NON-INTEGRABLE SUBRIEMANNIAN GEODESIC FLOW ON $J^2(\mathbb{R}^2, \mathbb{R})$

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Abstract. The space of 2-jets of a real function of two real variables, denoted by $J^2(\mathbb{R}^2,\mathbb{R})$, admits the structure of a metabelian Carnot group, so $J^2(\mathbb{R}^2,\mathbb{R})$ has a normal abelian sub-group $\mathbb{A}$. As any sub-Riemannian manifold, $J^2(\mathbb{R}^2,\mathbb{R})$ has an associated Hamiltonian geodesic flow. The Hamiltonian action of $\mathbb{A}$ on $T^*J^2(\mathbb{R}^2,\mathbb{R})$ yields the reduced Hamiltonian H_μ on $T^*\mathcal{H} \simeq T^*(J^2(\mathbb{R}^2,\mathbb{R})/\mathbb{A})$, where H_μ is a two-dimensional Euclidean space. The paper is devoted to proving that reduced Hamiltonian H_μ is non-integrable by meromorphic functions for some values of μ. This result suggests the sub-Riemannian geodesic flow on $J^2(\mathbb{R}^2,\mathbb{R})$ is not meromorphically integrable.

1. Introduction

Let $J^2(\mathbb{R}^2,\mathbb{R})$ be the space of 2-jets of a real function of two variables. $J^2(\mathbb{R}^2,\mathbb{R})$ is a Carnot group with step 3 and growth vector $(5,7,8)$. Let $\mathfrak{j}$ be the graded Lie algebra of $J^2(\mathbb{R}^2,\mathbb{R})$, that is,

$$\mathfrak{j} = \mathfrak{j}_1 \oplus \mathfrak{j}_2 \oplus \mathfrak{j}_3, \quad \text{such that } [\mathfrak{j}_i,\mathfrak{j}_j] \subseteq \mathfrak{j}_{i+j} \quad \text{and} \quad \mathfrak{j}_4 = \{0\}.$$

Let $\pi : J^2(\mathbb{R}^2,\mathbb{R}) \to \mathbb{R}^5 \simeq \mathfrak{j}_1$ be the canonical projection and let $\mathbb{R}^5$ be endowed with the Euclidean metric. Consider the sub-Riemannian metric on $J^2(\mathbb{R}^2,\mathbb{R})$ such that π is a sub-Riemannian submersion, see Definition 2.1 for the formal definition of a sub-Riemannian submersion, by construction the sub-Riemannian structure is left-invariant under the Carnot group multiplication. Like any sub-Riemannian structure, the cotangent bundle $T^*J^2(\mathbb{R}^2,\mathbb{R})$ is equipped with a Hamiltonian system whose underlying Hamiltonian H_{sR} is one whose solutions curves are sub-Riemannian geodesics on $J^2(\mathbb{R}^2,\mathbb{R})$. This Hamiltonian system is called the sub-Riemannian geodesic flow on $J^2(\mathbb{R}^2,\mathbb{R})$.

We say that a group $\mathbb{G}$ is metabelian if $[\mathbb{G},\mathbb{G}]$ is abelian. In [12], we considered the sub-Riemannian geodesics flow on a general metabelian Carnot group $\mathbb{G}$. Then, we performed the symplectic reduction of the cotangent bundle $T^*\mathbb{G}$ by the Hamiltonian action of the maximal

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normal abelian sub-group $\mathbb{A}$ containing $[\mathbb{G},\mathbb{G}]$, where $\mathbb{A}$ acts on $\mathbb{G}$ by left multiplication. This action is free and proper, so $\mathcal{H} := \mathbb{G}/\mathbb{A}$ is well defined. Let $\mathfrak{a}$ be the Lie algebra of $\mathbb{A}$ and let μ be in $\mathfrak{a}^*$, since $\mathbb{A}$ is abelian, the isotropic sub-group of $\mathbb{A}_\mu := \{g \in \mathbb{A} : Ad^*_g\mu = \mu\}$ is $\mathbb{A}$ and the symplectic reduced space is diffeomorphic to $T^*(\mathbb{G}/\mathbb{A}) \simeq T^*\mathcal{H}$. For more details about the symplectic reduction of the sub-Riemannian geodesics flow on a metabelian Carnot group, see [12], and for the general theory, see [8] or [11].

