrow \mathbb{R}^3$  where  $(a, b)$  corresponding to the two families defined in Section 2.2.1, namely:*

(Family 1):  $(a, 0)$  for all  $a \in \mathbb{R} \setminus \{0\}$ ,

(Family 2):  $(0, b)$  for all  $b \in \mathbb{R} \setminus \{0\}$ ,

Therefore,  $\Theta_{(a,b)}^\pm(\nu)$  is one-to-one for these families of parameters.

2.3.6. *Sequence of geodesics.* One of the main tools of the sequence method is a classic result in metric space theory.

**Proposition 25** (Proposition 2.26, [3]). *Let  $K \subset M$  be a compact subset of a sub-Riemannian manifold and let  $\mathcal{T}$  be a compact time interval. Consider the following set*

$$\text{Min}(K, \mathcal{T}) := \{c : \mathbb{R} \rightarrow M : c(\mathcal{T}) \subset K \text{ and } c(t) \text{ is minimizing in } \mathcal{T}\}.$$

Then,  $\text{Min}(K, \mathcal{T})$  is sequentially compact in the uniform norm.

Proposition 25 follows as a direct application of the Arzelà-Ascoli Theorem.

The following lemma is the last essential tool for the sequence method.

**Lemma 26** (Lemma 2.27, [3]). *Let  $c_1(t)$  and  $c_2(t) \in \mathbb{R}_F^{n+2}$  be two geodesics where  $c_1(t)$  in  $\text{Min}(K, \mathcal{T})$ , and  $\mathcal{T}' \subset \mathcal{T}$ . If  $\Phi(x, \mathbf{y}, \mathbf{z})$  is an isometry such that  $c_2(\mathcal{T}') \subset \Phi(c_1(\mathcal{T}))$ , then  $c_2(t) \in \text{Min}(\Phi(K), \mathcal{T}')$ .*

The following definition gives a condition for the  $x$ -component of a sequence of geodesics to be bounded.

**Definition 27.** *Let  $\text{Com}(r, x_0, C)$  be the set of pairs  $(c, \mathcal{T})$  satisfying the following conditions:*

- (1)  $c(t)$  is a minimizing geodesic in  $\mathcal{T}$ .
- (2)  $\text{Cost}(c, \mathcal{T})$  is uniformly bounded by  $C$  with respect to the supremum norm.
- (3)  $x(t) \in B(r, x_0)$  for all  $t \in \partial\mathcal{T}$ , where  $B(r, x_0)$  is the closed ball on  $\mathbb{R}$  with radius  $r$  and center at  $x_0$ , and  $\partial\mathcal{T}$  is the boundary of  $\mathcal{T}$ .

We say that a region  $B(r, x_0) \times \mathbb{R}^4$  is geodesically compact if for every sequence  $\{(c_n, \mathcal{T}_n)\}_{n \in \mathbb{N}} \in \text{Com}(r, x_0, C)$  satisfying:

$$\lim_{n \rightarrow \infty} \mathcal{T}_n = [-\infty, \infty],$$

there exists a compact subset  $K_x \subset \mathbb{R}$  such that  $x_n(\mathcal{T}_n) \subset K_x$  for all  $n$ . Furthermore, we say that a magnetic space is geodesically compact if, for every  $r$ , the region  $B(r, x_0) \times \mathbb{R}^2$  is geodesically compact.

**Lemma 28.** *The magnetic space  $\mathbb{R}_{(a,b)}^5$  is geodesically compact for all  $(a, b) \in \mathbb{R}^2$ .*

*Proof.* Without loss of generality, we can consider the ball  $B(r, 0)$  with an arbitrary radius  $r$ . We will proceed by contrapositive. We consider a sequence  $\{(c_n, \mathcal{T}_n)\}_{n \in \mathbb{N}} \in \text{Com}(r, x_0, C)$  and assume that  $x_n(\mathcal{T}_n)$  is unbounded, then we will prove that  $\text{Cost}(c_n, \mathcal{T}_n)$  is unbounded. Let  $h_n : [0, 1] \rightarrow x_n(\mathcal{T}_n)$  be the unique affine map sending  $[0, 1]$  to  $x_n(\mathcal{T}_n)$ . Consider the vector-valued function  $\widehat{G}_n(\tilde{x}) = G_{\nu_n}(h_n(\tilde{x}))$  where the geodesic  $c_n(t)$  has momentum  $\nu_n$ . It follows that  $\widehat{G}_n : [0, 1] \rightarrow \mathbb{R}^2$  is vector-valued function satisfying that  $\|\widehat{G}_n(\tilde{x})\|_{\mathbb{R}^2} \leq 1$  for all  $\tilde{x} \in [0, 1]$ . Therefore, there exists a subsequence  $\widehat{G}_{n_\ell}(\tilde{x})$  converging to  $\widehat{G}(\tilde{x})$ . Let us consider the cases  $\widehat{G}(\tilde{x}) \neq (1, 0)$  or  $\widehat{G}(\tilde{x}) = (1, 0)$ .

Case  $\widehat{G}(\tilde{x}) \neq (1, 0)$ : Let us consider the change of variable  $h_n(\tilde{x}) = x$ . It follows that:

$$\text{Cost}_t(c_n, \mathcal{T}_n) = |x_n(\mathcal{T}_n)| \int_0^1 (1 - (1, 0) \cdot \widehat{G}_{n_\ell}(\tilde{x})) h_n^*(d\phi_n),$$

where  $|x_n(\mathcal{T}_n)|$  is length of the interval  $x_n(\mathcal{T}_n)$ , and  $h_n^*(d\phi_n)$  is the pull-back. The condition  $\widehat{G}(\tilde{x}) \neq (1, 0)$  and Fatou's Lemma implies:

$$0 < \int_0^1 (1 - (1, 0) \cdot \widehat{G}(\tilde{x})) h^*(d\phi) \leq \liminf \int_0^1 (1 - (1, 0) \cdot \widehat{G}_{n_\ell}(\tilde{x})) h_n^*(d\phi_n).$$

Therefore,  $|x_n(\mathcal{T}_n)| \rightarrow \infty$  implies  $\text{Cost}_t(c_n, \mathcal{T}_n) \rightarrow \infty$  when  $n_\ell \rightarrow \infty$ .

Case  $\widehat{G}(\tilde{x}) = (1, 0)$ : A similar proof shows that  $\text{Cost}_y(c_n, \mathcal{T}_n) \rightarrow \infty$  when  $n_\ell \rightarrow \infty$ .  $\square$

The following definition associates a polynomial  $G_n(x)$  with a sequence of geodesics  $c_n(t)$ .

**Definition 29.** *We say a sequence of geodesics  $\{c_n(t)\}_{n \in \mathbb{N}}$  is strictly normal if  $c_n(t)$  is a normal geodesic for all  $n$  and every convergent subsequence converges to a normal geodesic.*

### 3. THE SEQUENCE METHOD

We devote this section to presenting the sequence method. Section 3.1 defines the sequence of sub-Riemannian geodesics employed in our proof, states Theorem B, applies the sequence method to prove it, and concludes with the proof of Theorem A. Section 3.2 establishes that the period map is one-to-one for the families of geodesics considered in Theorem A.

**3.1. The Sequence Method.** Let  $c_h(t)$  be the candidate geodesic with momentum  $\nu(1, 1)$  (Eq. (2.14)) and the form:

$$c_h(t) = (x_h(t), y_h^1(t), y_h^2(t), z_h^1(t), z_h^2(t)).$$

Our goal is to show that  $c_h(t)$  is minimizing in the interval  $[-T, T]$ , for an arbitrary  $T$ . The strategy is as follows: for all  $n \in \mathbb{N}$ , we will define a sequence of geodesics  $c_n(t)$  that minimize length over the interval  $[0, T_n]$ , with the property that  $c_n(t)$  connects the points  $c_h(-n)$  and  $c_h(n)$  (see Figure 3). We then construct a reparametrization  $\tilde{c}_n(t)$  of  $c_n(t)$ , and a set  $\text{Min}(K, \mathcal{T})$  such that  $\tilde{c}_n(t) \in \text{Min}(K, \mathcal{T})$ . We then extract a subsequence which converges to a geodesic  $c_\infty(t)$ . This limiting geodesic  $c_\infty(t)$  satisfies:

$$c_h([-T, T]) \subset \Phi_{(\mathbf{y}_0, \mathbf{z}_0)}(c_\infty(\mathcal{T}))$$

for some isometry  $\Phi_{(\mathbf{y}_0, \mathbf{z}_0)} : \mathbb{R}_{(a,b)}^5 \rightarrow \mathbb{R}_{(a,b)}^5$  defined by Eq. (2.6). By Lemma 26,  $c_h(t)$  is therefore minimizing in  $[-T, T]$ . Since  $T$  is arbitrary, it follows that  $c_h(t)$  is a metric line.

Without loss of generality, let us assume that:

$$c_h(0) = (x^+(\nu(1, 1)), 0, 0, 0, 0).$$

Indeed, using the translations in coordinates  $t$ ,  $\mathbf{y}$ , and  $\mathbf{z}$ , we can always find such an initial point. The time-reversibility of the Hamiltonian system for  $H_\mu$ , Eq. (2.3), implies that  $x(n) = x(-n)$ . So  $\Delta x(c_h, [-n, n]) := x_h(n) - x_h(-n) = 0$  for all  $n \in \mathbb{N}$ . Given the above initial point, there exists a time  $T_h^* > 0$  with the property that:

$$(3.1) \quad y_h^1(t) > 0, \quad \text{and} \quad y_h^1(-t) < 0 \quad \text{for all} \quad T_h^* < t.$$

Indeed, by construction, the candidate geodesic  $c_h(y)$  satisfies  $y_h^1(t) \rightarrow \infty$  as  $t \rightarrow \infty$ , and  $y_h^1(t) \rightarrow -\infty$  as  $t \rightarrow -\infty$ .

