ABELIAN INSTANCES OF NONABELIAN SYMPLECTIC REDUCTION

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Abstract. Consider a Lie group $\mathbb{G}$ with a normal abelian subgroup $\mathbb{A}$. Suppose that $\mathbb{G}$ acts on a Hamiltonian fashion on a symplectic manifold (M, ω). Such action can be restricted to a Hamiltonian action of $\mathbb{A}$ on M. This work investigates the conditions under which the (generally nonabelian) symplectic reduction of M by $\mathbb{G}$ is equivalent to the (abelian) symplectic reduction of M by $\mathbb{A}$. While the requirement that the symplectically reduced spaces share the same dimension is evidently necessary, we prove that it is, in fact, sufficient. We then provide classes of examples where such equivalence holds for generic momentum values. These examples include certain semi-direct products and a large family of nilpotent groups which includes some classical Carnot groups, like the Heisenberg group and the jet space $\mathcal{J}^k(\mathbb{R}^n, \mathbb{R}^m)$.

1. Introduction

Let $\mathbb{G}$ be a Lie group with regular, normal, abelian subgroup $\mathbb{A}$. Suppose that $\mathbb{G}$ acts on a Hamiltonian fashion on a symplectic manifold (M, ω). Such action can be restricted to a Hamiltonian action of the abelian subgroup $\mathbb{A}$. In this paper, we investigate conditions which lead to the equivalence of the symplectic reduction by the (generally nonabelian) full Lie group $\mathbb{G}$ and the abelian subgroup $\mathbb{A}$. Our inspiration was the observation that this equivalence holds for generic momentum values for a large number of low-dimensional Carnot groups (see Table 1 below).

1.1. Main contributions.

To formulate our result precisely, denote by $\mathfrak{g}$ and $\mathfrak{a}$ the Lie algebras of $\mathbb{G}$ and $\mathbb{A}$, and by $i : \mathfrak{a} \hookrightarrow \mathfrak{g}$ the natural inclusion map. Let $i^* : \mathfrak{g}^* \to \mathfrak{a}^*$ be the corresponding dual map (which is the restriction of elements of $\mathfrak{g}^*$ to $\mathfrak{a}^*$). We prove the following equivalence of symplectic reductions and give a wide class of examples.

Theorem A. *Let $\mathbb{G}$ be a Lie group with an abelian, normal, and regular subgroup $\mathbb{A}$. Suppose $\mathbb{G}$ defines a free and proper Hamiltonian action on the symplectic manifold (M, ω) with equivariant momentum map $J_{\mathbb{G}} : M \to \mathfrak{g}^*$. Let $\mu \in J_{\mathbb{G}}(M) \subset \mathfrak{g}^*$ and suppose that $\mathbb{G}_\mu$ is connected. Then, there is a*

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symplectic diffeomorphism between the symplectic reduced spaces $M/\!\!/_\mu \mathbb{G}$ *and* $M/\!\!/_{i^*(\mu)}\mathbb{A}$ *if and only if they have the same dimension.*

The condition that the spaces have the same dimension is obviously necessary for a diffeomorphism to exist, so our contribution is to show that it is sufficient. The proof of this is presented in section 2 and, as we explain below, relies heavily on the Theory of Reduction by Stages developed by Marsden et al [30].

The assumption that the action of $\mathbb{G}$ on M is free implies that the momentum maps $J_{\mathbb{G}} : M \to \mathfrak{g}^*$ and $J_{\mathbb{A}} : M \to \mathfrak{a}^*$ are submersions. As a consequence, $\dim J_{\mathbb{G}}^{-1}(\mu) = \dim M - \dim \mathbb{G}$ and $\dim J_{\mathbb{A}}^{-1}(i^*(\mu)) = \dim M - \dim \mathbb{A}$. Therefore,

$$\dim(M/\!\!/_\mu \mathbb{G}) = \dim(J_{\mathbb{G}}^{-1}(\mu)/\mathbb{G}_\mu) = \dim M - \dim \mathbb{G} - \dim \mathbb{G}_\mu,$$

and, since $\mathbb{A}$ is abelian,

$$\dim(M/\!\!/_{i^*(\mu)}\mathbb{A}) = \dim(J_{\mathbb{A}}^{-1}(i^*(\mu))/\mathbb{A}) = \dim M - 2\dim \mathbb{A}.$$

Therefore, the condition that the symplectic reduced spaces have the same dimension is

$$\dim \mathbb{G} + \dim \mathbb{G}_\mu = 2\dim \mathbb{A}. \tag{1.1}$$

In sections 3 and 4 we present examples of Lie groups $\mathbb{G}$, possessing a normal abelian subgroup $\mathbb{A}$, such that the above condition holds for generic momentum values $\mu \in \mathfrak{g}^*$. In contrast, note that eq. (1.1) holds for the exceptional value $\mu = 0$ only in the extreme case in which $\mathbb{G}$ and $\mathbb{A}$ have the same dimension.

The identity (1.1) can be restated in terms of the dimension of the coadjoint orbit $\mathcal{O}_\mu$ through μ as follows. Considering that $\dim \mathcal{O}_\mu = \dim \mathbb{G} - \dim \mathbb{G}_\mu$, we get that (1.1) is equivalent to

$$\dim \mathcal{O}_\mu = 2(\dim \mathbb{G} - \dim \mathbb{A}) = 2\dim(\mathbb{G}/\mathbb{A}). \tag{1.2}$$

Perhaps the simplest nontrivial example is $\mathbb{G} = \mathrm{SE}(2)$ and $\mathbb{A} = \mathbb{R}^2$. In this case, (1.1) holds if $\mathbb{G}_\mu$ is 1-dimensional which is equivalent to the condition that the coadjoint orbit through $\mu \in \mathfrak{se}(2)^*$ is generic. Theorem A implies, the somewhat surprising fact, that, for generic momentum values, symplectic reduction by the full euclidean group SE(2) is equivalent to the symplectic reduction by the translation part $\mathbb{R}^2$.[1] The applicability of the theorem to other semi-direct product Lie groups is clarified in Theorem B given in section 3.

Another special class of groups satisying (1.1) for generic $\mu \in \mathfrak{g}^*$ is that of $\mathbb{A}$-simple metabelian nilpotent Lie groups that we introduce in section 4 (Definition 5). These contain a large number of Carnot groups (see Table 1 below), including the Heisenberg group (of arbitrary dimension) and the jet space $\mathcal{J}^k(\mathbb{R}^n, \mathbb{R}^m)$. The precise result about the symplectic equivalence of

[1]Remark 1 below explains how this statement relates to the reduction of the planar 2-body problem.

the symplectic reduced spaces $M/\!\!/_{\mu}\mathbb{G}$ and $M/\!\!/_{i^*(\mu)}\mathbb{A}$ for $\mathbb{A}$-simple metabelian nilpotent Lie groups is given in Theorem C which is presented and proved in section 4.2.

R/D	4	5	6	7
2	$N_{4,2}$	$N_{5,1}$, $N_{5,2,1}$, $N_{5,2,2}$, $\boxed{N_{5,2,3}}$	$N_{6,1,1}$, $\boxed{N^*_{6,1,2}}$, $N_{6,1,3}$, $N_{6,2,1}$, $\boxed{N^*_{6,2,2}}$, $N_{6,2,5}$, $N_{6,2,5a}$, $N_{6,2,7}$	$2457L$, $2457L1$, $2457M$, $23457A$, $\boxed{2347B^*}$, $\boxed{2347C^*}$, $\boxed{12457H^*}$, $\boxed{12457L^*}$, $\boxed{12457L1^*}$, $12457A$
3		$N_{5,3,2}$	$N_{6,1,4}$, $N_{6,2,3}$, $N_{6,2,4}$, $N_{6,2,6}$, $N_{6,2,8}$, $N_{6,2,9}$, $N_{6,2,9a}$, $N_{6,2,10}$, $N_{6,3,1}$, $N_{6,3,1a}$, $N_{6,3,2}$, $N_{6,3,3}$, $N_{6,3,4}$, $N_{6,3,5}$, $\boxed{N_{6,3,6}}$, $N_{6,4,4a}$	$357A$, $357B$, $247A$, $247B$, $247C$, $\boxed{247D}$, $\boxed{247E}$, $\boxed{247E1}$, $\boxed{247F}$, $\boxed{247F1}$, $\boxed{247G}$, $\boxed{247H}$, $\boxed{247H1}$, $\boxed{247I}$, $\boxed{247J}$, $\boxed{247K}$, $247N$, $247P$, $247P1$, $2457A$, $2457B$, $147D$, $147E$, $147E1$,
4		$N_{5,3,1}$		$37A$, $37B$, $37B1$, $37C$, $37D$, $\boxed{37D1}$, $257B$, $137A$, $137A1$, $137C$
5				$27A$, $27B$
6				17

TABLE 1. All the stratified Lie algebras up to dimension 7 and nilpotent Lie algebras up to dimension 6 were classified in [29]. They are listed in the table, where the first column and the first row indicate the rank and dimension of the group, respectively. Our theory applies to those groups whose Lie algebras are *not* contained within a box; we refer to them as $\mathbb{A}$-**simple** (see Definition 5). Those which are not metabelian have an asterisk in addition to the box.

Remark 1. The assertion that the symplectic reduction by SE(2) is equivalent to the symplectic reduction by $\mathbb{R}^2$ for generic momentum values $\mu \in \mathfrak{se}(2)^*$ seems to be in contradiction with the standard procedure to reduce the planar two-body problem to a 1 degree of freedom Hamiltonian system by first 'reducing out the translations' and afterwards 'reducing out the rotations'. The point is that the second reduction coincides with the 'rotation piece' of SE(2) *only* if the the center of mass is located at the origin of the inertial frame, and this assumption implies that the momentum μ is non-generic (for such μ the coadjoint orbit $\mathcal{O}_\mu = \{\mu\}$ is zero-dimensional and $\mathbb{G}_\mu = \mathrm{SE}(2)$ is 3-dimensional). For other, generic, values of the momentum

μ, the second reduction corresponds to rotations about the *moving* center of mass, and this extra rotational-symmetry does *not* come from the SE(2) action. It instead comes from the invariance of the problem with respect to a larger family of symmetries, corresponding to the action of the Galilean group. In fact, it is the Galilean group symmetry which allows one to guarantee that the common assumption that the center of mass is located at the origin of the inertial frame can be made without loss of generality.

1.2. **Usefulness of the construction.** A clear application of our result is that abelian reduction is easier to implement. For instance, for the n-vortex problem on the plane with total vanishing circulation,[2] the full symplectic reduction by SE(2) can be done by reducing by $\mathbb{R}^2$.

Other applications are the study of sub-Riemannian geodesics on $\mathbb{A}$-simple metabelian nilpotent Lie groups, as was done for Engel-type groups in [16]. In fact, the paper [16], which was coathoured by the first author of this work, was a great motivation for our investigation.

Our work also allows us to understand the structure of the generic coadjoint orbits of $\mathbb{A}$-simple metabelian nilpotent groups as explained in Section 2.5.

1.3. **Relation with Reduction by Stages of Marsden, Misiołek, Perlmutter, Ratiu [31, 30].** Our proof of Theorem A is greatly inspired by the Theory of Reduction by Stages by Marsden et al [31, 30]. Let us give some details. Take $\mu \in \mathfrak{g}^*$ and consider the symplectic reduced space $M/\!\!/_{i^*(\mu)}\mathbb{A}$. Under our assumption that $\mathbb{A}$ is normal, the reduction by stages framework [30] guarantees that, under an appropriate so-called "stages-hypothesis", one may perform a second symplectic reduction of $M/\!\!/_{i^*(\mu)}\mathbb{A}$ by a certain group such that the resulting symplectic quotient is symplectomorphic to the "one-shot" symplectic reduction $M/\!\!/_{\mu}\mathbb{G}$. In particular, if the second step of the stages reduction is trivial, we conclude that $M/\!\!/_{\mu}\mathbb{G}$ and $M/\!\!/_{i^*(\mu)}\mathbb{A}$ are symplectomorphic, which is the case under consideration.