In the case $\mathbb{G} = J^2(\mathbb{R}^2,\mathbb{R})$, we have $[\mathfrak{j},\mathfrak{j}] = \mathfrak{j}_2 \oplus \mathfrak{j}_3$ and the Lie bracket relations in equations (2.2) and (2.3), see below, show $[\mathfrak{j}_2 \oplus \mathfrak{j}_3, \mathfrak{j}_2 \oplus \mathfrak{j}_3] = 0$ meaning $J^2(\mathbb{R}^2,\mathbb{R})$ is a metabelian Carnot group. Following the notation used in [12]: $\mathbb{A}$ is a 6-dimensional sub-group, whose Lie algebra is framed by $\{E^{\mathfrak{a}}_1, E^{\mathfrak{a}}_2, E^{\mathfrak{a}}_3, E^{\mathfrak{a}}_4, E^{\mathfrak{a}}_5, E^{\mathfrak{a}}_6\}$, see equations 2.2 and 2.3. Then, $\mathcal{H}$ is a 2-dimensional Euclidean space and the reduced Hamiltonian is a two degree of freedom system with polynomial potential, see equation 3.3.

The main Theorem of this paper is the following.

Theorem A. *Let $H_\mu : T^*\mathcal{H} \to \mathbb{R}$ be the reduced Hamiltonian given by the symplectic reduction of sub-Riemannian geodesic flow on $J^2(\mathbb{R}^2,\mathbb{R})$ under the action of $\mathbb{A}$, where μ is in $\mathfrak{a}^*$. Then, there exists a one parameter family in $\mathfrak{a}^*$ such that the reduced Hamiltonian H_μ is not meromorphically integrable.*

Theorem A suggests the sub-Riemannian geodesic flow on $J^2(\mathbb{R}^2,\mathbb{R})$ is not meromorphically integrable.

Examples of Carnot groups with a non-integrable geodesic flow are the following: One is the group of all 4 by 4 lower triangular matrices with 1's on the diagonal proved by R. Montgomery, M. Shapiro and A. Stolin, see [10]. Another is the Carnot group with growth vector $(3, 6, 14)$ showed by I. Bizyaev, A. Borisov, A. Kilin, and I. Mamaev, see [3]. Finally, there is the Carnot group with growth vector $(2, 3, 5, 8)$. Verified by L. V. Lokutsievskiy and Y. L. Sachkov, see [6].

Kruglikov, B., Vollmer, A. and Lukes-Gerakopoulos, G. made a classification of the integrable geodesic flow on Carnot groups of rank 2 and low dimension, see [4].

2. $J^2(\mathbb{R}^2,\mathbb{R})$ as a Carnot group

The 2-jet of a smooth function $f : \mathbb{R}^2 \to \mathbb{R}$ at a point $(x_0, y_0) \in \mathbb{R}^2$ is its 2-th order Taylor polynomial at x_0. We will encode the 2-jet as a 8-tuple of real numbers $(j^2 f)|_{(x_0,y_0)}$ as follows:

$$(j^2 f)|_{(x_0,y_0)} := \left(x_0, y_0, \frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, f \right) |_{(x_0,y_0)} \in \mathbb{R}^8$$

As f varies over smooth functions and (x_0, y_0) varies over $\mathbb{R}^2$, these 2-jets sweep out the 2-jet space, denoted by $J^2(\mathbb{R}^2, \mathbb{R})$. One can see that $J^2(\mathbb{R}^2, \mathbb{R})$ is diffeomorphic to $\mathbb{R}^8$ and its points are coordinatized according to

$$g = (x, y, u_{2,0}, u_{1,1}, u_{0,2}, u_{1,0}, u_{0,1}, u) \in \mathbb{R}^8.$$

Recall that if $u = f(x, y)$, then $u_{1,0} = \frac{\partial u}{\partial x}$, $u_{0,1} = \frac{\partial u}{\partial y}$, $u_{2,0} = \frac{\partial u_{1,0}}{\partial x}$, $u_{1,1} = \frac{\partial u_{1,0}}{\partial y} = \frac{\partial u_{0,1}}{\partial x}$ and $u_{0,2} = \frac{\partial u_{0,1}}{\partial y}$. We see that $J^2(\mathbb{R}^2, \mathbb{R})$ is endowed with a natural rank 5 distribution $\mathcal{D} \subset TJ^2(\mathbb{R}^2, \mathbb{R})$ characterized by the following Pfaffian equations

$$u_{1,0}dx + u_{0,1}dy - du = u_{2,0}dx + u_{1,1}dy - du_{1,0} = u_{1,1}dx + u_{0,2}dy - du_{0,1} = 0.$$