**3.1.1. Construction of the sequence.** Consider  $T > T_h^*$ , and the sequences of points  $\{c_h(-n)\}_{n \in \mathbb{N}}$  and  $\{c_h(n)\}_{n \in \mathbb{N}}$ , where  $n \in \mathbb{N}$  and  $T < n$ . Let us define a sequence of geodesics  $\{c_n(t)\}_{n \in \mathbb{N}}$  with the form:

$$c_n(t) = (x_n(t), y_n^1(t), y_n^2(t), y_n^2(t), y_n^2(t)),$$

where  $c_n : [0, T_n] \rightarrow \mathbb{R}_{(a,b)}^5$  is a minimizing geodesic and satisfying the following conditions"

$$(3.2) \quad c_n(0) = c_h(-n), \quad c_n(T_n) = c_h(n), \quad \text{and} \quad T_n \leq 2n.$$

We will refer to the above equations and inequality as the endpoint conditions and the shorter condition, respectively. By construction, the endpoint conditions imply the asymptotic conditions:

$$(3.3) \quad \begin{aligned} \lim_{n \rightarrow -\infty} (x_n(0), y_n^1(0), z_n^1(0)) &= (0, -\infty, -\infty), \\ \lim_{n \rightarrow \infty} (x_n(T_n), y_n^1(T_n), z_n^1(T_n)) &= (0, \infty, \infty). \end{aligned}$$

In addition, the following asymptotic period conditions also hold:

$$(3.4) \quad \begin{aligned} \lim_{n \rightarrow \infty} \Delta y^2(c_n, [0, T_n]) &= \Theta_{(a,b)}^1(1, 1), \\ \lim_{n \rightarrow \infty} \text{Cost}_y(c_n, [0, T_n]) &= \Theta_{(a,b)}^2(1, 1), \\ \lim_{n \rightarrow \infty} \Delta z^2(c_n, [0, T_n]) &= \Theta_{(a,b)}^3(1, 1). \end{aligned}$$

It is important to note that Eq. (3.4) only indicates convergence of the point sequences to the stated values; the convergence of the entire sequence of geodesics to a specific geodesic has not yet been proven.

Our second main result is the following.

**Theorem B.** *Let  $c_h(t)$  be a homoclinic geodesic with momentum  $\nu \in \Lambda_{(a,b)}$ . Assume that the following conditions hold:*

- (1)  *$\text{Cost}(c_h, \mathcal{T})$  is uniformly bounded for all compact intervals  $\mathcal{T}$  by a constant  $C_{\Theta}^{\text{Max}}$ .*
- (2) *The region  $B(r, 0) \times \mathbb{R}^2$  is geodesically compact for some  $r > 0$ .*
- (3) *The sequence of geodesics  $c_n(t)$  defining by Eq. (3.2) is normal.*
- (4) *The period map  $\Theta_{(a,b)}^{\pm} : \mathcal{A}_{(a,b)}^{\pm} \rightarrow [0, \infty) \times \mathbb{R}^2$  is one-to-one, and:*

$$\Theta_{(a,b)}^+(\mathcal{A}_{(a,b)}^+ \setminus \mathcal{A}_{(a,b)}^{\text{even}}) \cap \Theta_{(a,b)}^-(\mathcal{A}_{(a,b)}^- \setminus \mathcal{A}_{(a,b)}^{\text{even}}) = \{\emptyset\}.$$

*Then,  $c_h(t)$  is a metric line in  $\mathbb{R}_{(a,b)}^5$ . Consequently, if  $\gamma_h(t) \in \mathcal{J}^2(\mathbb{R}, \mathbb{R}^2)$  is the horizontal lift of  $c_h(t)$ , then  $\gamma(t)$  is a metric line in  $\mathcal{J}^2(\mathbb{R}, \mathbb{R}^2)$*

In the following subsection, we will apply the sequence method to prove Theorem B.

**3.1.2. A new initial point.** By construction, the initial point  $c_n(0)$  is unbounded. The first step of the sequence method is to find a re-parametrization  $\tilde{c}_n(t)$  with the property that  $\tilde{c}_n(0) \in K_0$  for all  $n > T$ , where  $K_0$  is a compact set in  $\mathbb{R}_{(a,b)}^5$ .

By construction of  $n > T > T_h^*$ , the Intermediate Value Theorem implies that there exist a  $t_n^*$  in  $(0, T_n)$  such that  $y_n^1(t_n^*) = 0$  for all  $n > T_h^*$ . Moreover, the conditions (1) and (2) from Theorem B imply that there exist a compact subset  $K_x \subset \mathbb{R}$  such that  $x_n([0, T_n]) \subset K_x$ , in particular  $x_n(t_n^*) \in K_x$ . Therefore, it is enough to build compact subsets  $K_y^2$ ,  $K_z^1$  and  $K_z^2$  such that  $y_n^2(t_n^*) \in K_y^2$ ,  $z_n^1(t_n^*) \in K_z^1$  and  $z_n^2(t_n^*) \in K_z^2$ . The existence of such compact sets is a consequence of the following result.

**Proposition 30.** *There exists  $\tilde{C}_y^2$ ,  $\tilde{C}_z^1$ , and  $\tilde{C}_z^2 \in (0, \infty)$  such that*

$$|\Delta y^2(c_n, [0, T_n])| \leq \tilde{C}_y^2, \quad |\text{Cost}_y(c_n, [0, T_n])| \leq \tilde{C}_z^1, \quad |\Delta z^2(c_n, [0, T_n])| \leq \tilde{C}_z^2,$$

*for all  $n > T$ .*

Let us first use Proposition 30 to build the set  $K_0$ , and then we will give its proof.

We define the compact set:

$$K_0 := K_x \times [-1, 1] \times K_y^2 \times K_z^1 \times K_z^2,$$

where  $K_y^2 := [-\tilde{C}_y^2, \tilde{C}_y^2]$  and  $K_z^2 := [-\tilde{C}_z^2, \tilde{C}_z^2]$ . The construction of  $K_z^1$  is as follows: Using the definition given in Section 2.3.4, we have that:

$$\begin{aligned} z_n(t_n^*) - z_n(0) &= \Delta z^1(c_n, [0, t_n^*]) = \Delta y_1(c_n, [0, t_n^*]) - \text{Cost}_y(c_n, [0, t_n^*]), \\ z_h(0) - z_h(-n) &= \Delta z^1(c_h, [-n, 0]) = \Delta y_1(c_h, [-n, 0]) - \text{Cost}_y(c_h, [-n, 0]). \end{aligned}$$

By construction,  $\Delta y_1(c_h, [-n, 0]) = \Delta y_1(c_n, [0, t_n^*])$ ,  $z_h(0) = 0$ , and  $z_n(0) = z_n(-n)$  for all  $n > T$ . It follows that:

$$(3.5) \quad \begin{aligned} |z_n(t_n^*)| &= |\Delta z^1(c_n, [0, t_n^*]) - \Delta z^1(c_h, [-n, 0])| \\ &\leq |\text{Cost}_y(c_n, [0, t_n^*])| + |\text{Cost}_y(c_h, [-n, 0])| \leq C_{\Theta}^{\text{Max}} + \tilde{C}_z^1, \end{aligned}$$

where  $C_{\Theta}^{\text{Max}}$  and  $\tilde{C}_y^1$  are given by the condition (1) from Theorem B, and Proposition 30, respectively. Therefore, the compact set  $K_z^1$  is defined by:

$$K_z^1 := [-C_{\Theta}^{\text{Max}} - \tilde{C}_z^1, C_{\Theta}^{\text{Max}} + \tilde{C}_z^1].$$

Summarizing, we build a compact set  $K_0$  with the property that  $c_n(t_n^*) \in K_0$  for all  $n > T$ . We consider, the re-parametrization  $\tilde{c}_n : \mathcal{T}_n \rightarrow \mathbb{R}_{(a,b)}^5$  given by  $\tilde{c}_n(t) := c_n(t + t_n^*)$ , where  $\mathcal{T}_n := [-t_n^*, T_n - t_n^*]$ . Therefore,  $\tilde{c}_n(t)$  is a sequence of minimizing geodesics on the interval  $\mathcal{T}_n$  with the property that  $\tilde{c}(0) \in K_0$ . Moreover, by construction, the points  $\tilde{c}_n(-t_n^*)$  and  $\tilde{c}_n(T_n - t_n^*)$  are unbounded. It follows that  $\mathcal{T}_n \rightarrow [-\infty, \infty]$  when  $n \rightarrow \infty$ , and we consider without loss of generality that  $\mathcal{T}_n$  is such that  $\mathcal{T}_n \subset \mathcal{T}_{n+1}$ .

*Proof of Proposition 30.* Let us find  $\tilde{C}_y^2$ . On one side, by condition (4) from Theorem B, the sequence of geodesics  $c_n(t)$  is normal, then we can associate a vector-valued function  $G_{\nu_i}(x)$ . Moreover, we can consider, the unique affine map  $h_n : [0, 1] \rightarrow x_n([0, T_n]) \subset K_x$  mapping  $[0, 1]$  to  $x_n([0, T_n])$ . It follows that we can rewrite  $\Delta y^2(c_n, [0, T_n])$  as follows

$$\begin{aligned} \Delta y^2(c_n, [0, \mathcal{T}_n]) &= \int_{h_n^{-1}(x_n([0, \mathcal{T}_n]))} h_n^*(G_1(x; \nu_n) d\phi_{\nu_n}) \\ &= u_n \int_{h_n^{-1}(x_n([0, \mathcal{T}_n]))} \frac{G_1(h_n(\tilde{x}); \nu_n)}{\sqrt{1 - \|\hat{G}_n(\tilde{x})\|^2}} d\tilde{x}. \end{aligned}$$

Where  $\tilde{x} \in [0, 1]$ ,  $u_n \in (0, \infty)$  is the length of the interval  $x_n([0, T_n])$  and  $\hat{G}_n(\tilde{x}) := G_{\nu_n}(h_n(\tilde{x}))$  is a vector-valued function defined on the interval  $[0, 1]$ . In addition,  $u_n$  and  $G_{\nu_n}(h_n(\tilde{x}))$  are uniformly bounded. On the other side, the sequence  $\Delta y^2(c_n, [0, T_n])$  is bounded because it converges by Eq. (3.4). It follows that one-form  $G_1(x; \nu_n) d\phi_{\nu_n}$  is smooth on the interval  $x_n([0, T_n])$ , as a consequence  $h_n^*(G_1(x; \nu_n) d\phi_{\nu_n})$  is smooth. Therefore, we find the upper bound  $\tilde{C}_y^2$  in two steps. First, we use the fact that  $h_n^*(G_1(x; \nu_n) d\phi_{\nu_n})$  is smooth to find the maximum of the integral with respect to the interval  $h_n^{-1}([x_n(0), x])$  where  $x \in x_n([0, T_n])$ . Second, by Remark 19, we rewrite

$\Delta y^2(c_n, [0, T_n])$  as a continuous function on a sequentially compact set given by the pairs  $(\widehat{G}_n(\tilde{x}), u_n)$ , to find a bound. In other words, we define  $\tilde{C}_y^2$  as follows

$$\tilde{C}_y^2 := \max_{(\widehat{G}_n(\tilde{x}), u_n)} \left\{ \max_{x \in x_n([0, T_n])} \left\{ \int_{h_n^{-1}([x_n(0), x])} |h_n^*(G_1(x; \nu_n) d\phi_{\nu_n})| \right\} \right\}.$$

In the same way, we can build the bounds  $\tilde{C}_z^1$  and  $\tilde{C}_z^2$  for  $\Delta z^2(c_n, [0, T_n])$  and  $\Delta z^2(c_n, [0, T_n])$ , respectively.  $\square$