A crucial part of our work was to notice that the hypotheses on μ made in Theorem A imply the validity of the stages-hypothesis, and also that the second symplectic reduction is trivial (see Remark 2 in Section 2.2 for more details). In fact, the proof of Theorem A that we present in Section 2 is an adaptation of the more general proof of "Point reduction by Stages" in Section 5.2 of the book [30]. We have included all details of the proof, with an effort to keep the same notation used in [30], to make our paper self-contained.

Structure of the paper. The proof of Theorem A is given in Section 2 after the necessary preliminaries on symplectic reduction and the notation have been introduced. Moreover, this section also contains a consequence of Theorem A (presented as Proposition 4) which is useful for the treatment of

[2] The condition of vanishing circulation is needed for the momentum map to be equivariant [37].

examples in Section 4. Finally, at the end of Section 2, we discuss the structure of the coadjoint orbits through the points $\mu \in \mathfrak{g}^*$ to which Theorem A applies. Section 3 briefly discusses the case in which $\mathbb{G}$ is a semi-direct product and Theorem B clarifies how Theorem A applies in this setting. This result is not new, since it is a particular case of the more general theory of symplectic reduction by semi-direct products developed in [31], but it is included to illustrate a simple application of Theorem A. In Section 4 we introduce $\mathbb{A}$-simple groups, which comprise a large class of examples for which Theorem A applies for generic $\mu \in \mathfrak{g}^*$. These are nilpotent, metabelian Lie groups satisfying a technical condition specified in Definition 5. The main result of this section is Theorem C. We also show in this section that large classes of positively graded Lie algebras are $\mathbb{A}$-simple, including classical Carnot groups like the Heisenberg group and the jet space $\mathcal{J}^k(\mathbb{R}^n, \mathbb{R}^m)$.

We finally mention that a preliminary version of the results of this paper is part of the Master's thesis of the third author [44], and there is a slight overlap with this work.

2. Equivalent symplectic reductions

This section contains the proof of Theorem A. We begin by recalling some preliminaries on symplectic reduction and introducing the notation in subsection 2.1. We then prove a useful Lemma 1 in subsection 2.2 and present the proof of the theorem in subsection 2.3. We then present a consequence of the theorem in subsection 2.4 that will be useful in our treatment of examples in section 4. Finally, in subsection 2.5 we give a discussion about the structure of the coadjoint orbits of the points $\mu \in \mathfrak{g}^*$ to which Theorem A applies.

2.1. Preliminaries.

Let (M, ω) be a symplectic manifold. We briefly recall the Marsden-Weinstein-Meyer symplectic reduction scheme [32, 34]. We refer the reader to the book [30] for details and proofs.

Let $\mathbb{G}$ be a Lie group and suppose that $\mathbb{G}$ defines free and proper smooth action on M that will be denoted by $(g, m) \mapsto g \cdot m$. We say that the action is ***Hamiltonian*** if the map $M \to M$, $m \mapsto g \cdot m$, is a symplectic diffeomorphism for all $g \in \mathbb{G}$, and the action possesses an equivariant momentum map

$$J_{\mathbb{G}} : M \to \mathfrak{g}^*,$$

where $\mathfrak{g} := \mathrm{lie}(\mathbb{G})$ is the Lie algebra of $\mathbb{G}$. We recall that the momentum map $J_{\mathbb{G}}$ is characterized by the property that, for any $\xi \in \mathfrak{g}$, the *infinitesimal generator vector field* ξ_M on M (defined by $\xi_M(m) := \frac{d}{dt}\big|_{t=0} \exp(\xi t) \cdot m$) is a Hamiltonian vector field, with Hamiltonian function $J_{\mathbb{G}}^{\xi} : M \to \mathbb{R}$ given by

$$J_{\mathbb{G}}^{\xi}(m) := \langle J_{\mathbb{G}}(m), \xi \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the duality pairing of $\mathfrak{g}^*$ and $\mathfrak{g}$. In other words, we have

$$\iota_{\xi_M} \omega = dJ_{\mathbb{G}}^{\xi}, \qquad \text{for all } \xi \in \mathfrak{g}.$$

Equivariance of $J_{\mathbb{G}}$ means that

$$J_{\mathbb{G}}(g\cdot m) = \mathrm{Ad}^*_{g^{-1}}(J_{\mathbb{G}}(m)), \qquad \text{for all } g\in\mathbb{G},\ m\in M,$$

where $\mathrm{Ad}^*_{g^{-1}}:\mathfrak{g}^*\to\mathfrak{g}^*$ is the coadjoint representation defined by the condition

$$\langle \mathrm{Ad}^*_{g^{-1}}(\mu),\xi\rangle = \langle \mu, \mathrm{Ad}_{g^{-1}}(\xi)\rangle, \quad \text{for all } \mu\in\mathfrak{g}^*,\ \xi\in\mathfrak{g}.$$

Under the above conditions, the ***symplectic reduction of*** M ***by the group*** $\mathbb{G}$ ***at*** $\mu\in\mathfrak{g}^*$, denoted $M/\!\!/_\mu\mathbb{G}$, is the symplectic manifold

$$(J_{\mathbb{G}}^{-1}(\mu)/\mathbb{G}_\mu, \omega_\mu).$$

Here $\mathbb{G}_\mu$ is the isotropy subgroup of μ by the coadjoint representation, i.e.,

$$\mathbb{G}_\mu := \{ g\in\mathbb{G} \,:\, \mathrm{Ad}^*_{g^{-1}}(\mu)=\mu\,\},$$

and the symplectic form ω_μ on the quotient space $J_{\mathbb{G}}^{-1}(\mu)/\mathbb{G}_\mu$ is characterized by the condition

$$\pi_\mu^*\omega_\mu = j_\mu^*\omega, \tag{2.1}$$

where $\pi_\mu : J^{-1}(\mu)\to J^{-1}(\mu)/\mathbb{G}_\mu$ is the orbit projection and $j_\mu : J^{-1}(\mu)\hookrightarrow M$ is the inclusion.

Now suppose that $\mathbb{A}\subset\mathbb{G}$ is a closed, abelian, normal subgroup. The Lie algebra $\mathfrak{a}$ of $\mathbb{A}$ is then an abelian ideal of $\mathfrak{g}$. The inclusion $\mathbb{A}\hookrightarrow\mathbb{G}$ induces a Lie algebra inclusion $i:\mathfrak{a}\hookrightarrow\mathfrak{g}$ whose dual,

$$i^*:\mathfrak{g}^*\to\mathfrak{a}^*, \tag{2.2}$$

plays an important role in our work. Note that for $\mu\in\mathfrak{g}^*$ we have $i^*(\mu)=\mu|_{\mathfrak{a}}$.

It is not difficult to show (see e.g. [30], [44]) that the the $\mathbb{G}$-action on M restricts to a free and proper $\mathbb{A}$-action on M, which is again Hamiltonian, and whose equivariant momentum map $J_{\mathbb{A}}:M\to\mathfrak{a}^*$ is given by

$$J_{\mathbb{A}} = i^*\circ J_{\mathbb{G}}. \tag{2.3}$$

In particular, we may consider the symplectic reduction of M by the group $\mathbb{A}$ at $\nu\in\mathfrak{a}^*$, denoted $M/\!\!/_\nu\mathbb{A}$. Given that $\mathbb{A}$ is abelian, the coadjoint representation of $\mathbb{A}$ on $\mathfrak{a}^*$ is trivial and $\mathbb{A}_\nu=\mathbb{A}$ for all $\nu\in\mathfrak{a}^*$. Therefore, $M/\!\!/_\nu\mathbb{A}$ is the symplectic manifold

$$(J_{\mathbb{A}}^{-1}(\nu)/\mathbb{A},\omega_\nu),$$

with ω_ν characterized in analogy with (2.1).

2.2. **A useful lemma.** The goal of this section is to prove Lemma 1 below which will be essential to prove Theorem A in section 2.3 ahead.

Lemma 1. *Suppose that $\mu\in\mathfrak{g}^*$ satisfies that $\mathbb{G}_\mu$ is connected and the dimension condition (1.1). Then,*

(a) $\mathbb{G}_\mu\subset\mathbb{A}$.
(b) *If $\tilde\mu\in\mathfrak{g}^*$ is such that $i^*(\mu)=i^*(\tilde\mu)$, then there exists $a\in\mathbb{A}$ such that $\mu=\mathrm{Ad}^*_{a^{-1}}(\tilde\mu)$.*

The proof of the lemma is presented in subsubsection 2.2.3, after we introduce some auxiliary results.

Remark 2. The conclusions of Lemma 1 allow us to apply the constructions of the Theory of Reduction by Stages [30] as mentioned in the introduction. More precisely, item (b) of Lemma 1 is a stronger version of the so-called "stages hypothesis" and item (a) allows us to conclude that the second reduction is trivial.

2.2.1. $\mathfrak{a}$ *as a maximal isotropic subspace.* Given $\mu \in \mathfrak{g}^*$, recall that the Lie subalgebra $\mathfrak{g}_\mu$ of $\mathbb{G}_\mu$ is given by

$$\mathfrak{g}_\mu = \{W \in \mathfrak{g} : \operatorname{ad}^*_W \mu = 0\}.$$

Denote by $\mathrm{Z}(\mathfrak{g})$ the center of $\mathfrak{g}$ and for further use, notice that

$$\mathrm{Z}(\mathfrak{g}) \subset \mathfrak{g}_\mu, \qquad \text{for all } \mu \in \mathfrak{g}^*. \tag{2.4}$$

For $\mu \in \mathfrak{g}^*$, consider the skew-symetric bilinear form

$$\Omega_\mu : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}, \qquad \Omega_\mu(W_1, W_2) := \langle \mu, [W_1, W_2]\rangle.$$

It is immediate to see that

$$\mathfrak{g}_\mu = \{W \in \mathfrak{g} : \Omega_\mu(W, \tilde{W}) = 0 \text{ for all } \tilde{W} \in \mathfrak{g}\},$$

and hence $\mathfrak{g}_\mu = \ker \Omega_\mu$.

Proposition 2. *If the dimension condition* (1.1) *holds, then* $\mathfrak{a}$ *is a maximal isotropic subspace of* Ω_μ*. As a consequence* $\mathrm{Z}(\mathfrak{g}) \subset \mathfrak{g}_\mu \subset \mathfrak{a}$.

Proof. Given a skew-symmetric bilinear form $B : V \times V \to \mathbb{R}$, maximal isotropic subspaces always exist and have the same dimension. Indeed, the rank of B is always even, and the dimension of maximal isotropic subspaces is half the rank of B plus the dimension of the kernel of B.