A sub-Riemannian structure on a manifold consists of a non-integrable distribution $\mathcal{D}$ together with a smooth inner product $(\cdot, \cdot)_{J^2(\mathbb{R}^2,\mathbb{R})}$ on $\mathcal{D}$. We arrive at the sub-Riemannian structure by observing that $\mathcal{D}$ is globally framed by

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x} + u_{1,0}\frac{\partial}{\partial u} + u_{2,0}\frac{\partial}{\partial u_{1,0}} + u_{1,1}\frac{\partial}{\partial u_{0,1}}, \\ X_2 &= \frac{\partial}{\partial y} + u_{0,1}\frac{\partial}{\partial u} + u_{1,1}\frac{\partial}{\partial u_{1,0}} + u_{0,2}\frac{\partial}{\partial u_{0,1}}, \\ Y_1 &= \frac{\partial}{\partial u_{2,0}}, Y_2 = \frac{\partial}{\partial u_{1,1}}, Y_3 = \frac{\partial}{\partial u_{0,2}}. \end{aligned} \tag{2.1}$$

The Canonical projection π is defined by

$$\pi(g) = g \;\; mod \; [J^2(\mathbb{R}^2, \mathbb{R}), J^2(\mathbb{R}^2, \mathbb{R})],$$

and in coordinates is given by $\pi(g) = (x, y, u_{2,0}, u_{1,1}, u_{0,2})$. Now the restrictions of the one-forms $dx, dy, du_{2,0}, du_{1,1}, du_{0,2}$ to $\mathcal{D}$ form a global co-frame for $\mathcal{D}^*$ which is dual to the frame from equation (2.1). Let us introduce the formal definition of a sub-Riemannian submersion.

Definition 2.1. *Let $(M, \mathcal{D}_M, (\cdot, \cdot)_M)$ and $(N, \mathcal{D}_N, (\cdot, \cdot)_N)$ be two sub-Riemannian manifolds and let $\phi : M \to N$ a submersion, we consider the case $dim(M) \geq dim(N)$. We say that ϕ is a sub-Riemannian submersion if $(\phi)_*\mathcal{D}_M = \mathcal{D}_N$ and $(\phi)^*(\cdot, \cdot)_N = (\cdot, \cdot)_M$.*

Therefore sub-Riemannian metric on $J^2(\mathbb{R}^2, \mathbb{R})$ making π a sub-Riemannian submersion is given in coordinates by

$$(\cdot, \cdot)_{J^2(\mathbb{R}^2,\mathbb{R})} = (dx^2 + dy^2 + du_{2,0}^2 + du_{1,1}^2 + du_{0,2}^2)|_{\mathcal{D}}.$$

An equivalent way to define the sub-Riemannian metric is to declare the left-invariant vector fields from equation (2.1) orthonormal. For more details about the jet space as Carnot group, see [14].

Let $\{E_1, E_2, E_1^{\mathfrak{a}}, E_2^{\mathfrak{a}}, E_3^{\mathfrak{a}}\}$ be the base for first layer $\mathfrak{j}_1$, where $X_i(g) = (L_g)_* E_i$ for $i = 1, 2$ and $Y_j(g) = (L_g)_* E_j^{\mathfrak{a}}$ for $j = 1, 2, 3$. The frame for $\mathfrak{j}_1$ generates the following Lie algebra:

$$E_4^{\mathfrak{a}} := [E_1, E_1^{\mathfrak{a}}] = [E_2, E_2^{\mathfrak{a}}], \quad E_5^{\mathfrak{a}} := [E_1, E_2^{\mathfrak{a}}] = [E_2, E_3^{\mathfrak{a}}], \tag{2.2}$$

equations (2.2) define the vector corresponding to the second layer $\mathfrak{j}_2$,

$$E_6^{\mathfrak{a}} := [E_1, E_4^{\mathfrak{a}}] = [E_2, E_5^{\mathfrak{a}}], \tag{2.3}$$

equations (2.3) define the vector corresponding to the third layer $\mathfrak{j}_3$. All the other brackets are zero. The Lie bracket relations in equations (2.2) and (2.3) imply that $\mathfrak{a}$ is framed by $\{E_1^{\mathfrak{a}}, E_2^{\mathfrak{a}}, E_3^{\mathfrak{a}}, E_4^{\mathfrak{a}}, E_5^{\mathfrak{a}}, E_6^{\mathfrak{a}}\}$. Let $\mathcal{H}$ be the 2-dimensional Euclidean space defined by quotient $J^2(\mathbb{R}^2, \mathbb{R})/\mathbb{A}$. Since $[E_1, E_2] = 0$, we can think $\mathcal{H}$ as a sub-group of $J^2(\mathbb{R}^2, \mathbb{R})$ such that $J^2(\mathbb{R}^2, \mathbb{R}) \simeq \mathbb{A} \rtimes \mathcal{H}$.