3.1.3. *Construction of  $\text{Min}(K, \mathcal{T})$ .* Consider  $N \in \mathbb{N}$  such that  $N > T$ , we will build a compact set  $K_N$  such that  $\tilde{c}_n(t) \in \text{Min}(K_N, \mathcal{T}_N)$  for all  $n > N$ . By the above construction, we have that  $\tilde{c}_n(t)$  is minimizing on  $\mathcal{T}_n$ , and  $\mathcal{T}_N \subset \mathcal{T}_n$ , so  $\tilde{c}_n(t)$  is minimizing on  $\mathcal{T}_N$ . Moreover,  $\|\tilde{c}_n(t)\| = 1$  implies that:

$$|\Delta x(\tilde{c}_n, \mathcal{T}_n)| \leq T_N, \quad |\Delta y_1(\tilde{c}_n, \mathcal{T}_n)| \leq T_N, \quad |\Delta y_2(\tilde{c}_n, \mathcal{T}_n)| \leq T_N.$$

Let us bound  $\Delta z_1(\tilde{c}_n, \mathcal{T}_n)$  and  $\Delta z_2(\tilde{c}_n, \mathcal{T}_n)$ . Using Eq. (2.8), we have:

$$C_z^i := T_N \max_{x \in K_x + [-T_N, T_N]} |P_i(x)| \geq \int_{\mathcal{T}_N} |P_i(x)| \geq |\Delta z_i(\tilde{c}_n, \mathcal{T}_n)|, \quad \text{for } i = 1, 2.$$

Therefore, if we define the compact set  $K_1 \subset \mathbb{R}_{(a,b)}^5$  as follows:

$$K_1 := [-T_N, T_N] \times [-T_N, T_N] \times [-T_N, T_N] \times [-C_z^1, C_z^1] \times [-C_z^2, C_z^2],$$

then  $\tilde{c}_n(\mathcal{T}_N) \subset K_N := K_0 + K_1$ . We conclude that  $\tilde{c}_n(t) \in \text{Min}(K_N, \mathcal{T}_N)$ .

3.1.4. *Proof of Theorem B.* We are ready to prove the second main theorem of this paper.

*Proof of Theorem B.* We just built a sequence of geodesics  $\tilde{c}_n(t) \in \text{Min}(K_N, \mathcal{T}_N)$ . By Proposition 25, there exists a subsequence  $\tilde{c}_{n_\ell}(t)$  converging to  $\tilde{c}_\infty(t)$ . The condition (3) implies that  $\tilde{c}_\infty(t)$  is a normal geodesic, so  $\tilde{c}(t)$  is associated to a vector-valued function  $G_\nu(x)$ . Moreover, Eqs. (3.3) and (3.4) imply that  $\nu \in \mathcal{A}_{(a,b)}$ , and:

$$\Theta_{(a,b)}^+(1, 1) = \lim_{n_\ell \rightarrow \infty} (\Delta y^2(c_{n_\ell}, \mathcal{T}_{n_\ell}), \text{Cost}_y(c_{n_\ell}, \mathcal{T}_{n_\ell}), \Delta z^2(c_{n_\ell}, \mathcal{T}_{n_\ell})).$$

Therefore, Condition (4) establishes that  $G_\nu(x) = P(x)$  and the hill interval of  $\tilde{c}(t)$  is  $I_\mu^+$ . Every two geodesics corresponding to the vector-valued function  $P(x)$  and sharing the same hill interval are related by isometry  $\Phi_{(\mathbf{y}_0, \mathbf{z}_0)}(x, \mathbf{y}, \mathbf{z})$ . Using that  $N$  is arbitrary and  $c_h([-T, T])$  is bounded, we can find a compact sets  $K = K_N$  and  $\mathcal{T} := \mathcal{T}_N$  such that  $c_h([-T, T]) \subset \Phi_{(\mathbf{y}_0, \mathbf{z}_0)}(\tilde{c}(\mathcal{T}))$ . Lemma 26 implies that  $c_h(t)$  is minimizing in  $[-T, T]$ , and  $T$  is arbitrary. Hence,  $c_h(t)$  is a metric lines in  $\mathbb{R}_{(a,b)}^5$ .  $\square$

3.1.5. *Proof of Theorem 12.* We are ready to prove Theorem 12, which implies that Theorem A is true along with the discussion from Section 2.2.1.

*Proof of Theorem 12.* It is enough to verify that the condition from Theorem B holds for each family of geodesics. Let us prove that Condition (1) holds in general, that is  $\text{Cost}(c_h, [-n, n])$  is uniformly bounded. Indeed, using that  $|G_1(x)| \leq 1$  in the hill interval  $I_\nu^+$ , and Proposition 18 we have that:

$$\Theta_{(a,b)}^{\text{Max}} := \Theta_{(a,b)}^+(1, 1) \geq \text{Cost}_t(c, [-n, n]) \geq |\text{Cost}_y(c, [-n, n])|.$$

Lemma 28 implies the Condition (2).

Proposition tells us that the abnormal geodesics are line geodesics. The sequence  $c_n(t)$  cannot converge to a geodesic  $c(t)$  whose tangent velocity is given by  $\dot{c}(t) = Y_1$  (this is the only geodesic with asymptotic velocity  $(0, 1, 0)$ ). Indeed,  $\text{Cost}_y(c, \mathcal{T}) = 0$  and  $\Delta(c, \mathcal{T}) = 0$  for al compact interval  $\mathcal{T}$ , therefore  $c(t)$  cannot satisfies the assymptotic condition from Eq since  $(0, 0, 0) \notin \Theta_{(a,b)}^+(\mathcal{A}_{(a,b)}^+)$  for all  $(a, b) \in \mathbb{R}^2 \setminus \{(0, 0)\}$ . (3.3). We concluded that for  $n$  big enough, the geodesic  $c_n(t)$  cannot be abnormal.

We will prove Condition (4) considering the different families.

**Case  $(a, 0)$  with  $a \in \mathbb{R} \setminus \{0\}$ :** Condition (4) is trivial beacuse  $G_\nu(x)$  is even for all  $\nu \in \mathcal{A}_{(a,0)}^\pm$  for all  $a \in \mathbb{R} \setminus \{0\}$ .

**Case  $(0, b)$  with  $b \in \mathbb{R} \setminus \{0\}$ :** For Condition (4),  $0 < G_\nu(x)$  if  $x \in I_\nu^+$  and  $G_\nu(x) < 0$  if  $x \in I_\nu^-$ . It follows that if  $(\tau, \eta) \in \mathcal{A}_{(0,b)}^+ \setminus \mathcal{A}_{(0,b)}^{\text{even}}$  then  $0 < \Theta_{(0,b)}^1(\tau, \eta, +)$ , and  $\Theta_{(0,b)}^1(\tau, \eta, +) = -\Theta_{(0,b)}^1(\tau, \eta, -)$ .

Thus, by establishing Theorem 12, we have proved Theorem A. To complete the argument of Theorem 12, it remains to prove Theorem 24. The following section is dedicated to this task.  $\square$

**3.2. The Period map II.** The goal of this section is to prove Theorem 24. We analyze two cases:  $0 < \eta$  and  $\eta = 0$ . Subsequently, by invoking Proposition 22, we will extend our consideration to the case  $\eta < 0$ . For the case  $\eta > 0$ , we will apply Hadamard's Global Diffeomorphism Theorem. We begin by presenting its formal statement.

**Theorem 31** (Hadamard's Global Diffeomorphism Theorem, [10]). *Let  $M$  and  $N$  be  $n$ -dimensional manifolds and  $F : M \rightarrow N$  be a differentiable map. If,*

- (1)  $M$  is open and connected,
- (2)  $N$  is open and simply connected,
- (3)  $DF|_p$  is nondegenerate for all  $p \in M$ ,
- (4)  $F$  is proper.

*Then  $F$  is a diffeomorphism.*

We cannot apply Hadamard's Global Diffeomorphism Theorem directly to the period map  $\Theta_{(a,b)}^+(\tau, \eta)$  because it is not proper and its target is not

two-dimensional. To address these obstacles, we introduce two auxiliary functions  $g : [-\frac{\pi}{2}, \frac{\pi}{2}]^2 \rightarrow \mathbb{R}^2$  and  $f : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  given by:

$$g(\theta_1, \theta_2) = (\tan(\theta_1), \tan(\theta_2)) \quad \text{and} \quad f(\Theta^1, \Theta^2, \Theta^3) = (\Theta^2, \Theta^3).$$

We now define the sets:

$$M_{(a,b)} := g^{-1}(\text{Int } \mathcal{A}_{(a,b)}^+|_{0 < \eta}) \quad \text{and} \quad N_{(a,b)} := (f \circ \Theta)(\text{Int } \mathcal{A}_{(a,b)}^+|_{0 < \eta}).$$

Therefore, we will consider the function:

$$(3.6) \quad F_{(a,b)} : M_{(a,b)} \rightarrow N_{(a,b)} \quad \text{given by} \quad F_{(a,b)} := f \circ \Theta_{(a,b)}^+ \circ g.$$

We notice that  $g$  is a bijection and  $f$  is surjective; it follows that if  $F_{(a,b)}$  is a bijection, then  $\Theta_{(a,b)}^+$  is one-to-one.

**Lemma 32.** *Consider the map  $f \circ \Theta_{(a,b)}^+ : \text{Int } \mathcal{A}_{(a,b)}^+|_{0 < \eta} \rightarrow N_{(a,b)}$ . Then,  $D(f \circ \Theta_{(a,b)}^+)|_{(\tau, \eta)}$  is nondegenerate for all  $(\tau, \eta) \in \text{Int } \mathcal{A}_{(a,b)}^+|_{0 < \eta}$ , and for all  $(a, b) \in \mathbb{R}^2 \setminus \{(0, 0)\}$ .*

The proof of Lemma 32 is in the following subsection.

**Lemma 33.** *Consider a function  $F_{(a,b)} : M_{(a,b)} \rightarrow N_{(a,b)}$ . Then:*

- (1)  $M_{(a,b)}$  is open and connected set.
- (2)  $N_{(a,b)}$  is open set. Moreover,  $N_{(a,b)}$  is simply connected set if  $(a, b) = (a, 0)$  or  $(a, b) = (0, b)$  for  $a, b \in \mathbb{R} \setminus \{0\}$ , respectively.
- (3)  $DF_{(a,b)}|_{(\theta_1, \theta_2)}$  is nondegenerate for all  $(\theta_1, \theta_2) \in M_{(a,b)}$ , and for all  $(a, b) \in \mathbb{R}^2 \setminus \{(0, 0)\}$ .
- (4)  $F_{(a,b)}$  is proper.

*Proof.* Item (1): the map  $g : [-\frac{\pi}{2}, \frac{\pi}{2}]^2 \rightarrow \mathbb{R}^2$  is a diffeomorphism and  $\text{Int } \mathcal{A}_{(a,b)}^+|_{0 < \eta}$  is open and connected set, so  $M_{(a,b)}^\pm$  is open and connected.