Therefore, the dimension of the maximal isotropic subspace of Ω_μ is given by

$$\begin{aligned}\frac{1}{2}\text{rank } \Omega_\mu + \dim\ker\Omega_\mu &= \frac{1}{2}(\dim\mathfrak{g} - \dim\mathfrak{g}_\mu) + \dim\mathfrak{g}_\mu \\ &= \frac{1}{2}(\dim\mathfrak{g} + \dim\mathfrak{g}_\mu).\end{aligned}$$

Now, $\mathfrak{a}$ being an abelian subalgebra implies that $\mathfrak{a}$ is an isotropic subspace. Moreover, the dimension assumption condition (1.1) implies that $\dim\mathfrak{a} = \frac{1}{2}(\dim\mathfrak{g} + \dim\mathfrak{g}_\mu)$ which shows that $\mathfrak{a}$ has the dimension of maximal isotropic subspaces. Finally, we must have $\mathfrak{g}_\mu \subset \mathfrak{a}$ since the kernel of Ω_μ is contained in any maximal isotropic subspace. This together with (2.4) completes the proof. □

Assume that $\mathrm{Z}(\mathfrak{g}) \subset \mathfrak{a}$ (which is true in the case that we care about in virtue of the above proposition). We will consider the quotient space $\mathfrak{a}/\mathrm{Z}(\mathfrak{g})$ and its dual $(\mathfrak{a}/\mathrm{Z}(\mathfrak{g}))^*$ which is especially important for our purposes. In order to realize this space in less abstract terms, we consider the decomposition

$$\mathfrak{g} = \mathfrak{X} \oplus \mathfrak{a}, \tag{2.5}$$

where $\mathfrak{X}$ is an arbitrary direct complement of $\mathfrak{a}$ in $\mathfrak{g}$. This induces a dual decomposition in terms of the annihilator spaces,[3]

$$\mathfrak{g}^* = \mathfrak{a}^\circ \oplus \mathfrak{X}^\circ, \tag{2.6}$$

and we may identify

$$\mathfrak{X}^* \cong \mathfrak{a}^\circ, \qquad \mathfrak{a}^* \cong \mathfrak{X}^\circ. \tag{2.7}$$

With these identifications we think of $\mathfrak{a}^* = \mathfrak{X}^\circ$ as a subspace of $\mathfrak{g}^*$ and there is a natural linear isomorphism

$$\mathrm{p} : (\mathrm{Z}(\mathfrak{g}))^\circ \cap \mathfrak{a}^* \to (\mathfrak{a}/\,\mathrm{Z}(\mathfrak{g}))^*.$$

If $\nu \in (\mathrm{Z}(\mathfrak{g}))^\circ \cap \mathfrak{a}^*$, then $\mathrm{p}(\nu) \in (\mathfrak{a}/\,\mathrm{Z}(\mathfrak{g}))^*$ is defined by

$$\langle \mathrm{p}(\nu), \overline{Y} \rangle_{\mathfrak{a}/\,\mathrm{Z}(\mathfrak{g})} := \langle \nu, Y \rangle_{\mathfrak{a}},$$

where $\overline{Y} \in \mathfrak{a}/\,\mathrm{Z}(\mathfrak{g})$ denotes the image of $Y \in \mathfrak{a}$ under the quotient projection $\mathfrak{a} \to \mathfrak{a}/\,\mathrm{Z}(\mathfrak{g})$.

2.2.2. *Definition and properties of* T_μ. We now define an auxiliary map depending on $\mu \in \mathfrak{g}^*$ that we denote by T_μ.

As in section 2.1, we continue to denote by $i : \mathfrak{a} \hookrightarrow \mathfrak{g}$ the inclusion and by $i^* : \mathfrak{g}^* \to \mathfrak{a}^*$ the dual projection. In terms of the decompositions, identifications, and notation introduced above, we define the linear map

$$\mathrm{T}_\mu : \mathfrak{X} \to (\mathfrak{a}/\,\mathrm{Z}(\mathfrak{g}))^*, \qquad \mathrm{T}_\mu(X) := \mathrm{p}(i^*(\mathrm{ad}^*_{-X}(\mu))). \tag{2.8}$$

Proposition 3. *For any $\mu \in \mathfrak{g}^*$, the linear map $\mathrm{T}_\mu : \mathfrak{X} \to (\mathfrak{a}/\,\mathrm{Z}(\mathfrak{g}))^*$ is well-defined and its adjoint map, $\mathrm{T}^*_\mu : (\mathfrak{a}/\,\mathrm{Z}(\mathfrak{g})) \to \mathfrak{X}^*$, is given by*

$$\mathrm{T}^*_\mu(\overline{Y}) = \mathrm{ad}^*_{i(Y)}(\mu).$$

Furthermore, if $\mu \in \mathfrak{g}^*$ is such that the dimension condition (1.1) holds, then T_μ is injective and T^*_μ is surjective.

Proof. By definition of i^*, it is clear that $i^*(\mathrm{ad}^*_{-X}(\mu)) \in \mathfrak{a}^*$ for any $X \in \mathfrak{X}$. Moreover, if $Z \in \mathrm{Z}(\mathfrak{g})$ is arbitrary then

$$\begin{aligned}\langle i^*(\mathrm{ad}^*_{-X}(\mu)), Z \rangle_{\mathfrak{a}} &= \langle \mathrm{ad}^*_{-X}(\mu), i(Z) \rangle_{\mathfrak{g}} \\ &= \langle \mu, \mathrm{ad}_{-X}(Z) \rangle_{\mathfrak{g}} = 0,\end{aligned}$$

where we have used $\mathrm{ad}_{-X}(Z) = 0$ since $Z \in \mathrm{Z}(\mathfrak{g})$. Therefore, the covector $i^*(\mathrm{ad}^*_{-X}(\mu))$ belongs to $(\mathrm{Z}(\mathfrak{g}))^\circ \cap \mathfrak{a}^*$ and we may apply p to it. In view of the definition of p, we conclude that T_μ is well-defined as a mapping from $\mathfrak{X}$ into $(\mathfrak{a}/\,\mathrm{Z}(\mathfrak{g}))^*$.

[3] If U is a subspace of a vector space V, its annihilator, U°, is the subspace of V^* formed by the elements $\alpha \in V^*$ such that $\langle \alpha, u \rangle = 0$ for all $u \in U$.

Next, if $X \in \mathfrak{X}$, and $\overline{Y} \in \mathfrak{a}/\operatorname{Z}(\mathfrak{g})$, then, by definition of T_μ,

$$\begin{aligned}\langle \operatorname{T}_\mu(X), \overline{Y}\rangle_{\mathfrak{a}/\operatorname{Z}(\mathfrak{g})} &= \langle i^*(\operatorname{ad}^*_{-X}(\mu)), Y\rangle_{\mathfrak{a}} = \langle \operatorname{ad}^*_{-X}(\mu), i(Y)\rangle_{\mathfrak{g}} \\ &= \langle \mu, \operatorname{ad}_{-X}(i(Y))\rangle_{\mathfrak{g}} = \langle \mu, \operatorname{ad}_{i(Y)}(X)\rangle_{\mathfrak{g}} \\ &= \langle \operatorname{ad}^*_{i(Y)}(\mu), X\rangle_{\mathfrak{g}}.\end{aligned}$$

To complete the proof that $\operatorname{T}^*_\mu(\overline{Y}) = \operatorname{ad}^*_{i(Y)}(\mu)$, we must verify that $\operatorname{ad}^*_{i(Y)}(\mu) \in \mathfrak{a}^\circ = \mathfrak{X}^*$. Indeed, let $Y_1 \in \mathfrak{a}$ be arbitrary, then

$$\langle \operatorname{ad}^*_{i(Y)}(\mu), Y_1\rangle_{\mathfrak{g}} = \langle \mu, [Y, Y_1]\rangle_{\mathfrak{g}} = 0,$$

where we used that $\mathfrak{a}$ is abelian.

Now suppose that $\mu \in \mathfrak{g}^*$ is such that the dimension condition (1.1) holds. Then, by Proposition 2, we know that $\mathfrak{a}$ is a maximal isotropic subspace with respect to Ω_μ. Suppose that $X \in \ker T_\mu$ is nontrivial, then

$$\begin{aligned}0 &= \langle \operatorname{T}_\mu(X), \overline{Y}\rangle_{\mathfrak{a}/\operatorname{Z}(\mathfrak{g})} = \langle i^*(\operatorname{ad}^*_X(\mu)), Y\rangle_{\mathfrak{a}} \\ &= \langle \operatorname{ad}^*_X(\mu), Y\rangle_{\mathfrak{g}} = \Omega_\mu(X, Y),\end{aligned}$$

for all $Y \in \mathfrak{a}$. So $\langle X\rangle \oplus \mathfrak{a}$ is an isotropic subspace of Ω_μ, contradicting that $\mathfrak{a}$ is maximal. Hence $\ker \operatorname{T}_\mu = \{0\}$ implying that T_μ is injective and T^*_μ is surjective. □

2.2.3. *Proof of Lemma 1.* The conclusion that $\mathbb{G}_\mu \subset \mathbb{A}$ follows at once from $\mathfrak{g}_\mu \subset \mathfrak{a}$ (Proposition 2) and the assumption that $\mathbb{G}_\mu$ is connected.

In order to show item (b), first note that for $Y \in \mathfrak{a}$, putting $a = \exp(i(Y)) \in \mathbb{A} \subset \mathbb{G}$ we have

$$\begin{aligned}\operatorname{Ad}^*_{a^{-1}}(\mu) &= e^{\operatorname{ad}^*_{-i(Y)}}(\mu) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}(\operatorname{ad}^*_{i(Y)})^n(\mu) \\ &= \mu - \operatorname{T}^*_\mu(\overline{Y}),\end{aligned} \tag{2.9}$$

where we have used that $(\operatorname{ad}^*_{i(Y)})^n(\mu) = 0$ if $n \geq 2$. Indeed, if $X \in \mathfrak{X}$ is an arbitrary element, we have $[Y, [Y, X]] = 0$ since $\mathfrak{a}$ is an abelian ideal. So

$$\langle (\operatorname{ad}^*_Y)^2(\mu), X\rangle = \langle \mu, [Y, [Y, X]]\rangle = 0.$$

Now, the condition that $i^*(\mu) = i^*(\tilde{\mu})$, implies $\mu - \tilde{\mu} \in \ker i^* = \mathfrak{a}^\circ = \mathfrak{X}^*$. Hence, by surjectivity of T^*_μ (Proposition 3), there exists $Y \in \mathfrak{a}$ such that $\operatorname{T}^*_\mu(\overline{Y}) = \mu - \tilde{\mu}$. Putting $a = \exp(i(Y))$ and using eq. (2.9), we conclude that

$$\operatorname{Ad}^*_{a^{-1}}(\mu) = \mu - \operatorname{T}^*_\mu(\overline{Y}) = \tilde{\mu},$$

as required.

□

Remark 3. The proof of item (b) of Lemma 1 only uses that the hypotheses made on μ imply that T^*_μ is surjective. Hence, item (b) of the Lemma remains valid under the condition that T^*_μ is surjective, or equivalently, that T_μ is injective. (Obviously, we also need $\operatorname{Z}(\mathfrak{g}) \subset \mathfrak{a}$ in order to define T_μ.)

2.3. **Proof of Theorem A.** The proof that we present is an adaptation of the more general Reduction by Stages Theory of Marsden et al [30]. We follow closely the presentation of "Point reduction by stages" in section 5.2 of the book [30] attempting to use the same notation. Many parts of the proof simplify in our setting since we work with less generality.

Let $\mu \in \mathfrak{g}^*$ satisfying the hypothesis of the theorem. The proof that we present only exploits that for such μ items (a) and (b) of Lemma 1 hold (especially item (b) will be used at several steps of the proof). This observation will be exploited in Section 2.4 below to obtain a useful reformulation of the theorem.