2.1. **The exponential coordinates of the second kind.** The jet space $J^2(\mathbb{R}^2, \mathbb{R})$ has a natural definition using the coordinates x, y, and u's; however, these coordinates do not easily show the symmetries of the system. The exponential coordinates of the second kind exhibit the symmetries:

We recall that the exponential map $\exp : \mathfrak{j} \to J^2(\mathbb{R}^2, \mathbb{R})$ is a global diffeomorphism, this allow us to endow $J^2(\mathbb{R}^2, \mathbb{R})$ with coordinates $(x, y, \theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6)$ in the following way: a point g in $J^2(\mathbb{R}^2, \mathbb{R})$ is given by

$$g := \prod_{i=1}^{6} \exp(\theta_i E_i^{\mathfrak{a}}) * \exp(y E_2) * \exp(x E_1).$$

Then the horizontal left-invariant vector fields are given by

$$X_1 := \frac{\partial}{\partial x}, \qquad X_2 := \frac{\partial}{\partial y}, \tag{2.4}$$

and

$$\begin{aligned} Y_1 &:= \frac{\partial}{\partial \theta_1} + x \frac{\partial}{\partial \theta_4} + \frac{x^2}{2!} \frac{\partial}{\partial \theta_6}, \\ Y_2 &:= \frac{\partial}{\partial \theta_2} + y \frac{\partial}{\partial \theta_4} + x \frac{\partial}{\partial \theta_5} + xy \frac{\partial}{\partial \theta_6}, \\ Y_3 &:= \frac{\partial}{\partial \theta_3} + y \frac{\partial}{\partial \theta_5} + \frac{y^2}{2!} \frac{\partial}{\partial \theta_6}. \end{aligned} \tag{2.5}$$

The left-invariant vector fields from equation (2.4) and (2.5) just depend on the independent variables x and y. All the metabelian Carnot groups have this property, which is the heart of the symplectic reduction. For more details, see [12].

The change from the coordinates $(x, y, u_{2,0}, u_{1,1}, u_{0,2}, u_{1,0}, u_{0,1}, u)$ to the exponential coordinates of the second kind $(x, y, \theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6)$ is the following

$$\begin{pmatrix} x \\ y \\ \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \\ \theta_5 \\ \theta_6 \end{pmatrix} = \begin{pmatrix} x \\ y \\ u_{2,0} \\ u_{1,1} \\ u_{0,2} \\ xu_{2,0} + yu_{1,1} - u \\ xu_{1,1} + yu_{0,2} - u \\ \frac{x^2}{2} + xyu_{1,1} + \frac{y^2}{2}u_{0,2} - xu_{1,0} - yu_{0,1} + u \end{pmatrix}$$

3. Geodesic flow on $J^2(\mathbb{R}^2,\mathbb{R})$

Let us consider the traditional coordinates on $T^*J^2(\mathbb{R}^2,\mathbb{R})$, that is, (p, g) where $p := (p_x, p_y, p_1, p_2, p_3, p_4, p_5, p_6)$ are the momentum associated with exponential coordinates of the second type, see [2] and [5] for more details about the traditional coordinates. Let P_{X_1}, P_{X_2}, P_{Y_1}, P_{Y_2} and P_{Y_3} be the momentum functions associated with the left-invariant vector fields on the first layer $\mathfrak{j}_1$ are given by

$$P_{X_1} = p_x, \qquad P_{X_2} = p_y \qquad Y_1 = p_1 + xp_4 + \frac{x^2}{2!}p_6,$$
$$Y_2 = p_2 + yp_4 + xp_5 + xyp_6, \qquad Y_3 = p_3 + yp_5 + \frac{y^2}{2!}p_6, \tag{3.1}$$

see [9], or [1] for more details about the momentum functions. Then, the Hamiltonian governing the sub-Riemannian geodesic flow on $J^2(\mathbb{R}^2,\mathbb{R})$ is

$$H_{sR} := \frac{1}{2}(P_{X_1}^2 + P_{X_2}^2 + P_{Y_1}^2 + P_{Y_2}^2 + P_{Y_3}^2). \tag{3.2}$$

See [9], or [1] for more details about the definition of H_{sR}.