Item (2): Lemma 32 implies  $f \circ \Theta_{(a,b)}^+ : \text{Int } \mathcal{A}_{(a,b)}^+|_{0 < \eta} \rightarrow N_{(a,b)}$  is an open map, then  $N_{(a,b)}$  is an open set. By definition of the period map, we have that  $N_{(a,b)} \subset \mathbb{R} \times (0, \infty)$ .

Let us prove that  $N_{(a,0)}$  is simply connected. Consider the action of  $\mathbb{R}^+$  on  $\mathcal{A}_{(a,0)}^+$  defined as follows: for every  $\lambda \in \mathbb{R}^+$ , let  $\lambda \cdot (\tau, \eta) := (\frac{\tau}{\lambda^2}, \frac{\eta}{\lambda^2})$ . It follows from Eqs. (2.10) and (2.12) that  $\lambda \cdot (\tau, \eta) \in \mathcal{A}_{(a,0)}^+$  for all  $\lambda > 0$ . This action on  $\mathcal{A}_{(a,0)}^+$  induces an action on  $N_{(a,0)}$ , given by  $\lambda \cdot (\Theta^2, \Theta^3) = (\lambda^3 \Theta^2, \lambda^3 \Theta^3)$ . Now, let  $\sigma : [0, 1] \rightarrow N_{(a,b)}^\pm$  be arbitrary loop. To show that  $\sigma(s)$  is contractible, it is enough to demonstrate that  $\sigma(s)$  is homotopic equivalent to an arc  $\omega$ , where:

$$\omega \subset \{(\Theta^2, \Theta^3) = (\cos(\psi), \sin(\psi)) : \psi \in [0, \pi]\}.$$

Because  $\omega$  is homotopic equivalent to a point, this proves  $\sigma(s)$  is contractible. To formalize this argument, we construct a homotopy  $H : [0, 1]^2 \rightarrow N_{(a,b)}$  with the following properties:  $H(0, s) = \sigma(s)$ ,  $H(\rho, 0) = H(\rho, 1)$ , and

$H(1, s) = \omega$ . To define  $H(\rho, s)$ , note that  $(0, 0) \notin N_{(a,b)}^+$ , which ensures that  $\|\sigma(s)\| \neq 0$  for all  $s \in [0, 1]$ . Observing the action, we see that  $\lambda \cdot \sigma(s) \rightarrow (0, 0)$  as  $\lambda \rightarrow 0$ , and  $\lambda \cdot \sigma(s)$  is unbounded as  $\lambda \rightarrow \infty$ . Therefore, for each  $s \in [0, 1]$ , there exists  $\lambda(s) \in \mathbb{R}^+$  such that  $\|\lambda(s) \cdot \sigma(s)\| = 1$ . We define the homotopy by:

$$H(\rho, s) = (\rho\lambda(s) + (1 - \rho)) \cdot \sigma(s).$$

Thus,  $\sigma(s)$  is homotopic equivalent to  $\omega$ , and hence contractible to a point.

The proof for the case  $N_{(0,b)}$  proceeds similarly. Here, we consider the action of  $\mathbb{R}^+$  on  $\mathcal{A}_{(0,b)}$  given by  $\lambda \cdot (\tau, \eta) = (\frac{\tau}{\lambda^2}, \frac{\eta}{\lambda})$ , which induces the corresponding action on  $N_{(0,b)}$  given by  $\lambda \cdot (\Theta_2, \Theta_3) = (\lambda^3\Theta_2, \lambda^2\Theta_3)$ .

Item (3): direct calculation shows that

$$DF_{(a,b)}|_{(\theta_1, \theta_2)} = D(f \circ \Theta_{(a,b)}^+)|_{g(\theta_1, \theta_2)} \circ Dg|_{(\theta_1, \theta_2)}.$$

The linear map  $Dg$  is non-degenerate, and  $D(f \circ \Theta_{(a,b)}^+)$  is not degenerate by Lemma 32. Therefore,  $DF_{(a,b)}|_{(\theta_1, \theta_2)}$  is non-degenerate for all  $(\theta_1, \theta_2)$  and  $(a, b) \in \mathbb{R}^2 \setminus \{(0, 0)\}$ .

Item (4): The map  $F_{(a,b)}$  is proper since its domain  $M_{(a,b)} \subset [-\frac{\pi}{2}, \frac{\pi}{2}]^2$  is a subset of a compact set.  $\square$

The following result shows that to show  $\Theta_{(a,b)}^\pm : \text{Int } \mathcal{A}_{(a,b)}^+ \rightarrow N_{(a,b)}$  is one-to-one is enough to consider only the domain  $\text{Int } \mathcal{A}_{(a,b)}^+|_{0 < \eta}$ .

**Lemma 34.** *Consider  $(a, b) \in \mathbb{R}^2 \setminus \{(0, 0)\}$ . If  $\Theta_{(a,b)}^+ : \mathcal{A}_{(a,b)}^+|_{0 < \eta} \rightarrow N_{(a,b)}$  is one-to-one, then  $\Theta_{(a,b)}^\pm : \mathcal{A}_{(a,b)}^\pm \rightarrow N_{(a,b)}$  is one-to-one.*

*Proof.* First let us consider  $\Theta_{(a,b)}^+ : \mathcal{A}_{(a,b)}^+ \rightarrow N_{(a,b)}$ . By Proposition 22, it is enough to consider the cases  $0 \leq \eta$ , and  $\eta < 0$ :

**Case  $0 \leq \eta$ :** By construction of the period map (Definition 21), it follows that  $\Theta_{(a,b)}^3(\tau, \eta) = 0$  if and only if  $\eta = 0$ , and  $\Theta_{(a,b)}^+(\tau, 0) = (0, \Theta_{(a,b)}^2(\tau, 0), 0)$ . Therefore, we have that  $\Theta_{(a,b)}^+(\mathcal{A}_{(a,b)}^+|_{0 < \eta}) \cap \Theta_{(a,b)}^+(\mathcal{A}_{(a,b)}^+|_{0 = \eta}) = \{\emptyset\}$ , and it is enough to show that  $\Theta_{(a,b)}^2(\tau, 0)$  is one-to-one with respect to  $\tau$ . Using a change of variable  $x = \sqrt{\frac{2}{\tau}}u$  and integration by parts shows that:

$$\begin{aligned} \Theta_{(a,b)}^2(\tau, 0) &= \int_0^{\sqrt{\frac{2}{\tau}}} \frac{x^2(1 - \tau x^2)}{\sqrt{1 - (1 - \tau x^2)^2}} dx \\ (3.7) \quad &= -\frac{1}{\sqrt{2}\tau^{\frac{2}{3}}} \left( \int_0^1 \sqrt{1 - (1 - 2u^2)^2} du \right) = -\frac{\sqrt{2}}{3\tau^{\frac{3}{2}}} < 0. \end{aligned}$$

**Case  $\eta < 0$ :** By construction,  $\Theta_{(a,b)}^3(\tau, \eta, +) > 0$  for all  $\eta > 0$ . Using the symmetry from Proposition 22 we have that:

$$\Theta_{(a,b)}^+(\tau, -\eta) = (-\Theta_{(a,b)}^1(\tau, \eta, +), \Theta_{(a,b)}^2(\tau, \eta, +), -\Theta_{(a,b)}^3(\tau, \eta, +)).$$

It follows that  $\Theta_{(a,b)}^+ : \mathcal{A}_{(a,b)}^+|_{0 < \eta} \rightarrow N_{(a,b)}$  is one-to-one, which implies that  $\Theta_{(a,b)}^+ : \mathcal{A}_{(a,b)}^+|_{\eta < 0} \rightarrow \mathbb{R}^3$  is one-to-one. Moreover,

$$\Theta_{(a,b)}^+(\mathcal{A}_{(a,b)}^+|_{\eta < 0}) \cap \Theta_{(a,b)}^+(\mathcal{A}_{(a,0)}^+|_{0 \leq \eta}) = \{\emptyset\}.$$

Therefore,  $\Theta_{(a,b)}^+ : \mathcal{A}_{(a,b)}^+ \rightarrow \mathbb{R}^3$  is one-to-one for all  $(a, b) \in \mathbb{R}^2 \setminus \{\emptyset\}$ . Moreover, Proposition 22 establishes that  $\Theta_{(a,b)}^-(\tau, \eta) = \Theta_{(a,-b)}^+(\tau, \eta)$ . Consequently,  $\Theta_{(a,b)}^- : \mathcal{A}_{(a,b)}^- \rightarrow \mathbb{R}^3$  is one-to-one.  $\square$

3.2.1. *Proof of Theorem 24.* We are ready to prove the theorem.

*Proof of Theorem 24.* We start by considering the case  $(a, 0)$ . Let  $F_{(a,0)} : M_{(a,0)} \rightarrow N_{(a,0)}$  denote the map defined in Eq. (3.6). By Lemma 33,  $F_{(a,0)}$  satisfies the conditions of Hadamard's Global Diffeomorphism Theorem. Consequently,  $F_{(a,b)}$  is a global diffeomorphism, which in turn implies that  $F_{(a,b)}$  is one-to-one. From the preceding discussion, we have that  $\Theta_{(a,0)}^+ : \text{Int } \mathcal{A}_{(a,0)}^+|_{0 < \eta} \rightarrow \mathbb{R}^3$  is one-to-one, and Lemma 34 implies that  $\Theta_{(a,0)}^\pm : \text{Int } \mathcal{A}_{(a,0)}^\pm \rightarrow \mathbb{R}^3$  is one-to-one. Since  $\text{Int } \mathcal{A}_{(a,0)}^\pm = \mathcal{A}_{(a,0)}^\pm$ , we concluded that  $\Theta_{(a,0)}^\pm$  is one-to-one.

A similar argument proves the result for the case  $(0, b)$ .

3.2.2. *Proof of Lemma 32.* Before presenting the proof, we outline the strategy for the case  $(a, b) \neq (0, 0)$  and introduce several auxiliary functions that will be employed. Instead of computing the determinant of  $Df \circ \Theta_{(a,b)}^+$  directly, we consider the composition  $f \circ \Theta_{(a,b)}^+ = F_2 \circ F_1$ . Let us construct the functions  $F_1$  and  $F_2$ : first, we use the following factorization:

$$\begin{aligned} 1 - V_\nu(x) &= x^2(C(\tau, \eta) - B(\tau, \eta)x - A(\tau, \eta)x^2) \\ &= \vartheta_3(\tau, \eta)x^2(1 - (\vartheta_1(\tau, \eta)x - \vartheta_2(\tau, \eta))^2), \end{aligned}$$

where:

$$A(\tau, \eta) = a^2\eta^2 + \tau^2, \quad B(\tau, \eta) = 2ab\eta^2, \quad C(\tau, \eta) = 2\tau - b^2\eta^2,$$

and

$$(3.8) \quad (\vartheta_1, \vartheta_2, \vartheta_3) = \left( \frac{2A(\eta, \tau)}{\sqrt{\Delta(\tau, \eta)}}, \frac{B(\eta, \tau)}{\sqrt{\Delta(\tau, \eta)}}, \frac{\Delta(\tau, \eta)}{4A(\eta, \tau)} \right).$$

Here,  $\Delta(\tau, \eta) := B^2(\tau, \eta) + 4A(\tau, \eta)C(\tau, \eta)$  is the discriminant of the quadratic polynomial  $C - Bx - Ax^2$ . By the definition of  $\Lambda_{(a,b)}$  this discriminant is positive. By construction, we have that  $\vartheta_1 > 0$ ,  $\vartheta_2 \in [-1, 1]$ , and  $\vartheta_3 > 0$ .