Write $\nu := i^*(\mu) \in \mathfrak{a}^*$. In view of (2.3), we have $J_{\mathbb{G}}^{-1}(\mu) \subset J_{\mathbb{A}}^{-1}(\nu)$. Denote by $s_\mu : J_{\mathbb{G}}^{-1}(\mu) \hookrightarrow J_{\mathbb{A}}^{-1}(\nu)$ the inclusion map. Then s_μ is a smooth embedding. Indeed, $J_{\mathbb{G}}^{-1}(\mu)$ is the zero level set of the smooth map

$$C_\mu : J_{\mathbb{A}}^{-1}(\nu) \to \mathfrak{a}^\circ, \qquad C_\mu(m) = J_{\mathbb{G}}(m) - \mu,$$

which can be checked to be submersive (see [44]).

Denote by

$$P_\mu := J_{\mathbb{G}}^{-1}(\mu)/\mathbb{G}_\mu \quad \text{and} \quad P_\nu := J_{\mathbb{A}}^{-1}(\nu)/\mathbb{A},$$

the symplectic reduced manifolds, and by $\pi_\mu : J_{\mathbb{G}}^{-1}(\mu) \to P_\mu$ and $\pi_\nu : J_{\mathbb{A}}^{-1}(\nu) \to P_\nu$ the orbit projection maps (which, under our assumptions are surjective submersions).

Let $F : P_\mu \to P_\nu$ be given by

$$F(\pi_\mu(m)) := \pi_\nu \circ s_\mu(m), \qquad \text{where } m \in J_{\mathbb{G}}^{-1}(\mu). \tag{2.10}$$

We prove below that F is (i) well-defined, (ii) bijective and (iii) smooth.

(i) Suppose that $\pi_\mu(m) = \pi_\mu(\tilde{m})$ for $m, \tilde{m} \in J_{\mathbb{G}}^{-1}(\mu)$. Given that $\mathbb{G}_\mu \subset \mathbb{A}$ (item (a) of Lemma 1), there exists $a \in \mathbb{G}_\mu \subset \mathbb{A}$ such that $\tilde{m} = a \cdot m$, which implies

$$\pi_\nu \circ s_\mu(m) = \pi_\nu \circ s_\mu(\tilde{m}).$$

In other words, the value of F does not depend on the orbit representative of $\pi_\mu(m)$ and is well-defined.

(ii) *F is injective.* Suppose that $F(\pi_\mu(m)) = F(\pi_\mu(\tilde{m}))$ for $m, \tilde{m} \in J_{\mathbb{G}}^{-1}(\mu)$. By definition of F this implies the existence of $a \in \mathbb{A}$ such that $\tilde{m} = a \cdot m$. In particular, we conclude that[4]

$$\tilde{m} \in J_{\mathbb{G}}^{-1}(\mu) \cap (\mathbb{G} \cdot m).$$

Now recall the well-known identity (see e.g. item (ii) of the "Reduction Lemma" in [30]):

$$J_{\mathbb{G}}^{-1}(\mu) \cap (\mathbb{G} \cdot m) = \mathbb{G}_\mu \cdot m, \tag{2.11}$$

[4] $\mathbb{G} \cdot m$ denotes the $\mathbb{G}$-orbit through m. Similar obvious interpretations apply to $\mathbb{G}_\mu \cdot m$ and $\mathbb{A} \cdot m$ below.

to conclude that $\tilde{m} = g \cdot m$ for some $g \in \mathbb{G}_\mu$. Hence $\pi_\mu(m) = \pi_\mu(\tilde{m})$ showing that F is injective.

F is surjective. Let $m \in J_{\mathbb{A}}^{-1}(\nu)$ and accept for the moment that

$$(\mathbb{A} \cdot m) \cap J_{\mathbb{G}}^{-1}(\mu) \neq \emptyset. \tag{2.12}$$

Then, there exists a representative $\tilde{m} \in J_{\mathbb{G}}^{-1}(\mu)$ of $\pi_\nu(m) \in P_\nu$, and we have,

$$F(\pi_\mu(\tilde{m})) = \pi_\nu(\tilde{m}) = \pi_\nu(m).$$

Therefore, the surjectivity of F may be established by showing that (2.12) holds for all $m \in J_{\mathbb{A}}^{-1}(\nu)$. To prove this, let $\tilde{\mu} := J_{\mathbb{G}}(m) \in \mathfrak{g}^*$. We have,

$$i^*(\tilde{\mu}) = i^* \circ J_{\mathbb{G}}(\tilde{\mu}) = J_{\mathbb{A}}(\tilde{\mu}) = \nu = i^*(\mu), \tag{2.13}$$

where we have used $J_{\mathbb{A}} = i^* \circ J_{\mathbb{G}}$ (eq. (2.3)). Therefore, by item (b) of Lemma 1, there exists $a \in \mathbb{A}$ such that $\mu = \mathrm{Ad}^*_{a^{-1}} \tilde{\mu}$. Let $\hat{m} := a \cdot m$. Then obviously, $\hat{m} \in \mathbb{A} \cdot m$. On the other hand, by the equivariance of $J_{\mathbb{G}}$,

$$J_{\mathbb{G}}(\hat{m}) = J_{\mathbb{G}}(a \cdot m) = \mathrm{Ad}^*_{a^{-1}}(J_{\mathbb{G}}(m)) = \mathrm{Ad}^*_{a^{-1}}(\tilde{\mu}) = \mu,$$

proving that $\hat{m} \in (\mathbb{A} \cdot m) \cap J_{\mathbb{G}}^{-1}(\mu)$ and hence the intersection (2.12) is non-empty as claimed.

(iii) The smoothness of F follows from the smoothness of $\pi_\nu \circ s_\mu$ and the fact that π_μ is a surjective submersion.

We now show that $F : P_\mu \to P_\nu$ is a symplectic map, i.e.,

$$F^* \omega_\nu = \omega_\mu,$$

where we recall that the symplectic reduced forms $\omega_\mu \in \Lambda^2(P_\mu)$ and $\omega_\nu \in \Lambda^2(P_\nu)$ are characterized by the conditions

$$\pi_\mu^* \omega_\mu = j_\mu^* \omega, \qquad \pi_\nu^* \omega_\nu = j_\nu^* \omega,$$

where $j_\mu : J_{\mathbb{G}}^{-1}(\mu) \hookrightarrow M$ and $j_\nu : J_{\mathbb{A}}^{-1}(\nu) \hookrightarrow M$ are the inclusion maps. For this, it is sufficient to show that

$$\pi_\mu^* F^* \omega_\nu = j_\mu^* \omega.$$

Using that $F \circ \pi_\mu = \pi_\nu \circ s_\mu$ by definition of F, we have,

$$\begin{aligned} \pi_\mu^* F^* \omega_\nu &= (F \circ \pi_\mu)^* \omega_\nu = (\pi_\nu \circ s_\mu)^* \omega_\nu = s_\mu^* \pi_\nu^* \omega_\nu = s_\mu^* j_\nu^* \omega \\ &= (j_\nu \circ s_\mu)^* \omega = j_\mu^* \omega, \end{aligned}$$

where, in the last identity, we have used $j_\mu = j_\nu \circ s_\mu$ which follows from the definition of these inclusion maps.

To finalize the proof of the theorem, it remains only to show that F has a smooth inverse, which requires substantial work. We first propose what such an inverse, that we denote as $\phi : P_\nu \to P_\mu$, should be. Let $\pi_\nu(m) \in P_\nu$ for some $m \in J_{\mathbb{A}}^{-1}(\nu)$ and let $\tilde{\mu} := J_{\mathbb{G}}(m) \in \mathfrak{g}^*$. Repeating the calculation

(2.13), shows that $i^*(\tilde\mu) = \nu = i^*(\mu)$ and therefore, by item (b) of Lemma 1, there exists $a \in \mathbb{A}$ such that $\mu = \mathrm{Ad}^*_{a^{-1}} \tilde\mu$. We define,

$$\phi(\pi_\nu(m)) = \pi_\mu(a \cdot m). \tag{2.14}$$

We first observe that the right hand side of this formula makes sense, since, by equivariance of $J_{\mathbb{G}}$, we have $J_{\mathbb{G}}(a\cdot m) = \mathrm{Ad}^*_{a^{-1}} J_{\mathbb{G}}(m) = \mathrm{Ad}^*_{a^{-1}} \tilde\mu = \mu$. Hence, the point $a \cdot m$ belongs to $J_{\mathbb{G}}^{-1}(\mu)$ and we may apply π_μ to it.

Similarly to what we did with F, below we show that (i) ϕ is well-defined, (ii) ϕ is the inverse of F, and (iii) ϕ is smooth.

(i) Suppose that $\pi_\nu(\hat m) = \pi_\nu(m)$ for $\hat m, m \in J_{\mathbb{A}}^{-1}(\nu)$ and let $a_0 \in \mathbb{A}$ be such that $\hat m = a_0 \cdot m$. We repeat the construction for $\hat m$ that we did for m in the definition of ϕ. Let $\hat\mu := J_{\mathbb{G}}(\hat m) \in \mathfrak{g}^*$. Again, since $i^*(\hat\mu) = i^*(\mu)$, by item (b) of Lemma 1, there exists $\hat a \in \mathbb{A}$ such that $\hat\mu = \mathrm{Ad}^*_{\hat a^{-1}} \mu$. As a consequence, we may write,

$$\phi(\pi_\nu(\hat m)) = \pi_\mu(\hat a \cdot \hat m) = \pi_\mu(\hat a a_0 \cdot m). \tag{2.15}$$

Proving that ϕ is well-defined amounts to checking that $\phi(\pi_\nu(m)) = \phi(\pi_\nu(\hat m))$, which in view of (2.14) and (2.15), is equivalent to:

$$\pi_\mu(a \cdot m) = \pi_\mu(\hat a a_0 \cdot m).$$

In other words, we must show the existence of $g \in \mathbb{G}_\mu$ such that $a\cdot m = (g\hat a a_0)\cdot m$. By freeness of the $\mathbb{G}$-action on M, this is equivalent to showing that $a^{-1} g \hat a a_0$ is the identity element of $\mathbb{G}$. Therefore, to complete the proof, we only need to check that $g := a a_0^{-1} \hat a^{-1}$ belongs to $\mathbb{G}_\mu$. But this is true since both points $a\cdot m$ and $\hat a a_0 \cdot m$ belong to $J_{\mathbb{G}}^{-1}(\mu)$ and also to the orbit $\mathbb{G}\cdot m$. As a consequence, they belong to the intersection $J_{\mathbb{G}}^{-1}(\mu) \cap \mathbb{G}\cdot m$ which by (2.11) equals $\mathbb{G}_\mu \cdot m$. Therefore, we conclude that a and $\hat a a_0$ belong to $\mathbb{G}_\mu$, and therefore also $g = a(\hat a a_0)^{-1} \in \mathbb{G}_\mu$.

(ii) Let $m \in J_{\mathbb{A}}^{-1}(\nu)$ and let $a \in \mathbb{A}$ be the group element which defines $\phi(\pi_\nu(m))$ according to (2.14). In view of the definition (2.10) of F, we have,

$$F\circ\phi(\pi_\nu(m)) = F(\pi_\mu(a\cdot m)) = \pi_\nu(s_\mu(a\cdot m)) = \pi_\nu(a\cdot m) = \pi_\nu(m).$$

This shows that $F = \phi^{-1}$ since we know that F is bijective.