The Hamiltonian function H_{sR} does not depend on the coordinates θ_1, θ_2, θ_3, θ_4, θ_5 and θ_6, so they are cyclic coordinates, in other words, p_1, p_2, p_3, p_4, p_5 and p_6 are constants of motion, see [5] or [2] for more details about the cyclic coordinates. Moreover, since H_{sR} is invariant under the action of $\mathbb{A}$, these constants of motion correspond to the momentum map $J : T^*J^2(\mathbb{R}^2,\mathbb{R}) \to \mathfrak{a}^*$ given by

$$J(p, g) = \mu := (a_1, a_2, a_3, a_4, a_5, a_6) \;\text{ where }\; p_i = a_i, \;\; 1 \le i \le 6.$$

See [9] or [11] for the formal definition of the momentum map. See [12] for the construction of the momentum map in the context of metabelian Carnot group.

3.1. **The reduced Hamiltonian.** By the symplectic theory, the reduced space is diffeomorphic to $T^*(G/A) \simeq T^*\mathcal{H}$, and the reduced Hamiltonian is a two-degree-of-freedom system with a polynomial potential of degree four in the variables x and y, and depending on the parameters $\mu := (a_1, a_2, a_3, a_4, a_5, a_6)$ in $\mathfrak{a}^*$, given by

$$H_\mu(p_x, p_y, x, y) := \frac{1}{2}\left(p_x^2 + p_2^2 + \phi_\mu(x, y)\right), \tag{3.3}$$

where $\phi_\mu(x, y)$ is the following potential

$$(a_1 + a_4 x + \frac{x^2}{2!}a_6)^2 + (a_2 + a_5 x + a_4 y + a_6 xy)^2 + (a_3 + a_5 y + a_6\frac{y^2}{2!})^2. \tag{3.4}$$

Let $\pi_{\mathbb{A}} : J^2(\mathbb{R}^2, \mathbb{R}) \to \mathcal{H}$ be the canonical projection given by

$$\pi_{\mathbb{A}}(g) = (x, y).$$

Let $\Pi_{\mathbb{A}} : T^*G \to T^*\mathcal{H}$ be co-lift projection associated to $\pi_{\mathbb{A}}$, that is,

$$\Pi_{\mathbb{A}}(p, g) = (p_x, p_y, x, y).$$

Then symplectic reduction implies

$$H_{sR}|_{J^{-1}(\mu)} = H_\mu \circ \Pi_{\mathbb{A}}.$$

3.2. **Background Theorem.** Here we introduce the **Background Theorem**, which provides a complete classification of the Yang-Mills Hamiltonian system by S. Shi and W. Li, in [13].

Background Theorem. *Let H be the Hamiltonian system given by*

$$H = \frac{1}{2c}(p_x^2 + p_y^2) + \frac{1}{2c}(ax^2 + by^2) + \frac{1}{4c^2}(cx^2 + dy^2 + 2ex^2y^2), \tag{3.5}$$

where a, b, $c \neq 0$, d and e are in $\mathbb{R}$. Then H is meromorphically integrable in the Liouvillian sense (i.e., the existence of an additional meromorphic integral) if and only if one of the following conditions hold:

(A) $e = 0$,
(B) $c = d = e$,
(C) $a = b$ and $e = 3c = ed$,
(D) $b = 4a$, $e = 3c$ and $d = 8c$,
(E) $b = 4a$, $e = 6c$ and $d = 16c$,
(F) $b = 4a$, $e = 3d$ and $c = 8d$,
(G) $b = 4a$, $e = 6d$ and $c = 16d$.

A consequence of the **Background Theorem** is the following.

Corollary 3.1. *Let H be the Hamiltonian given by*

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{4c^2}(cx^4 + 4cx^2y^2 + cy^4). \tag{3.6}$$

Then H is not meromorphically integrable in the Liouvillian sense.

Proof. Following the notation from the **Background Theorem**, we have that $c = d$, $e = 2c$, and $a = b = 0$, so H is not meromorphically integrable. □

An alternative proof of corollary 3.1 is given by Maciejewski, A. J. and Przybylska, M., in [7].

3.3. Proof of Theorem A.

Now we are ready to prove Theorem A.

Proof. If $\mu = \left(0, 0, 0, 0, 0, 0, 0, \sqrt{\frac{2}{c}}\right)$ with c in $(0, \infty)$, then equation (3.4) implies the potential $\phi_\mu(x, y)$ is $\frac{1}{4c^2}(cx^4 + 4cx^2y^2 + cy^4)$. Let H_μ be given by equation (3.3), then H_μ is equal to the Hamiltonian given by equation (3.6), so by Corollary 3.1 H_μ is not integrable by meromorphic functions. □

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