Next, we examine the composition  $f \circ \Theta_{(a,b)}^+ = F_2 \circ F_1$  where  $F_1 : \mathcal{A}_{(a,b)}^+ \rightarrow \mathbb{R}^3$  is defined as:

$$(3.9) \quad F_1(\tau, \eta) = (\vartheta_1, \vartheta_2, \vartheta_3)$$

and  $F_2 : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  is given by:

$$(3.10) \quad \begin{aligned} F_2(\vartheta_1, \vartheta_2, \vartheta_3) &:= (\tilde{\Theta}_2(\vartheta_1, \vartheta_2, \vartheta_3), \tilde{\Theta}_3(\vartheta_1, \vartheta_2, \vartheta_3)), \\ \tilde{\Theta}_2(\vartheta_1, \vartheta_2, \vartheta_3) &= \frac{1}{\sqrt{\vartheta_3}} \int_0^{\frac{1-\vartheta_2}{\vartheta_1}} \frac{x(1 - (\frac{\vartheta_3}{2}(1 + \frac{b}{a}\vartheta_1\vartheta_2 - \vartheta_2^2))x^2)}{\sqrt{1 - (\vartheta_1x + \vartheta_2)^2}} dx, \\ \tilde{\Theta}_3(\vartheta_1, \vartheta_2, \vartheta_3) &= \sqrt{\frac{\vartheta_1\vartheta_2}{ab}} \int_0^{\frac{1-\vartheta_2}{\vartheta_1}} \frac{x(b + ax)^2}{\sqrt{1 - (\vartheta_1x + \vartheta_2)^2}} dx. \end{aligned}$$

Where we used that  $\eta > 0$  to write  $(\tau, \eta)$  in terms of  $(\vartheta_1, \vartheta_2, \vartheta_3)$  as follows:

$$\tau = \frac{\vartheta_3}{2} \left(1 + \frac{b}{a}\vartheta_1\vartheta_2 - \vartheta_2^2\right), \quad \eta = \sqrt{\frac{\vartheta_1\vartheta_2\vartheta_3}{ab}}.$$

We present the explicit expression of the  $\tilde{\Theta}_2(\vartheta_1, \vartheta_2, \vartheta_3)$  and  $\tilde{\Theta}_3(\vartheta_1, \vartheta_2, \vartheta_3)$  in the Appendix C.

We conclude that, in order to establish that  $Df \circ \Theta_{(a,b)}^+$  is non-degenerate, it suffices to show that  $DF_1|_{(\tau,\eta)}$ , and  $DF_2|_{F_1(\tau,\eta)}$  are both of full rank. The following two results are dedicated to this objective.

**Proposition 35.** *Let  $F_1 : \mathcal{A}_{(a,b)}^+ \rightarrow \mathbb{R}^3$  be the function defined by Eqs. (3.8) and (3.9). If  $(a, b) \neq (0, 0)$ , then the differential  $DF_1|_{(\tau,\eta)}$  has full rank for all  $(\tau, \eta) \in \text{Int } \mathcal{A}_{(a,b)}^+|_{0 < \eta}$ .*

*Proof.* To demonstrate that  $DF_1|_{(\tau,\eta)}$  has full rank, it suffices to identify a two-by-two submatrix with a nonzero determinant. Proceeding with this approach, and noting that  $2\tau - b^2\eta^2 > 0$  for  $(\tau, \eta) \in \text{Int } \mathcal{A}_{(a,b)}^+$ , we obtain:

$$\frac{\partial\vartheta_1}{\partial\tau} \frac{\partial\vartheta_2}{\partial\eta} - \frac{\partial\vartheta_1}{\partial\eta} \frac{\partial\vartheta_2}{\partial\tau} = \frac{2ab\eta(a^2\eta^2 + \tau^2)}{(2\tau^2 - b^2\eta^2\tau + 2a^2\eta^2)^2} \neq 0,$$

for all  $(\tau, \eta) \in \mathcal{A}_{(a,b)}^+$  with  $\eta \neq 0$ .  $\square$

Before proving that  $DF_2|_{F_1(\tau,\eta)}$  has full rank, let us outline the strategy and present some auxiliary results. First, we observe that  $\tilde{\Theta}_3$  is independent of  $\vartheta_3$ . This observation, together with the following result, will be instrumental in the proof.

**Proposition 36.** *Let  $\tilde{\Theta}_2$ , and  $\tilde{\Theta}_3$  be the functions defined by Eq. (3.10). Then,*

$$\frac{\partial\tilde{\Theta}_2}{\partial\vartheta_3}|_{F_1(\tau,\eta)} < 0, \quad \text{and} \quad \frac{\partial\tilde{\Theta}_3}{\partial\vartheta_3} = 0.$$

*Proof.* The expression for  $\tilde{\Theta}_2(\vartheta_1, \vartheta_2, \vartheta_3)$  from Appendix C implies:

$$\frac{\partial\tilde{\Theta}_2}{\partial\vartheta_3}|_{F_1(\tau,\eta)} = -\frac{2\alpha^2\rho_1(\vartheta_2) + \tau\rho_2(\vartheta_2)}{4\alpha^4(\vartheta_3)^{\frac{3}{2}}},$$

Here  $\varrho_1(\vartheta_2)$  and  $\varrho_2(\vartheta_2)$  are auxiliary functions given by:

$$(3.11) \quad \begin{aligned} \varrho_1(\vartheta_2) &:= \sqrt{1 - \vartheta_2^2} - \vartheta_2 \arccos(\vartheta_2), \\ \varrho_2(\vartheta_2) &:= (11\vartheta_2^2 + 4)\sqrt{1 - \vartheta_2^2} - (6\vartheta_2^3 + 9\vartheta_2) \arccos(\vartheta_2). \end{aligned}$$

Therefore, it suffices to show that  $\varrho_1(\vartheta_2)$  and  $\varrho_2(\vartheta_2)$  have the same sign for all  $\vartheta_2 \in (-1, 1)$ ; note that  $|\vartheta_2| = 1$  if and only if  $C(\tau, \eta) = 0$ , which occurs precisely when  $(\tau, \eta) \in \partial\mathcal{A}_{(a,b)}^+$ . This claim is verified by examining the graphs of both functions, as shown in Figure 3.1. Furthermore, as shown in Eq. (3.10),  $\tilde{\Theta}_3$  is independent of  $\vartheta_3$ .  $\square$

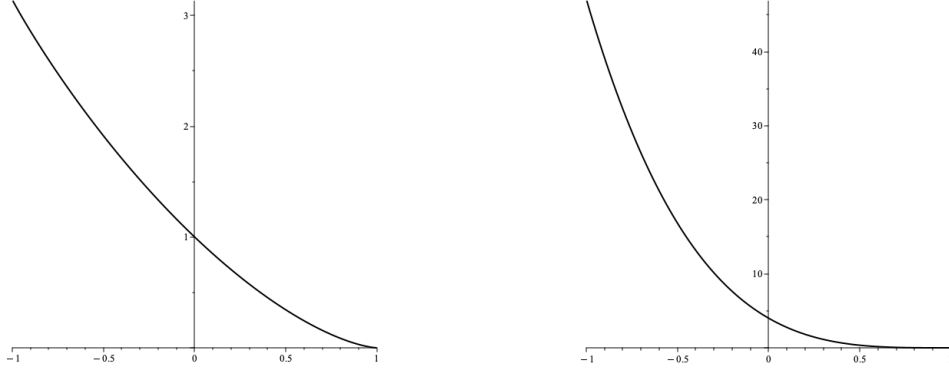


FIGURE 3.1. The panels display the graphs of  $\varrho_1(\vartheta_2)$  on the left and  $\varrho_2(\vartheta_2)$  on the right, with  $\vartheta_2$  varying over the interval  $[-1, 1]$ .

**Proposition 37.** *Let  $F_2 : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be the function defined by Eq. (3.10). If  $(a, b) \neq (0, 0)$ , then the differential  $DF_2|_{F_1(\tau, \eta)}$  has full rank for all  $(\tau, \eta) \in \text{Int } \mathcal{A}_{(a,b)}^+|_{0 < \eta}$ .*

*Proof.* We must show  $DF_2|_{F_1(\tau, \eta)}$  has rank two. To do so, we consider the cases  $0 < ab$  and  $ab < 0$ .

**Case  $0 < ab$ :** We consider the following determinant and apply Proposition 36 to obtain:

$$\frac{\partial \tilde{\Theta}_2}{\partial \vartheta_1} \frac{\partial \tilde{\Theta}_3}{\partial \vartheta_3} - \frac{\partial \tilde{\Theta}_2}{\partial \vartheta_3} \frac{\partial \tilde{\Theta}_3}{\partial \vartheta_1} = - \frac{\partial \tilde{\Theta}_2}{\partial \vartheta_3} \frac{\partial \tilde{\Theta}_3}{\partial \vartheta_1}.$$

Thus, it suffices to show that  $\frac{\partial \tilde{\Theta}_3}{\partial \vartheta_1} \neq 0$ . The expression for  $\tilde{\Theta}_3(\vartheta_1, \vartheta_2, \vartheta_3)$  from Appendix C implies:

$$\frac{\partial \tilde{\Theta}_3}{\partial \vartheta_1} \Big|_{F_1(\tau, \eta)} = - \frac{\vartheta_2(7a^2\varrho_2(\vartheta_2) + 30ab\vartheta_1\varrho_3(\vartheta_2) + 18b^2\vartheta_1^2\varrho_1(\vartheta_2))}{12\vartheta_1^4\sqrt{ab\vartheta_1\vartheta_2}}.$$

Here  $\varrho_1(\vartheta_2)$  and  $\varrho_2(\vartheta_2)$  are defined in Eq. (3.11). Additionally,  $\varrho_3(\vartheta_2)$  is given by:

$$\varrho_3(\vartheta_2) := (2\vartheta_2^2 + 1) \arccos(\vartheta_2) - 3\vartheta_2 \sqrt{1 - \vartheta_2^2}.$$

The condition  $0 < ab$  implies that  $0 < \beta < 1$ , and by construction  $0 < \vartheta_1$ . Therefore, it suffices to verify that  $\varrho_3(\vartheta_2) > 0$ . This is confirmed by graphing the function, as shown in Figure 3.2.