(iii) Let $m_0 \in J_{\mathbb{A}}^{-1}(\nu)$ and consider a local section $\Gamma : U \to \pi_\nu^{-1}(U) \subset J_{\mathbb{A}}^{-1}(\nu)$ where $U \subset P_\nu$ is a neighbourhood of $\pi_\nu(m_0)$. Consider the smooth map

$$J_{\mathbb{G}} \circ \Gamma : U \to \mathfrak{g}^*.$$

We claim that it takes values on $\mathbb{A}\cdot\mu$, which denotes the $\mathbb{A}$-orbit through μ of the restriction of the coadjoint action of $\mathbb{G}$ on $\mathfrak{g}^*$ to $\mathbb{A}\subset\mathbb{G}$. Actually, we will show that $J_{\mathbb{G}}|_{J_{\mathbb{A}}^{-1}(\nu)}$ does, which is enough. Indeed, let $m \in J_{\mathbb{A}}^{-1}(\nu)$, then $i^*(J_{\mathbb{G}}(m)) = \nu$ by (2.3) so by item (b) of Lemma 1 there exists $a \in \mathbb{A}$ such that $\mathrm{Ad}^*_{a^{-1}}(J_{\mathbb{G}}(m)) = \mu$ and our claim holds.

Now, item (a) of Lemma 1 states that $\mathbb{G}_\mu \subset \mathbb{A}$ so there is a natural diffeomorphism between the orbit $\mathbb{A}\cdot\mu$ and the quotient $\mathbb{A}/\mathbb{G}_\mu$. We denote such diffeomorphism by $\mathcal{C}_\mu : \mathbb{A}\cdot\mu \to \mathbb{A}/\mathbb{G}_\mu$, which is characterised by the condition,

$$\mathcal{C}_\mu(\mathrm{Ad}^*_{a^{-1}}(\mu)) = \tau(a), \qquad a \in \mathbb{A}, \tag{2.16}$$

where $\tau : \mathbb{A} \to \mathbb{A}/\mathbb{G}_\mu$ is the principal bundle projection.

Consider the (smooth) composition

$$\mathcal{C}_\mu \circ J_{\mathbb{G}} \circ \Gamma : U \to \mathbb{A}/\mathbb{G}_\mu.$$

Restricting the neighbourhood U if necessary, we may consider a smooth lift

$$(\mathcal{C}_\mu \circ J_{\mathbb{G}} \circ \Gamma)^\ell : U \to \mathbb{A},$$

which we use to construct the smooth map

$$\alpha : U \to \mathbb{A}, \qquad \alpha(\pi_\nu(m)) = \Big((\mathcal{C} \circ J_{\mathbb{G}} \circ \Gamma)^\ell(\pi_\nu(m))\Big)^{-1},$$

where $^{-1}$ denotes the group inverse. Finally, we consider the smooth map

$$\sigma : U \to P_\mu, \qquad \sigma(\pi_\nu(m)) = \pi_\mu\left(\alpha(\pi_\nu(m)) \cdot \Gamma(\pi_\nu(m))\right). \tag{2.17}$$

Our proof that ϕ is smooth consists in showing that ϕ coincides with σ on U, namely

$$\sigma(\pi_\nu(m)) = \phi(\pi_\nu(m)), \qquad \text{for all } \pi_\nu(m) \in U. \tag{2.18}$$

Fix $\pi_\nu(m) \in U$. Given that $J_{\mathbb{G}} \circ \Gamma(\pi_\nu(m)) \in \mathbb{A}\cdot\mu$, there exists $a_0 \in \mathbb{A}$ such that

$$\mu = \mathrm{Ad}^*_{a_0^{-1}}\left(J_{\mathbb{G}} \circ \Gamma(\pi_\nu(m))\right).$$

On the one hand, in view of the definition (2.14) of ϕ, the above identity implies

$$\phi(\pi_\nu(m)) = \pi_\mu(a_0 \cdot \Gamma(\pi_\nu(m))). \tag{2.19}$$

On the other hand, we can write $\mathrm{Ad}^*_{a_0}\mu = J_{\mathbb{G}} \circ \Gamma(\pi_\nu(m))$, and using the definition of the lift of a map, we get

$$\begin{aligned}\mathcal{C}_\mu\left(\mathrm{Ad}^*_{a_0}\mu\right) &= \mathcal{C}_\mu \circ J_{\mathbb{G}} \circ \Gamma\left(\pi_\nu(m)\right)) \\ &= \tau\Big((\mathcal{C}_\mu \circ J_{\mathbb{G}} \circ \Gamma)^\ell(\pi_\nu(m))\Big).\end{aligned}$$

By (2.16), the above equality implies the existence of $g \in \mathbb{G}_\mu$ such that

$$a_0^{-1}g^{-1} = (\mathcal{C}_\mu \circ J_{\mathbb{G}} \circ \Gamma)^\ell(\pi_\nu(m)).$$

Therefore, $\alpha(\pi_\nu(m)) = ga_0$. Following the definition (2.17) of σ, and using that $g \in \mathbb{G}_\mu$, we have

$$\sigma(\pi_\nu(m)) = \pi_\mu\left(ga_0 \cdot \Gamma(\pi_\nu(m))\right) = \pi_\mu\left(a_0 \cdot \Gamma(\pi_\nu(m))\right).$$

Comparing this with (2.19) proves (2.18) as required.

2.4. **Alternative formulation of the Theorem.** As mentioned before, the proof of Theorem A given in the previous section only uses the hypotheses on μ through the properties (a) and (b) of Lemma 1, so the proof remains valid under the assumption that these conditions hold. Moreover, in view of Remark 3, we conclude that the proof goes through under the following alternative hypotheses on μ:

(a) $\mathbb{G}_\mu \subset \mathbb{A}$,
(b) The mapping T_μ defined by (2.8) is injective.

These observations lead to the following proposition that will be used in section 4.

Proposition 4. *Let $\mathbb{G}$ be a Lie group with an abelian, normal and regular subgroup $\mathbb{A}$. Suppose $\mathbb{G}$ defines a free and proper Hamiltonian action on the symplectic manifold (M, ω) with equivariant momentum map $J_\mathbb{G} : M \to \mathfrak{g}^*$. Let $\mu \in J_\mathbb{G}(M) \subset \mathfrak{g}^*$ and suppose that $\mathbb{G}_\mu \subset \mathbb{A}$. Choose any complement $\mathfrak{X}$ of $\mathfrak{a}$ in $\mathfrak{g}$ and consider the map*[5] *$\mathrm{T}_\mu : \mathfrak{X} \to (\mathfrak{a}/\,\mathrm{Z}(\mathfrak{g}))^*$ defined by (2.8). If T_μ is injective, then there is a symplectic diffeomorphism between the symplectic reduced spaces $M/\!\!/_\mu\mathbb{G}$ and $M/\!\!/_{i^*(\mu)}\mathbb{A}$.*

2.5. **Calculation of coadjoint orbits.** Take $M = T^*\mathbb{G}$ in Theorem A and consider the action of $\mathbb{G}$ on $T^*\mathbb{G}$ by the cotangent lift of left multiplication. This action is free, proper and Hamiltonian, and has surjective momentum map $J_\mathbb{G} : T^*\mathbb{G} \to \mathfrak{g}^*$ given by the right trivialisation. As is well-known, for $\mu \in \mathfrak{g}^*$, the symplectic reduced space $M/\!\!/_\mu\mathbb{G}$ is symplectically diffeomorphic to the coadjoint orbit $(\mathcal{O}_\mu, \omega_\mu)$ where ω_μ denotes the (minus) Kostant-Kirillov-Soriau symplectic form [32]. Therefore, if $\mathbb{G}_\mu$ is connected, and the regular, abelian, normal subgroup $\mathbb{A} \subset \mathbb{G}$ satisfies (4.10), then Theorem A implies that $T^*\mathbb{G}/\!\!/_\mu\mathbb{A}$ is symplectically diffeomorphic to $(\mathcal{O}_\mu, \omega_\mu)$. On the other hand, by the theory of cotangent bundle reduction (see e.g. [1, page 300] or [42, Theorem 6.6.3]), we know that $T^*\mathbb{G}/\!\!/_\mu\mathbb{A}$ is symplectically diffeomorphic to the cotangent bundle $T^*(\mathbb{G}/\mathbb{A})$ equipped with the symplectic form $\omega_{\mathbb{G}/\mathbb{A}} - B_\mu$, where $\omega_{\mathbb{G}/\mathbb{A}}$ is the canonical symplectic form and B_μ is a *magnetic term*. Therefore, we conclude that the coadjoint orbits of those $\mu \in \mathfrak{g}^*$ for which Theorem A applies are symplectically diffeomorphic to $T^*(\mathbb{G}/\mathbb{A})$ (equipped with the symplectic form $\omega_{\mathbb{G}/\mathbb{A}} - B_\mu$).

We mention that conclusions of the above type are well-known for the coadjoint orbits of semidirect product groups, without requiring that the condition (1.1) holds. For instance, see the "Semi-direct product reduction theorem" in [31]. This type of correspondence is due to the relationship of our work with the Theory of Reduction by Stages described in Section 1.3 in the introduction.

In collaboration with Le Donne and Paddeu, the first author established a related result in the setting of metabelian nilpotent groups, see [16, Theorem

[5] note that the quotient space $\mathfrak{a}/\,\mathrm{Z}(\mathfrak{g})$ is well-defined since $\mathrm{Z}(\mathfrak{g}) \subset \mathfrak{a}$. This is true because $\mathrm{Z}(\mathfrak{g}) \subset \mathfrak{g}_\mu$ always holds and $\mathfrak{g}_\mu \subset \mathfrak{a}$ by the assumption that $\mathbb{G}_\mu \subset \mathbb{A}$.

1.2]. In this case, the quotient $\mathbb{G}/\mathbb{A}$ has a vector space structure, which, using an additional "shift map", allowed the authors to demonstrate that $T^*\mathbb{G}/\!\!/_\mu\mathbb{A}$ is symplectically diffeomorphic to the cotangent bundle $T^*(\mathbb{G}/\mathbb{A})$ endowed with its canonical symplectic form.

3. Examples I: Semidirect products

As a first class of examples, suppose that $\mathbb{A}$ is a real vector space and $\mathbb{G}$ is the semi-direct product group $\mathbb{G} = \mathbb{H} \ltimes \mathbb{A}$ for a certain connected Lie group $\mathbb{H}$ that acts by linear maps on $\mathbb{A}$. As a manifold, $\mathbb{G} = \mathbb{H} \times \mathbb{A}$ and the group operation on $\mathbb{G}$ is

$$(h_1, a_1)(h_2, a_2) = (h_1h_2, a_1 + h_1a_2).$$

Then $\mathbb{A}$ is a regular, normal, abelian subgroup of $\mathbb{G}$.

Symplectic reduction by this class of groups has been extensively studied (see e.g. [30, section 4.2] for a comprehensive list of references). The result that we present below as Theorem B is a particular instance of the "Reduction by Stages for Semidirect Product Actions" in [31, Theorem 3.2]. In this sense, our discussion here does not give any new results but attempts only to illustrate how Theorem A applies in the simple context of semidirect products.

The Lie algebra $\mathfrak{g}$ is the semidirect product Lie algebra $\mathfrak{h} \ltimes \mathbb{A}$, where $\mathfrak{h}$ is the Lie algebra of $\mathbb{H}$ and we have identified $\mathfrak{a} = \mathbb{A}$. The bracket is given by

$$(\xi_1, a_1)(\xi_2, a_2) = ([\xi_1, \xi_2], \xi_1a_2 - \xi_2a_1),$$

where ξa denotes the induced action of $\xi \in \mathfrak{h}$ on $a \in \mathbb{A}$.