**Case  $ab < 0$ :** We consider the following determinant and apply Proposition 36 to obtain:

$$\frac{\partial \tilde{\Theta}_2}{\partial \vartheta_2} \frac{\partial \tilde{\Theta}_3}{\partial \vartheta_3} - \frac{\partial \tilde{\Theta}_2}{\partial \vartheta_3} \frac{\partial \tilde{\Theta}_3}{\partial \vartheta_2} = - \frac{\partial \tilde{\Theta}_2}{\partial \vartheta_3} \frac{\partial \tilde{\Theta}_3}{\partial \vartheta_2}.$$

Thus, it suffices to show that  $\frac{\partial \tilde{\Theta}_3}{\partial \vartheta_2} \neq 0$ . The expression for  $\tilde{\Theta}_3(\vartheta_1, \vartheta_2, \vartheta_3)$  from Appendix C implies:

$$(3.12) \quad \frac{\partial \tilde{\Theta}_3}{\partial \vartheta_2} \Big|_{F_1(\tau, \eta)} = - \frac{a^2 \varrho_4(\vartheta_2) - 6ab \varrho_5(\vartheta_2) \vartheta_1 + 6b^2 \varrho_6(\vartheta_2) \vartheta_1^2}{12\vartheta_1^3 \sqrt{ab} \vartheta_1 \vartheta_2}.$$

Here  $\varrho_3(\vartheta_2)$ ,  $\varrho_5(\vartheta_2)$ , and  $\varrho_6(\vartheta_2)$  are functions given by

$$\begin{aligned} \varrho_4(\vartheta_2) &:= (42\vartheta_2^3 + 27\vartheta_2) \arccos(\vartheta_2) - (65\vartheta_2^2 + 4) \sqrt{1 - \vartheta_2^2}, \\ \varrho_5(\vartheta_2) &:= (10\vartheta_2^2 + 1) \arccos(\vartheta_2) - 11\vartheta_2 \sqrt{1 - \vartheta_2^2}, \\ \varrho_6(\vartheta_2) &:= 3\vartheta_2 \arccos(\vartheta_2) - \sqrt{1 - \vartheta_2^2}. \end{aligned}$$

The numerator in Eq. (3.12) is a quadratic function of  $\vartheta_1$ . To demonstrate that this numerator is nonzero, it suffices to show that the discriminant of the quadratic is negative. The discriminant is given by

$$(3.13) \quad \Delta(\vartheta_2) = 6a^2b^2(6\varrho_5^2(\vartheta_2) - 4\varrho_4(\vartheta_2)\varrho_6(\vartheta_2)).$$

Therefore, to conclude, it is enough to verify that the expression in parentheses from Eq. (3.11) is negative. This fact is confirmed by graphing the function, as shown in Figure 3.2.  $\square$

*Proof of Lemma 32.* Consider three cases:  $(a, 0)$  with  $a \neq 0$ ,  $(0, b)$  with  $b \neq 0$ , and  $(a, b) \neq (0, 0)$ .

**Case  $(a, 0)$  with  $a \neq 0$ :** The expression for  $\Theta_{(a,b)}^+(\tau, \eta)$  is:

$$(f \circ \Theta_{(a,b)}^+)(\tau, \eta) = \left( \frac{\sqrt{2\tau}(3a^2\eta^2 - \tau^2)}{3(a^2\eta^2 + \tau^2)^2}, \frac{2a^2\nu(2\tau)^{\frac{3}{2}}}{3(a^2\eta^2 + \tau^2)^2} \right).$$

A direct calculation shows that  $\det D(f \circ \Theta_{(a,b)}^+) = \frac{4a^2\tau(3a^2\eta^2 + \eta^2)}{3(a^2\eta^2 + \tau^2)^4}$ . It follows that  $\det D(f \circ \Theta_{(a,b)}^+) \neq 0$ , since  $a \neq 0$ , and  $(\tau, \nu) \neq (0, 0)$ .

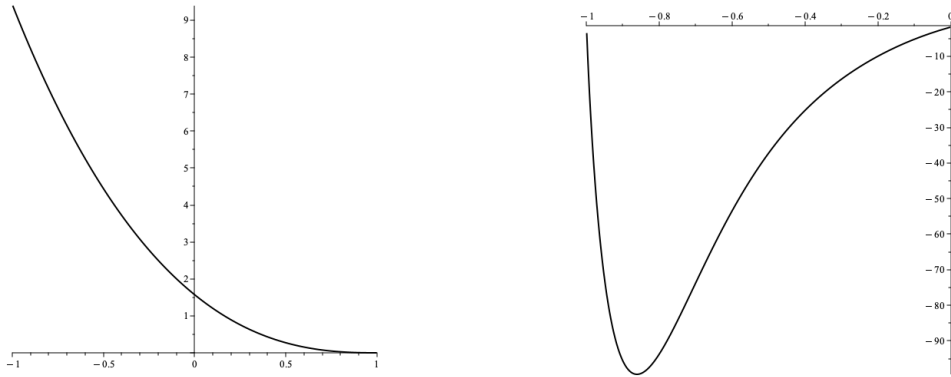


FIGURE 3.2. The panels display the graphs of  $\varrho_3(\vartheta_2)$  on the left with  $\vartheta_2$  ranging over  $(-1, 1)$ , and the graph  $\frac{\Delta(\vartheta_2)}{6a^2b^2}$  on the right with  $\vartheta_2$  ranging over  $(-1, 0)$ .

**Case  $(0, b)$  with  $b \neq 0$ :** The expression for  $\Theta_{(a,b)}^+(\tau, \eta)$  is given by:

$$(f \circ \Theta_{(a,b)}^+)(\tau, \eta) = \left( \frac{\sqrt{2\tau - b^2\eta^2}(2b^2\eta^2 - \tau)}{3\tau^3}, \frac{b^2\eta\sqrt{2\tau - b^2\eta^2}}{\tau^2} \right).$$

A direct computation shows that  $\det D(f \circ \Theta_{(a,b)}^+) = \frac{b^2}{\tau^4}$ . It follows that  $\det D(f \circ \Theta_{(a,b)}^+) \neq 0$  since  $b \neq 0$  and  $(\tau, \eta) \neq (0, 0)$ .

**Case  $(a, b) \neq (0, 0)$ :** Since  $(a, b) \neq (0, 0)$  the functions  $F_1$  and  $F_2$  are well defined and we write  $f \circ \Theta_{(a,b)}^+ = F_2 \circ F_1$ . Propositions 35 and 37 implies that  $D(F_2 \circ F_1)|_{(\tau, \eta)}$  is non-degenerate. Therefore,  $D(f \circ \Theta_{(a,b)}^+)|_{(\tau, \eta)}$  is non-degenerate.  $\square$

## CONCLUSION AND FUTURE WORK

We investigated the sub-Riemannian geodesics within the 2-jet space of plane curves. Using the symmetries: rotations, Carnot translations, and dilatations, we characterized the family of sub-Riemannian geodesics satisfying the asymptotic condition outlined in Conjecture 5. These geodesics were determined by a momentum parameter  $\mu \in \Lambda$ , where  $\Lambda$  is parametrized by the parameters  $(a, b) \in \mathbb{R}^2$  together with the dilatation.

Theorem A established that two families of geodesics with momentum  $\mu \in \Lambda$  are metric lines, namely the cases  $(a, 0)$ , and  $(0, b)$ . To prove Theorem A, we constructed a sub-Riemannian manifold  $\mathbb{R}_{(a,b)}^5$  and defined a period map that encodes the asymptotic behavior of these geodesics.

Theorem B extended the analysis to a general geodesic with momentum  $\mu \in \Lambda$ , providing conditions which make this sub-Riemannian geodesic a metric line. One key condition required is proving that the period map is one-to-one. To tackle this, we invoked Hadamard's Global Diffeomorphism

Theorem, which in turn depends on the image of the period map being simply connected. At present, we have established this property only for the cases  $(a, 0)$  and  $(0, b)$ .

Further work is needed to complete the proof of Conjecture 1. The objectives for future research are:

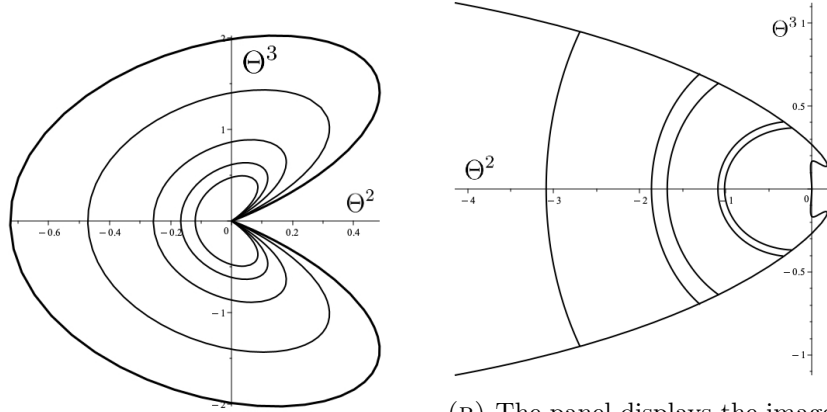
(a) Prove that the set  $N_{(a,b)}$ , as defined in Section 3.2, is simply connected for all  $(a, b) \in \mathbb{R}^2 \setminus \{(0, 0)\}$ .

In the cases  $(a, 0)$  and  $(0, b)$ ,  $\mathcal{A}_{(a,b)}^+$  is an open set, but in the general case, attention must be paid to the boundary  $\partial\mathcal{A}_{(a,b)}^+$ .

(b) Prove that  $\Theta_{(a,b)}^+ : \partial\mathcal{A}_{(a,b)}^+|_{0 < \eta} \rightarrow \mathbb{R}^2$  is one-to-one, and further, leverage the fact that  $\Theta_{(a,b)}^+ : \mathcal{A}_{(a,b)}^+ \rightarrow \mathbb{R}^2$  is an open map to show that  $\Theta_{(a,b)}^+ : \mathcal{A}_{(a,b)}^+|_{0 < \eta} \rightarrow \mathbb{R}^2$  is one-to-one.

(c) Establish Condition (4) from Theorem B for the general case  $(a, b)$ .

Figure 3.3 presents numerical evidence supporting the validity of tasks (a), (b), and (c).



(A) The panel displays the image of  $F_{(1,1)}$ . The curves are the values of  $F_{(1,1)}$  when we fix  $\tau$ , and vary  $\eta \in (-\sqrt{2\tau}, \sqrt{2\tau})$ .

(B) The panel displays the image of  $F_{(1,-1)}$ . The curves are the values of  $F_{(1,-1)}$  when we fix  $\tau$ , and vary  $\eta \in [-\sqrt{2\tau}, \sqrt{2\tau}]$  along with the boundary of  $N_{(1,-1)}$ .

FIGURE 3.3. The panels present numerical evidence supporting that  $N_{(a,b)}$  is simply connected. Our results further indicate that  $N_{(a,b)} = \mathbb{R}^2 \setminus [0, \infty] \times \{0\}$  for  $0 < ab$ .

#### APPENDIX A. THE JET SPACE AS CARNOT GROUP

The distribution  $\mathcal{D}$  generates a Lie algebra of vector fields via the bracket iteration:

$$Y_\ell^i := [X, Y_\ell] = \frac{\partial}{\partial u_\ell^{k-i}}, \quad \text{for all } \ell = 1, \dots, n \text{ and } i = 1, \dots, k.$$

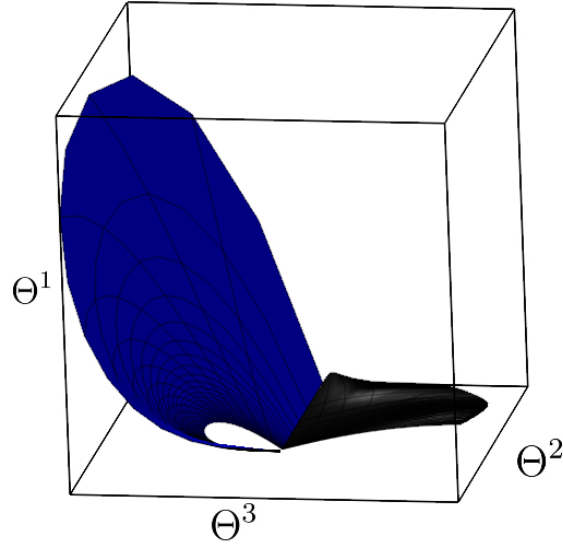


FIGURE 3.4. The panel display the image of  $\Theta_{(1,1)}^+(\tau, \eta)$  on blue and  $\Theta_{(1,1)}^-(\tau, \eta) = \Theta_{(1,-1)}^+(\tau, \eta)$  on black showing that they do only intercept of the values for  $\eta = 0$ .

Note that the bracket relations above do not depend on the point  $j_{x_0}^k(f)$ , which implies the vector fields are the left translation of the Lie algebra  $\mathfrak{j}^k(\mathbb{R}, \mathbb{R}^n)$ . Indeed, the restriction of these vector fields to the origin defines a stratified Lie algebra whose layers are:

$$(A.1) \quad \begin{aligned} V_1 &= \{X, Y_1, \dots, Y_n\} \\ V_i &= \{Y_1^{i-1}, \dots, Y_n^{i-1}\}, \text{ where } i = 2, \dots, k+1. \end{aligned}$$

Therefore, the Lie algebra is stratified as follows:

$$\mathfrak{j}^k(\mathbb{R}, \mathbb{R}^n) = V_1 \oplus \dots \oplus V_k.$$

For the case  $\mathcal{J}^2(\mathbb{R}, \mathbb{R}^2)$ , Its Lie algebra is generated by:

$$\mathfrak{j}^2(\mathbb{R}, \mathbb{R}^2) = \text{span}\{X, Y_1, Y_2, Y_1^1 Y_2^1, Y_1^2, Y_2^2\}.$$

The normal subgroup  $\mathcal{N}$  from the discussion in Section 2.2 is defined as  $\mathcal{N} = \exp(\mathfrak{n})$ , where  $\mathfrak{n} := \text{span}\{Y_2, Y_2^1, Y_2^2\}$  is an ideal by the above discussion. It follows that  $\mathfrak{j}^2(\mathbb{R}, \mathbb{R}^1) \simeq \mathfrak{j}^2(\mathbb{R}, \mathbb{R}^2)/\mathfrak{n}$  as we mentioned early.

The Lie bracket relations show that  $V_{i+1} = [V_1, V_i]$ , for  $i = 1, \dots, k$ , and  $[\mathfrak{j}^k(\mathbb{R}, \mathbb{R}^n), \mathfrak{j}^k(\mathbb{R}, \mathbb{R}^n)]$  is an abelian ideal of  $\mathfrak{j}^k(\mathbb{R}, \mathbb{R}^n)$ . Consequently,  $\mathcal{J}^k(\mathbb{R}, \mathbb{R}^n)$  is a metabelian Carnot group, a concept we will formally define below.

A group  $\mathbb{G}$  is metabelian if  $[\mathbb{G}, \mathbb{G}]$  is abelian [15, Chapter 5]. Equivalently, a group  $\mathbb{G}$  is metabelian if and only if it contains a normal abelian subgroup  $\mathbb{A} \triangleleft \mathbb{G}$  such that  $\mathbb{G}/\mathbb{A}$  is abelian. Therefore, we can think of every metabelian

Carnot group as an extension of  $\mathbb{G}/\mathbb{A}$  by  $\mathbb{A}$  where  $\mathbb{A}$  is the maximal abelian normal subgroup containing  $[\mathbb{G}, \mathbb{G}]$ ; this is a pivotal aspect in the symplectic reduction of  $T^*\mathbb{G}$  by  $\mathbb{A}$ . For further details on the construction of the reduced Hamiltonian  $H_\mu$  and the symplectic reduction, consult [4].

In the case of  $\mathcal{J}^k(\mathbb{R}, \mathbb{R}^n)$ , the maximal abelian subgroup  $\mathbb{A} = \exp(\mathfrak{a})$  where  $\mathfrak{a} = \text{span}\{Y_1, \dots, Y_n, Y_1^1, \dots, Y_n^k\}$  is maximal abelian ideal. Hence,  $\mathcal{J}^k(\mathbb{R}, \mathbb{R}^n)/\mathbb{A} \simeq \mathbb{R}$ .

**A.1. Symmetries.** The symmetries admitted by  $\mathcal{J}^k(\mathbb{R}, \mathbb{R}^n)$  are the following:

**(Carnot Translations)** By construction, the sub-Riemannian manifold is left-invariant under group translations.

**(Dilatations)** Carnot groups possess a natural family of dilatations, forming a one-parameter group of automorphisms of  $\mathbb{G}$ . For every  $\lambda \in \mathbb{R} \setminus \{0\}$ , we denote the dilatation by  $\lambda$  by  $\delta_\lambda$ . If  $\mathbb{G}$  is endowed with a left-invariant sub-Riemannian metric, then dilatations are compatible with the sub-Riemannian metric. i.e.,

$$\text{dist}_{\mathbb{G}}(\delta_\lambda g_1, \delta_\lambda g_2) = |\lambda| \text{dist}_{\mathbb{G}}(g_1, g_2).$$

In addition, if  $\gamma(t)$  is sub-Riemannian geodesic parameterized by arc length, so is  $\gamma_\lambda(t) = \delta_{\frac{1}{\lambda}} \gamma(\lambda t)$ . Lemma 2.5 from [2] shows that it is enough to classify metric lines up to dilatations and translations.

**(Rotations)** Let us consider the special orthonormal group  $SO(n)$ , and let  $Q \in SO(n)$ . We define an automorphism  $\psi_Q : \mathfrak{j}^k(\mathbb{R}, \mathbb{R}^n) \rightarrow \mathfrak{j}^k(\mathbb{R}, \mathbb{R}^n)$ , by defining its value in the basis  $\{X, Y_1, \dots, Y_n, Y_1^1, \dots, Y_n^k\}$ :

$$\begin{aligned} \psi_Q(X) &= X, \quad \psi_Q(Y_\ell) = \sum_{k=1}^n Q_{jk} Y_k \quad \text{for all } \ell = 1, \dots, n, \\ \psi_Q(Y_\ell^i) &= \sum_{k=1}^n Q_{jk} Y_k^i \quad \text{for all } \ell = 1, \dots, n, \quad \text{and } i = 2, \dots, k. \end{aligned}$$

The action  $\psi_Q$  induces an isometric action  $\Psi_Q : \mathcal{J}^k(\mathbb{R}, \mathbb{R}^n) \rightarrow \mathcal{J}^k(\mathbb{R}, \mathbb{R}^n)$ . Indeed, by construction,  $\psi_Q(Y_\ell^i) = [\psi_Q(X), \psi_Q(Y_\ell^{i-1})]$ , so  $\psi_Q$  defines a linear Lie automorphism. Since  $\mathcal{J}^k(\mathbb{R}, \mathbb{R}^n)$  is connected and simply connected, the exponential map defines an action  $\Psi_Q$  on  $\mathcal{J}^k(\mathbb{R}, \mathbb{R}^n)$  [8, Theorem 5.6]. This action preserves the distribution  $\mathcal{D}$ . In addition, the norm of a vector  $v \in \mathcal{D}$  is the same as the vector  $(\Psi_Q)_* v$ .

## APPENDIX B. ABNORMAL GEODESICS

We will prove the results corresponding to abnormal geodesics.

### B.1. Proof of Proposition 10.

*Proof of Proposition 10.* We compute the abnormal curves using Pontryagin's maximum principle. Consider the following optimal control problem:

$$(B.1) \quad \dot{\gamma}(t) = v_x(t)X + v_1(t)Y_1 + \cdots + v_n(t)Y_n$$

subject to the boundary conditions  $\gamma(0)$  and  $\gamma(T)$ . Note that Eq. (B.1) implies  $\dot{x} = v_x(t)$ . By the Cauchy-Schwarz inequality, minimizing the sub-Riemannian length is equivalent to minimizing the following action:

$$(B.2) \quad \frac{1}{2} \int_0^T v_x^2 + v_1^2 + \cdots + v_n^2 dt \rightarrow \min,$$

subject to the constraint  $v_x^2 + v_1^2 + \cdots + v_n^2 = 1$ . Accordingly, the Hamiltonian associated with the optimal control problem (B.1) and (B.2) for the abnormal case is:

$$(B.3) \quad H(P, j_x^k(f), v) = v_x(t)P_X + v_1(t)P_{Y_1} + \cdots + v_n(t)P_{Y_n},$$

where  $P_X, P_{Y_1}, \dots, P_{Y_n}$  denote the left-invariant momentum functions corresponding to the frame of left-invariant vector fields defining the distribution  $\mathcal{D}$ . Applying Pontryagin's maximum principle yields a Hamiltonian system for the left-invariant momentum functions:

$$(B.4) \quad \begin{aligned} \dot{P}_x &= -v_1(t)P_{Y_1^1} - \cdots - v_n(t)P_{Y_n^1}, \\ \dot{P}_{Y_\ell} &= v_x(t)P_{Y_\ell^1} \text{ for all } \ell = 1, \dots, n \\ \dot{P}_{Y_\ell^i} &= -v_x(t)P_{Y_\ell^{i+1}} \text{ for all } \ell = 1, \dots, n \text{ and } i = 2, \dots, k-1 \\ \dot{P}_{Y_\ell^k} &= 0 \text{ for all } \ell = 1, \dots, n. \end{aligned}$$

In addition, the maximal condition is given by:

$$(B.5) \quad \max_{u \in \mathbb{R}^{n+1}} H(P(t), j_x^k(f)(t), v) = H(P(t), \hat{\gamma}(t), \hat{u}(t)),$$

where  $\hat{\gamma}(t)$  is the optimal process, subject to the non-triviality condition  $P(t) \neq 0$ . Consequently, the maximum condition yields:

$$(B.6) \quad 0 = P_X, \text{ and } 0 = P_{Y_\ell} \text{ for } i = 1, \dots, n.$$

Thus, the maximum condition implies:

$$0 = -v_1(t)P_{Y_1^1} - \cdots - v_n(t)P_{Y_n^1}, \text{ and } 0 = v_x(t)P_{Y_\ell^1} \text{ for all } \ell = 1, \dots, n.$$

If  $v_x(t) \neq 0$ , then  $P_{Y_\ell^1} = 0$  for all  $\ell = 1, \dots, n$ . It follows that  $\dot{P}_{Y_\ell^1} = 0$  for all  $\ell = 1, \dots, n$ , which in turn implies  $\dot{P}_{Y_\ell^2} = 0$  for all  $j$ , and so forth. Thus, the condition  $v_x(t) \neq 0$  results in all left-invariant momentum functions vanishing, yielding the trivial case  $P(t) = 0$ .