For the formulation of Theorem B note that, as a vector space, the dual Lie algebra $\mathfrak{g}^* = \mathfrak{h}^* \times \mathbb{A}^*$. Moreover, the mapping $i^* : \mathfrak{g}^* = \mathfrak{h}^* \times \mathbb{A}^* \to \mathbb{A}^*$ is given by $i^*(\gamma, \nu) = \nu$. Also note that there is an induced action of $\mathbb{H}$ on $\mathbb{A}^*$, by the action of the transpose of h^{-1} on $\mathbb{A}$. The corresponding isotropy group of $\nu \in \mathbb{A}^*$ will be denoted by $\mathbb{H}_\nu$.

Theorem B. *As above, let $\mathbb{G}$ be the semidirect product $\mathbb{H} \ltimes \mathbb{A}$, with $\mathbb{H}$ connected and $\mathbb{A}$ a vector space. Suppose that $\mathbb{G}$ defines a free and proper Hamiltonian action on the symplectic manifold (M, ω) with equivariant momentum map $J_\mathbb{G} : M \to \mathfrak{g}^*$. Let $\mu = (\gamma, \nu) \in \mathfrak{h}^* \times \mathbb{A}^*$. There exists a symplectic diffeomorphism between the symplectic reduced spaces $M/\!\!/_\mu\mathbb{G}$ and $M/\!\!/_\nu\mathbb{A}$ if and only if $\mathbb{H}_\nu = \{e_\mathbb{H}\}$.*

The proof goes by showing that the condition that $\mathbb{H}_\nu = \{e_\mathbb{H}\}$ is equivalent to the simultaneous validity of the following two conditions: 1. the dimension of the coadjoint orbit $\mathcal{O}_\mu$ satisfies (1.2), 2. the subgroup $\mathbb{G}_\mu$ is connected. After establishing this equivalence, the proof follows from Theorem A. We omit the details.[6]

[6] In fact, Theorem 3.1 in [31] shows that if $\mathbb{H}_\nu = \{e_\mathbb{H}\}$ then $\mathcal{O}_\mu$ is symplectically diffeomorphic to $T^*\mathbb{H}$ and in particular its dimension satisfies (1.2).

As indicated in the introduction, Theorem A applies to the Euclidean group $\mathbb{G} = \mathrm{SE}(2)$ (for generic elements $\mu \in \mathfrak{g}^*$). Let us see why this is true in the light of Theorem B. We have $\mathbb{G} = \mathrm{SE}(2) = \mathrm{SO}(2) \ltimes \mathbb{R}^2$ where $\mathbb{H} = \mathrm{SO}(2)$ acts on $\mathbb{A} = \mathbb{R}^2$ by rotations. Upon identification of $\mathbb{A}^*$ with $\mathbb{R}^2$ via the dot product, the resulting action of $\mathrm{SO}(2)$ on $\mathbb{A}^*$ is again by rotations. A nonzero $\nu \in \mathbb{R}^2 = \mathbb{A}^*$ is not fixed by any nontrivial rotation so $\mathrm{SO}(2)_\nu = \{e_{\mathrm{SO}(2)}\}$ and Theorem B can be used to conclude the equivalence between the abelian and nonabelian symplectic reductions.

4. Examples II: A class of metabelian nilpotent groups

In this section, we present a class of nilpotent groups for which Theorem A holds for generic $\mu \in \mathfrak{g}^*$. We begin by recalling the necessary preliminaries, including the definitions of nilpotent, Carnot, and metabelian groups in subsection 4.1. Then, in subsection 4.2 we give the definition of $\mathbb{A}$-simple group (Definition 5) and we state and prove Theorem C, which is the main result of this section. The theorem establishes that, for an $\mathbb{A}$-simple group $\mathbb{G}$, there exists an open and dense subset $\mathfrak{g}^*_{reg} \subset \mathfrak{g}$ with the property that the reduced spaces $M/\!\!/_\mu\mathbb{G}$ and $M/\!\!/_{i^*(\mu)}\mathbb{A}$ are symplectomophic for all $\mu \in \mathfrak{g}^*_{reg}$. Finally, in subsection 4.3 we show that $\mathbb{A}$-simple groups are plentiful within the class of metabelian nilpotent groups. Low-dimensional examples are indicated in Table 1 in the introduction. Other notable examples include the Heisenberg group, Carnot groups whose center is one-dimensional, and the jet space $\mathcal{J}^k(\mathbb{R}^n, \mathbb{R}^m)$.

Throughout this section, we assume that $\mathbb{G}$ is a connected and simply connected metabelian nilpotent Lie group with Lie algebra $\mathfrak{g}$. As a consequence, the exponential map $\exp : \mathfrak{g} \to \mathbb{G}$ is a global diffeomorphism (see e.g. [18, Theorem 1.2.1]). This will allow us to deduce several properties of $\mathbb{G}$ from properties of its Lie algebra $\mathfrak{g}$.

4.1. **Preliminaries.**

4.1.1. *Nilpotent groups.* Given a subgroup $\mathbb{H}$ of $\mathbb{G}$, we denote by $[\mathbb{H}, \mathbb{G}]$ the subgroup generated by all the commutators, i.e.,

$$[\mathbb{H}, \mathbb{G}] := \{hgh^{-1}g^{-1} : h \in \mathbb{H} \text{ and } g \in \mathbb{G}\}.$$

The *descending central series* of $\mathbb{G}$ is given by

$$\mathbb{G}^1 := \mathbb{G}, \quad \mathbb{G}^{i+1} := [\mathbb{G}, \mathbb{G}^i] \text{ for all } i = 1, 2, \dots.$$

We say that $\mathbb{G}$ is ***nilpotent*** if there exists $s \in \mathbb{N}$ such that $\mathbb{G}^{s+1} = \{e_\mathbb{G}\}$. If $\mathbb{G}^s \neq \{e_\mathbb{G}\}$ then we call s the ***step*** of $\mathbb{G}$. Let us denote the group's center by $\mathrm{Z}(\mathbb{G})$. If $\mathbb{G}$ is a nilpotent group with step s, it follows that $\mathbb{G}^s \subseteq \mathrm{Z}(\mathbb{G})$. Therefore, the center $\mathrm{Z}(\mathbb{G})$ is not empty for every nilpotent group.

The connectedness of $\mathbb{G}$ allows us to characterize the nilpotency condition in terms of its Lie algebra $\mathfrak{g}$. Given a subalgebra $\mathfrak{h} \subset \mathfrak{g}$, we denote by $[\mathfrak{h}, \mathfrak{g}]$ the subspace generated by all the Lie brackets, i.e.,

$$[\mathfrak{h}, \mathfrak{g}] = \mathrm{span}\{[X, Y] : X \in \mathfrak{h} \text{ and } Y \in \mathfrak{g}\}.$$

The *descending central series* of $\mathfrak{g}$ is defined by

$$\mathfrak{g}^1 := \mathfrak{g}, \qquad \mathfrak{g}^{i+1} := [\mathfrak{g}, \mathfrak{g}^i], \quad \text{for all } i = 1, 2, \ldots.$$

It is well-known that $\mathfrak{g}^i$ is the Lie algebra of the i^{th} element, $\mathbb{G}^i$, of the descending central series of $\mathbb{G}$ [22, Theorem 12.3.1]. In particular, if the Lie group $\mathbb{G}$ is nilpotent of step s, we have $\{0\} \neq \mathfrak{g}^s \subset \mathrm{Z}(\mathfrak{g})$ and $\mathfrak{g}^{s+1} = \{0\}$. In this case, we say that the Lie algebra $\mathfrak{g}$ is ***nilpotent*** of ***step*** s. As in the group case, since $\mathfrak{g}^s \subset \mathrm{Z}(\mathfrak{g})$, we conclude that every nilpotent Lie algebra has a non-trivial center. Conversely, since $\mathbb{G}$ is assumed to be be connected, the condition that $\mathfrak{g}$ is nilpotent of step s implies that the $\mathbb{G}$ is nilpotent of step s.

A particular example of a nilpotent group is a ***Carnot*** group. A Carnot group is a connected and simply connected nilpotent Lie group whose Lie algebra is stratified. A stratification of a nilpotent Lie algebra $\mathfrak{g}$ of step s is a direct-sum decomposition

$$\mathfrak{g} = \mathrm{V}_1 \oplus \mathrm{V}_2 \oplus \cdots \oplus \mathrm{V}_s,$$

where $\mathrm{V}_s \neq \{0\}$ and $[\mathrm{V}_1, \mathrm{V}_a] = \mathrm{V}_{1+a}$ for $a = 1, \ldots, s$, and $\mathrm{V}_{s+1} = \{0\}$. We call the subspaces V_a the ***layers*** of $\mathfrak{g}$. We say a Lie algebra is ***stratifiable*** if a stratification exists. Although stratifications are not unique, any two stratifications of $\mathfrak{g}$ differ by a Lie algebra automorphism of $\mathfrak{g}$ (see [26, Proposition 2.17]).

Carnot groups constitute a fundamental subclass of nilpotent groups. In the framework of sub-Riemannian geometry, they play a role analogous to that of Euclidean spaces in Riemannian geometry, as the tangent space of a sub-Riemannian manifold carries the structure of a Carnot group, see [20] and also [36, 11].

A more general class of nilpotent algebras consists of those which are positively gradable. A ***positive grading*** of a Lie algebra is a family $(\mathrm{V}_a)_{a\in\mathbb{N}}$ of linear subspaces of $\mathfrak{g}$ with the property that only a finite number of the V_a are not the zero vector space, and

$$\mathfrak{g} = \bigoplus_{a\in\mathbb{N}} \mathrm{V}_a,$$

where $[\mathrm{V}_a, \mathrm{V}_b] \subset \mathrm{V}_{a+b}$ for all $a, b \in \mathbb{N}$. We say that a Lie algebra is ***positively gradable*** if there exists a positive grading of it. Every stratifiable algebra is positively gradable, but the converse is not true [28]. We say that a subspace $\mathrm{V} \subset \mathfrak{g}$ is bracket generating if every vector in $\mathfrak{g}$ can be written as a linear combination of a finite number, $k \geq 0$, of bracket iterations of elements in V. A subspace V is bracket generating if and only if

$$\mathrm{V} + [\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}.$$

The ***rank*** of a positively gradable algebra is the dimension of the smallest bracket generating subspace. In particular, for Carnot groups, the first layer, V_1, is bracket generating and the rank of the group is $\dim \mathrm{V}_1$. In [29], Le Donne and Tripaldi classified, in terms of their dimension and their rank, the stratifiable Lie algebras of dimension up to 7, and the positive gradable Lie algebras of dimension up to 6 (see Table 1).

4.1.2. *Metabelian nilpotent groups.* To the best of our knowledge, the notion of metabelian groups was introduced by Robinson (see e.g. [46]). A group $\mathbb{G}$ is ***metabelian*** if $[\mathbb{G},\mathbb{G}]$ is abelian. Considering that $[\mathbb{G},\mathbb{G}]$ is a normal subgroup of $\mathbb{G}$ and $\mathbb{G}/[\mathbb{G},\mathbb{G}]$ is abelian, it follows that metabelian groups always possess an abelian normal subgroup $\mathcal{N}$ such that $\mathbb{G}/\mathcal{N}$ is abelian. Conversely, if a group $\mathbb{G}$ posssesses an abelian normal subgroup $\mathcal{N}$ such that $\mathbb{G}/\mathcal{N}$ is abelian then necessarily $\mathcal{N}\supset[\mathbb{G},\mathbb{G}]$ which implies that $\mathbb{G}$ is metabelian. In other words, metabelian groups are characterized by the existence of a normal abelian subgroup $\mathcal{N}$, containing $[\mathbb{G},\mathbb{G}]$, such that $\mathbb{G}/\mathcal{N}$ is abelian. We mention that several previous studies of nilpotent Lie groups were done under the additional assumption that the group is metabelian, even if this extra structure was not explicitly recognised by the authors (e.g. [3, 17, 24, 41, 6, 47, 27, 10, 15] and others). We refer the reader to [46, Chapter 5] for more details and properties of metabelian groups.