On the other hand, if  $v_x(t) = 0$ , then  $\dot{\gamma}(t) \in \text{span}\{Y_1, \dots, Y_n\}$ , meaning the  $x$ -component of the curve remains constant. Since the vector fields  $Y_1, \dots, Y_n$  commute, the singular curves are tangent to an integrable distribution. Consequently, these curves lie on integral submanifolds, namely a hyperplane whose tangent spaces are spanned by  $\{Y_1, \dots, Y_n\}$ . As a result, these abnormal geodesics are straight lines.  $\square$

## B.2. Proof of Proposition 3.1.5.

*Proof of Proposition 3.1.5.* Adopting the approach outlined in Section B.1, consider the following optimal problem:

$$(B.7) \quad \dot{c}(t) = v_x(t)\tilde{X} + v_1(t)\tilde{Y}_1 + v_2(t)\tilde{Y}_2,$$

with its corresponding action. The Hamiltonian associated with the abnormal case for this control problem is:

$$(B.8) \quad H(p_x, p_y, p_z, x, y, z, u) = v_x(t)p_x + \sum_{\ell=1}^2 v_\ell(t)(p_{y^\ell} + p_{z^\ell}P_\ell(x)),$$

Applying Pontryagin's maximum principle yields a Hamiltonian system for the variables  $p_x$ ,  $p_y$ , and  $p_z$ :

$$(B.9) \quad \dot{p}_{x_i} = \sum_{\ell=1}^2 v_\ell(t)p_{z^\ell} \frac{\partial P_\ell}{\partial x}, \quad \dot{p}_{y^j} = 0 \quad \text{and} \quad \dot{p}_{z^j} = 0 \quad \text{for } i = 1, 2.$$

Thus, the maximum condition yields:

$$(B.10) \quad 0 = p_x, \quad \text{and} \quad 0 = p_{y^i} + p_{z^i}P_i(x) \quad \text{for } i = 1, 2.$$

Observe that the non-triviality condition requires  $(p_y, p_z) \neq (\mathbf{0}, \mathbf{0})$ . If  $p_z = \mathbf{0}$ , then  $p_y = \mathbf{0}$ , so we must have  $p_x \neq 0$ . Furthermore, the Hamiltonian equations imply  $v_x(t) = p_x$ , and therefore  $v_x(t) = 0$ . To obtain a non-trivial curve, it is necessary that  $(v_1(t), v_2(t)) \neq (0, 0)$ . By combining Eqs. (B.9) and (B.10), we obtain:

$$0 = (v_1(t), v_2(t)) \cdot (p_{z^1} \frac{\partial P_1}{\partial x}, p_{z^2} \frac{\partial P_2}{\partial x}).$$

□

## APPENDIX C. PERIOD MAP III

We present the explicit expression of the auxiliary function  $F_2 : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ , defined by Eq. 3.10, used in the proof of Lemma 32. After the integration,  $F_2(\vartheta_1, \vartheta_2, \vartheta_3)$  is given by:

$$\begin{aligned} \tilde{\Theta}_2(\vartheta_1, \vartheta_2, \vartheta_3) &= \frac{1}{a\sqrt{\vartheta_3}\vartheta_1^4} (\sigma_1(\vartheta_1, \vartheta_2, \vartheta_3)\sqrt{1-\vartheta_2^2} - \sigma_2(\vartheta_1, \vartheta_2, \vartheta_3) \arccos \vartheta_2), \\ \tilde{\Theta}_3(\vartheta_1, \vartheta_2, \vartheta_3) &:= \frac{-\sqrt{\vartheta_2}}{\sqrt{ab}\vartheta_1^{\frac{7}{2}}} (\sigma_3(\vartheta_1, \vartheta_2, \vartheta_3)\sqrt{1-\vartheta_2^2} + \sigma_4(\vartheta_1, \vartheta_2, \vartheta_3) \arccos \vartheta_2). \end{aligned}$$

Here  $\sigma_1(\vartheta_1, \vartheta_2, \vartheta_3)$ ,  $\sigma_2(\vartheta_1, \vartheta_2, \vartheta_3)$ ,  $\sigma_3(\vartheta_1, \vartheta_2, \vartheta_3)$ , and  $\sigma_4(\vartheta_1, \vartheta_2, \vartheta_3)$  are auxiliary functions defined as:

$$\begin{aligned}\sigma_1(\vartheta_1, \vartheta_2, \vartheta_3) &:= \frac{11}{12}(\vartheta_2^2 + \frac{3}{11})(a\vartheta_2^2 - b\vartheta_1\vartheta_2 - a)\vartheta_3^2 + a\vartheta_1^2, \\ \sigma_2(\vartheta_1, \vartheta_2, \vartheta_3) &:= \frac{\vartheta_2}{2}((\vartheta_2^2 + \frac{3}{2})(a\vartheta_2^2 - b\vartheta_1\vartheta_2 - a)\vartheta_3^2 + 2a\vartheta_1^2), \\ \sigma_3(\vartheta_1, \vartheta_2, \vartheta_3) &:= 3ab\vartheta_1\vartheta_2 - b^2\vartheta_1^2 - a^2(\frac{11\vartheta_2^2}{6} + \frac{2}{3}), \\ \sigma_4(\vartheta_1, \vartheta_2, \vartheta_3) &:= b^2\vartheta_1^2\vartheta_2 + (\vartheta_2^3 + \frac{3}{2}\vartheta_2) - 2ab\vartheta_1(\vartheta_2^2 + \frac{1}{2}).\end{aligned}$$

#### ACKNOWLEDGEMENT

The authors wish to express their gratitude to the Department of Mathematics at the University of Michigan for organizing the 2025 REU project, during which this research was conducted. In particular, we thank Prof. Asaf Cohen for his time and dedication in making the project possible. We would also like to thank Annie Winkler for her support and help before and during the project. Additionally, Daniella Catalá gratefully acknowledges the support of Prof. Anthony Bloch through his NSF grant ([DMS-2103026](#)), and Miriam Vollmayr-Lee gratefully acknowledges the support of Prof. Ralf Spatzier through his NSF grant ([DMS-2404309](#)).

**Funding information.** This paper was developed with the support of the NSF grants [DMS-2103026](#) and [DMS-2404309](#).

**Author contribution.** The authors confirm sole responsibility for the study's conception, the presentation of the results, and the preparation of the manuscript.

**Conflict of interest.** The author states no conflict of interest.

#### REFERENCES

- [1] A. Agrachev, D. Barilari, and U. Boscain. *A comprehensive introduction to sub-Riemannian geometry*, volume 181. Cambridge University Press, 2019.
- [2] A. Bravo-Doddoli. Metric lines in the jet space. *Analysis and Geometry in Metric Spaces*, 12(1):20240016, 2024.
- [3] A. Bravo-Doddoli. Metric lines in engel-type groups, 2025.
- [4] A. Bravo-Doddoli, E. Le Donne, and N. Paddeu. Symplectic reduction of the subriemannian geodesic flow for metabelian nilpotent groups. *Geometric Mechanics*, 01(01), March 2024.
- [5] A. Bravo-Doddoli and R. Montgomery. Geodesics in jet space. *Regular and Chaotic Dynamics*, 27(2):151–182, 2022.
- [6] M. Gromov. Carnot-carathéodory spaces seen from within. In *Sub-Riemannian geometry*, pages 79–323. Springer, 1996.
- [7] E. Hakavuori and E. Le Donne. Blowups and blowdowns of geodesics in carnot groups. *Journal of Differential Geometry*, 123(2):267–310, 2023.
- [8] B. Hall. *Lie Groups, Lie Algebras, and Representations: An Elementary Introduction*. Graduate Texts in Mathematics. Springer International Publishing, 2015.

- [9] J. Hilgert and K.H. Neeb. *Structure and geometry of Lie groups*. Springer Science & Business Media, 2011.
- [10] S. G. Krantz and H. R. Parks. *The implicit function theorem: history, theory, and applications*. Springer Science & Business Media, 2002.
- [11] D. F. Lawden. *Elliptic functions and applications*, volume 80. Springer Science & Business Media, 2013.
- [12] E. Le Donne and F. Tripaldi. A cornucopia of carnot groups in low dimensions. *Analysis and Geometry in Metric Spaces*, 10(1):155–289, 2022.
- [13] Enrico Le Donne. *Metric Lie groups. Carnot-Carathéodory spaces from the homogeneous viewpoint*, volume 306 of *Grad. Texts Math.* Cham: Springer, 2025.
- [14] R. Montgomery. *A tour of subriemannian geometries, their geodesics and applications*. Number 91. American Mathematical Soc., 2002.
- [15] D.J.S. Robinson. *A Course in the Theory of Groups*, volume 80. Springer Science & Business Media, 2012.
- [16] B. Warhurst. Jet spaces as nonrigid carnot groups. *Journal of Lie theory*, 15(1):341–356, 2005.

DANIELLA CATALÁ: DEPARTMENT OF MATHEMATICS, MASSACHUSETTS INSTITUTE OF TECHNOLOGY, CAMBRIDGE, MA 02139, U.S.

*Email address:* `dcatala@mit.edu`

MIRIAM VOLLMAYR-LEE: DEPARTMENT OF MATHEMATICS, GRINNELL COLLEGE, GRINNELL, IA 50112, U.S.

*Email address:* `vollmayr@grinnell.edu`

ALEJANDRO BRAVO-DODDOLI: DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MICHIGAN, ANN ARBOR, MI 48109, U.S.

*Email address:* `Abravodo@umich.edu`