Let $\mathbb{G}$ be a metabelian nilpotent group. A central role for us in what follows is played by the ***maximal abelian normal subgroups*** of $\mathbb{G}$, which will be denoted by $\mathbb{A}$. It is not difficult to see that that there always exists a maximal abelian normal subgroup $\mathbb{A}\subset\mathbb{G}$ containing $[\mathbb{G},\mathbb{G}]$, but it may not be unique. Furthermore, there may exist other maximal abelian normal subgroups of $\mathbb{G}$ which do not contain $[\mathbb{G},\mathbb{G}]$.[7] Given a maximal abelian normal subgroup $\mathbb{A}\subset\mathbb{G}$, one may think of $\mathbb{G}$ as an extension of $\mathbb{G}/\mathbb{A}$ by $\mathbb{A}$, and it is not difficult to check that the group's center $\mathrm{Z}(\mathbb{G})\subset\mathbb{A}$.

In general, a Lie algebra $\mathfrak{g}$ is called ***metabelian*** if the ideal $[\mathfrak{g},\mathfrak{g}]$ is abelian. Our connectedness assumption on $\mathbb{G}$ implies that the Lie group $\mathbb{G}$ is metabelian if and only if its Lie algebra $\mathfrak{g}$ is metabelian. Furthermore, if $\mathbb{A}\subset\mathbb{G}$ is a maximal abelian normal subgroup, then its Lie algebra $\mathfrak{a}$ is a ***maximal abelian ideal*** of $\mathfrak{g}$, and our topological assumptions on $\mathbb{G}$ imply that the correspondence also goes in the other direction. Namely, there is a natural correspondence between maximal abelian normal subgroups of $\mathbb{G}$ and maximal abelian ideals of $\mathfrak{g}$. We will systemetically denote maximal abelian ideals of $\mathfrak{g}$ by $\mathfrak{a}$ (even if the underlying subgroup $\mathbb{A}$ is not specified). As in the group case, maximal abelian ideals may not be unique and may not necessarily contain $[\mathfrak{g},\mathfrak{g}]$. However, it is not difficult to see that any maximal abelian ideal $\mathfrak{a}\subset\mathfrak{g}$ contains the Lie algebra center $\mathrm{Z}(\mathfrak{g})$.

4.2. **$\mathbb{A}$-simple metabelian nilpotent groups.** We introduce the following class of groups for which we will show that Theorem A applies for generic $\mu\in\mathfrak{g}^*$.

Definition 5. *Let $\mathfrak{g}$ be a metabelian nilpotent Lie algebra and let $\mathfrak{a}\subset\mathfrak{g}$ be a maximal abelian ideal. We say that $\mathfrak{g}$ is **$\mathfrak{a}$-simple** if there exists a*

[7] the simplest metabelian group that we found for which this holds is the 7-dimensional group whose Lie algebra is denoted by $147D$ in [29].

basis $\{X_1, \dots, X_n\}$ of a direct complement $\mathfrak{X}$ of $\mathfrak{a}$ in $\mathfrak{g}$ and a set of linearly independent vectors $Y_1, \dots, Y_n \in \mathfrak{a}$ satisfying

$$0 \neq [X_i, Y_i] \in \mathrm{Z}(\mathfrak{g}) \;\; \textit{for all} \;\; i = 1, \cdots, n.$$

*Let $\mathbb{G}$ be a connected and simply connected, metabelian nilpotent group, and let $\mathbb{A} \subset \mathbb{G}$ be a maximal abelian normal subgroup. We say that $\mathbb{G}$ is **$\mathbb{A}$-simple** if its Lie algebra $\mathfrak{g}$ is $\mathfrak{a}$-simple, where $\mathfrak{a}$ is the Lie algebra of $\mathbb{A}$.*

The following theorem states that, for generic momentum values, the symplectic reduction by an $\mathbb{A}$-simple Lie group $\mathbb{G}$ is equivalent to the abelian symplectic reduction by the maximal abelian normal subgroup $\mathbb{A}$.

Theorem C. *Let $\mathbb{G}$ be a metabelian nilpotent $\mathbb{A}$-simple Lie group. There exists an open dense subset $\mathfrak{g}^*_{reg} \subset \mathfrak{g}^*$ with the following property. Suppose that $\mathbb{G}$ defines a free and proper Hamiltonian action on the symplectic manifold (M, ω) with equivariant momentum map $J_{\mathbb{G}} : M \to \mathfrak{g}^*$. If $\mu \in \mathfrak{g}^*_{reg} \cap J_{\mathbb{G}}(M)$ then there is a symplectic diffeomorphism between the symplectic reduced spaces $M/\!\!/_{\mu}\mathbb{G}$ and $M/\!\!/_{i^*(\mu)}\mathbb{A}$.*

Note that Theorem C, together with the discussion of Section 2.5, implies that the generic coadjoint orbits of an $\mathbb{A}$-simple Lie group $\mathbb{G}$ are symplectomorphic to $T^*(\mathbb{G}/\mathbb{A})$ (equipped with a magnetic modification of the canonical symplectic form).

Theorem C is an immediate consequence of Proposition 4 and the following Lemma.

Lemma 6. *Let $\mathbb{G}$ be a metabelian nilpotent $\mathbb{A}$-simple group. There exists an open dense subset $\mathfrak{g}^*_{reg} \subset \mathfrak{g}^*$ such that, for all $\mu \in \mathfrak{g}^*_{reg}$, we have*

(a) $\mathbb{G}_\mu \subset \mathbb{A}$,
(b) *For the direct complement $\mathfrak{X}$ of $\mathfrak{a}$ in $\mathfrak{g}$ given in Definition 5, the mapping $\mathrm{T}_\mu : \mathfrak{X} \to (\mathfrak{a}/\,\mathrm{Z}(\mathfrak{g}))^*$ defined by (2.8) is injective.*

The proof of Lemma 6 relies on Proposition 7 given below that allows us to work with a convenient basis of an $\mathfrak{a}$-simple Lie algebra $\mathfrak{g}$. In what follows, we will use the following index convention:

- The lower case indices i, j, k run from 1 to $n = \dim \mathfrak{X}$ (as in Definition 5).
- The upper case index I runs from 1 to $\dim(\mathrm{Z}(\mathfrak{g})) - 1$.
- The lower case index a runs from 1 to $\dim \mathfrak{g} - \dim(\mathrm{Z}(\mathfrak{g})) - 2n$.

Proposition 7. *Let $\mathfrak{g}$ be an $\mathfrak{a}$-simple Lie algebra and let $\mathfrak{X}$ be the direct complement of $\mathfrak{a}$ in $\mathfrak{g}$ given in Definition 5. There exists a basis of $\mathfrak{g}$ of the form*

$$\{Z_0, Z_I, Y_j, Y_a, X_i\},$$

such that

$$\mathrm{Z}(\mathfrak{g}) = \langle Z_0, Z_I \rangle, \qquad \mathfrak{a} = \langle Z_0, Z_I, Y_j, Y_a \rangle, \qquad \mathfrak{X} = \langle X_i \rangle,$$

and

$$[X_i, Y_j] = \delta_{ij} Z_0 + \sum_I C^I_{ij} Z_I + \sum_a C^a_{ij} Y_a, \tag{4.1}$$

where δ_{ij} is Kronecker's delta and C^I_{ij}, C^a_{ij} are some structure coefficients.

We present a proof of Lemma 6 and give the proof of Proposition 7 afterwards.

Proof of Lemma 6. Work with the basis of Proposition 7 and denote by

$$\{Z_0^*, Z_I^*, Y_j^*, Y_a^*, X_i^*\},$$

the dual basis for $\mathfrak{g}^*$. Introduce linear coordinates $(c, \epsilon_I, \beta_k, \gamma_a, \alpha_k)$ on $\mathfrak{g}^*$ by writing $\mu \in \mathfrak{g}^*$ as

$$\mu = cZ_0^* + \sum_I \epsilon_I Z_I^* + \sum_k (\alpha_k X_k^* + \beta_k Y_k^*) + \sum_a \gamma_a Y_a^*. \tag{4.2}$$

Now, for $\mu \in \mathfrak{g}^*$, let $\mathcal{M}(\mu)$ be the $n \times n$ matrix with entries

$$\mathcal{M}(\mu)_{ij} := \langle \mathrm{ad}^*_{-X_i} \mu, Y_j \rangle.$$

It is clear that the entries of $\mathcal{M}(\mu)$ depend linearly on the coordinates $(c, \epsilon_I, \beta_k, \gamma_a, \alpha_k)$.

Now consider $X = \sum_i \alpha_i X_i \in \mathfrak{X}$. Using the definition (2.8) of T_μ, and considering that $Y_j \in \mathfrak{a}$, we have

$$\begin{aligned} \langle \mathrm{T}_\mu(X), \overline{Y_j} \rangle &= \langle \mathrm{ad}^*_{-X} \mu, Y_j \rangle \\ &= \sum_i \alpha_i \mathcal{M}(\mu)_{ij}. \end{aligned}$$

This shows that if $X \in \ker \mathrm{T}_\mu$ then the column vector $(\alpha_1, \dots, \alpha_n)^T \in \mathbb{R}^n$ is a null-vector of $\mathcal{M}(\mu)^T$. In particular, we conclude that T_μ is injective whenever the matrix $\mathcal{M}(\mu)$ is invertible.

Consider the mapping $\psi : \mathfrak{g}^* \to \mathbb{R}$ given by

$$\psi(\mu) = \det(\mathcal{M}(\mu)).$$

Then ψ is a polynomial function of degree n on the coordinates $(c, \epsilon_I, \beta_k, \gamma_a, \alpha_k)$ and from the discussion above, we know that T_μ is injective on the open subset $\mathfrak{g}^*_{\mathrm{reg}}$ of $\mathfrak{g}^*$ on which $\psi(\mu) \neq 0$.

Next, using (4.1), we find that if $\epsilon_I = 0$ for all I and $\gamma_a = 0$ for all a, then

$$\begin{aligned} \mathcal{M}(\mu)_{ij} &= -\langle \mu, [X_i, Y_j] \rangle \\ &= -\left\langle cZ_0^* + \sum_k (\alpha_k X_k^* + \beta_k Y_k^*)\, ,\, \delta_{ij} Z_0 + \sum_I C^I_{ij} Z_I + \sum_a C^a_{ij} Y_a \right\rangle \\ &= -c\delta_{ij}. \end{aligned}$$

In other words, if $\epsilon_I = 0$ for all I and $\gamma_a = 0$ for all a, then $\mathcal{M}(\mu)$ is a scalar multiple of the $n \times n$ identity matrix by the factor $-c$. In particular

$$\psi(\mu) = (-1)^n c^n.$$

This shows that ψ is not identically zero. Given that ψ is polynomial, it follows that ψ cannot identically vanish on an open set of $\mathfrak{g}^*$. Therefore, the set $\mathfrak{g}^*_{\mathrm{reg}} \subset \mathfrak{g}^*$ on which $\psi \neq 0$, apart from being open, is dense in $\mathfrak{g}^*$. This proves item (b) of Lemma 6.

To prove item (a) suppose that $\mu \in \mathfrak{g}^*_{\mathrm{reg}}$, so T_μ is injective, and, by contradiction, suppose that $\mathfrak{g}_\mu$ is not contained in $\mathfrak{a}$. Given that $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{X}$, there exists a nontrivial $X \in \mathfrak{X} \cap \mathfrak{g}_\mu$. Since $X \in \mathfrak{g}_\mu$, we have $\mathrm{ad}^*_X \mu = 0$ which implies $\mathrm{T}_\mu(X) = 0$, contradicting the injectivity of T_μ. Hence, we must have $\mathfrak{g}_\mu \subset \mathfrak{a}$. As we mentioned earlier, our topological assumptions on $\mathbb{G}$ imply that the exponential map $\exp : \mathfrak{g} \to \mathbb{G}$ is a global diffeomorphism. As a consequence $\exp(\mathfrak{g}_\mu) = \mathbb{G}_\mu$, see e.g. [18, Lemma 1.3.1]. Exponentiating the inclusion $\mathfrak{g}_\mu \subset \mathfrak{a}$, we conclude that $\mathbb{G}_\mu \subset \mathbb{A}$ as required.

□

For the proof of Proposition 7 we will need the following propositions.

Proposition 8. *Let $\mathfrak{g}$ be an $\mathfrak{a}$-simple Lie algebra and let $\mathfrak{X}$ be the direct complement of $\mathfrak{a}$ in $\mathfrak{g}$ given in Definition 5. Suppose that* $\dim \mathrm{Z}(\mathfrak{g}) = 1$. *There exists a basis of $\mathfrak{g}$ of the form*

$$\{Z_0, Y_j, Y_a, X_i\},$$

such that

$$\mathrm{Z}(\mathfrak{g}) = \langle Z_0 \rangle, \qquad \mathfrak{a} = \langle Z_0, Y_j, Y_a \rangle, \qquad \mathfrak{X} = \langle X_i \rangle,$$

satisfying

$$[X_i, Y_j] = \delta_{ij} Z_0.$$

Proof. Let $\widetilde{X}_i \in \mathfrak{X}$, and $\widetilde{Y}_j \in \mathfrak{a}$ be the vectors in Definition 5 of $\mathfrak{a}$-simple, in other words they satisfy $0 \neq [\widetilde{X}_i, \widetilde{Y}_i] \in \mathrm{Z}(\mathfrak{g})$. We have $\mathfrak{X} = \langle \widetilde{X}_i \rangle$ and we define $\mathfrak{D} := \langle \widetilde{Y}_j \rangle \subset \mathfrak{a}$. If $Z_0 \in \mathrm{Z}(\mathfrak{g})$ is a non-zero element, then we may define a duality pairing $\mathrm{B} : \mathfrak{X} \times \mathfrak{D} \to \mathbb{R}$ by the equation

$$[X, Y] = \mathrm{B}(X, Y) Z_0, \text{ where } X \in \mathfrak{X} \text{ and } Y \in \mathfrak{D}.$$

The condition $[\widetilde{X}_i, \widetilde{Y}_i] \neq 0$ implies that B is non-degenerate. Using a standard linear algebra Gram-Schmidt-type procedure it is not difficult to construct bases $\{X_i\}$ of $\mathfrak{X}$ and $\{Y_j\}$ of $\mathfrak{D}$ with the property that $\mathrm{B}(X_i, Y_j) = \delta_{ij}$. The set $\{Y_j\}$ is linearly independent, so we may complete a basis for the ideal $\mathfrak{a}$ to find the desired basis for $\mathfrak{g}$. □

Proposition 9. *Given an $\mathfrak{a}$-simple Lie algebra $\mathfrak{g}$, there exists a metabelian Lie algebra $\tilde{\mathfrak{g}}$ with* $\dim(\mathrm{Z}(\tilde{\mathfrak{g}})) = 1$, *and a surjective Lie algebra homomorphism $\pi : \mathfrak{g} \to \tilde{\mathfrak{g}}$ satisfying the following properties:*

(a) $\ker \pi \subset \mathfrak{a}$,

(b) $\pi(\mathrm{Z}(\mathfrak{g})) = \mathrm{Z}(\tilde{\mathfrak{g}})$.

Moreover, if we define $\tilde{\mathfrak{a}} := \pi(\mathfrak{a})$ *then* $\tilde{\mathfrak{g}}$ *is* $\tilde{\mathfrak{a}}$*-simple.*

Proof. We will proceed by induction on the dimension of $\mathrm{Z}(\mathfrak{g})$. If $\dim \mathrm{Z}(\mathfrak{g}) = 1$, then the Lie algebra homomorphism is the identity map, and the result follows trivially.

Now assume that the result is true for $\mathfrak{a}$-simple Lie algebras whose center has dimension $\ell \geq 1$ and let $\mathfrak{g}_0$ be an $\mathfrak{a}_0$-simple Lie algebra with $\dim \mathrm{Z}(\mathfrak{g}_0) = \ell + 1$.

Consider the decomposition $\mathfrak{g}_0 := \mathfrak{X}_0 \oplus \mathfrak{a}_0$, and the vectors $X_i \in \mathfrak{X}_0, Y_i \in \mathfrak{a}_0$ given by the Definition 5 of $\mathfrak{a}$-simple algebra. Let $0 \neq Z \in \mathrm{Z}(\mathfrak{g}_0)$ satisfying

$$Z \in \left(\bigcup_i \langle [X_i, Y_i] \rangle \right)^c \cap \mathrm{Z}(\mathfrak{g}_0). \tag{4.3}$$

Such non-zero Z always exists by our assumption that $\dim \mathrm{Z}(\mathfrak{g}_0) = \ell + 1 \geq 2$. Consider the ideal $\mathfrak{z}_0 := \langle Z \rangle \subset \mathrm{Z}(\mathfrak{g}_0)$ of $\mathfrak{g}_0$ and the Lie alegebra $\mathfrak{g}_1 := \mathfrak{g}_0/\mathfrak{z}_0$. The canonical projection $\pi_1 : \mathfrak{g}_0 \to \mathfrak{g}_1$ is a surjective Lie algebra homomorphism satisfying

$$\ker \pi_1 = \mathfrak{z}_0 \subset \mathrm{Z}(\mathfrak{g}_0) \subset \mathfrak{a}_0, \qquad \dim(\pi_1(\mathrm{Z}(\mathfrak{g}_0)) = \ell. \tag{4.4}$$

Moreover, given that π_1 is a surjective Lie algebra homomorphism, we have

$$\pi_1(\mathrm{Z}(\mathfrak{g}_0)) \subset \mathrm{Z}(\mathfrak{g}_1), \tag{4.5}$$

and we claim that this implies that $\mathfrak{g}_1$ is $\mathfrak{a}_1 := \pi_1(\mathfrak{a}_0)$-simple. To show this, first note that, since π_1 is a surjective Lie algebra homomorphism with $\ker \pi_1 \subset \mathfrak{a}_0$, then $\mathfrak{g}_1$ is metabelian and $\mathfrak{a}_1 \subset \mathfrak{g}_1$ is a maximal abelian ideal. Next, we may write $\mathfrak{g}_1 = \mathfrak{X}_1 \oplus \mathfrak{a}_1$ with $\mathfrak{X}_1 := \pi_1(\mathfrak{X}_0)$. Consider the vectors $\pi_1(X_i) \in \mathfrak{X}_1$, $\pi_1(Y_i) \in \mathfrak{a}_1$. Since π_1 is a Lie algbra homomorphism we have

$$[\pi_1(X_i), \pi_1(Y_i)]_{\mathfrak{g}_1} = \pi_1 \, [X_i, Y_i]_{\mathfrak{g}_0} .$$

By construction, $[X_i, Y_i]_{\mathfrak{g}_0} \notin \mathfrak{z}_0 = \ker \pi_1$ so $[\pi_1(X_i), \pi_1(Y_i)]_{\mathfrak{g}_1} \neq 0$, Moreover, since $[X_i, Y_i]_{\mathfrak{g}_0} \in \mathrm{Z}(\mathfrak{g}_0)$, we conclude that $[\pi_1(X_i), \pi_1(Y_i)]_{\mathfrak{g}_1} \in \pi_1(\mathrm{Z}(\mathfrak{g}_0)) \subset \mathrm{Z}(\mathfrak{g}_1)$ which proves that $\mathfrak{g}_1$ is $\mathfrak{a}_1$-simple as claimed.

Now, in view of (4.5), one of the following possibilities holds:

1. $\pi_1(\mathrm{Z}(\mathfrak{g}_0)) = \mathrm{Z}(\mathfrak{g}_1)$.
2. $\pi_1(\mathrm{Z}(\mathfrak{g}_0)) \subsetneq \mathrm{Z}(\mathfrak{g}_1)$.

In the first case, we may apply the induction hypothesis to $\mathfrak{g}_1$ (since $\dim(\mathrm{Z}(\mathfrak{g}_1)) = \ell$) to conclude the existence of a Lie algebra $\tilde{\mathfrak{g}}$, with $\dim(\mathrm{Z}(\tilde{\mathfrak{g}})) = 1$ and a surjective Lie algebra homomorphism $\tilde{\pi} : \mathfrak{g}_1 \to \tilde{\mathfrak{g}}$ satisfying items (a), (b) in the statement of the Lemma and such that $\tilde{\mathfrak{g}}$ is $\tilde{\mathfrak{a}} := \tilde{\pi}(\mathfrak{a}_1)$-simple. We claim that the proof of the proposition follows by taking $\pi := \tilde{\pi} \circ \pi_1 : \mathfrak{g}_0 \to \tilde{\mathfrak{g}}$. Indeed, we have

$$\ker \pi = \ker \pi_1 \oplus \pi_1^{-1}(\ker \tilde{\pi}).$$

Hence, using that $\ker \pi_1 \subset \mathfrak{a}_0$ (in view of (4.4)) and $\ker \tilde{\pi} \subset \mathfrak{a}_1 = \pi_1(\mathfrak{a}_0)$ (by the induction hypothesis and the definition of $\mathfrak{a}_1$) we conclude that $\ker \pi \subset$

$\mathfrak{a}_0$ and item (a) holds. Also, by the induction hypothesis we have $\mathrm{Z}(\tilde{\mathfrak{g}}) = \tilde{\pi}(\mathrm{Z}(\mathfrak{g}_1))$ which together with the condition 1. above implies $\pi(\mathrm{Z}(\mathfrak{g}_0)) = \mathrm{Z}(\tilde{\mathfrak{g}})$ showing that item (b) also holds.

On the other hand, if condition 2. above holds, there exists a nontrivial subspace $\mathfrak{z}_1 \subset \mathrm{Z}(\mathfrak{g}_1)$ such that

$$\mathrm{Z}(\mathfrak{g}_1) = \pi_1(\mathrm{Z}(\mathfrak{g}_0)) \oplus \mathfrak{z}_1.$$

Then $\mathfrak{z}_1$ is an ideal in $\mathfrak{g}_1$ and we consider the Lie algebra $\mathfrak{g}_2 := \mathfrak{g}_1/\mathfrak{z}_1$. The canonical projection $\pi_2 : \mathfrak{g}_1 \to \mathfrak{g}_2$ is a surjective Lie algebra homomorphism, which, in analogy with (4.4), satisfies,

$$\ker \pi_2 = \mathfrak{z}_1 \subset \mathrm{Z}(\mathfrak{g}_1) \subset \mathfrak{a}_1, \qquad \dim(\pi_2(\mathrm{Z}(\mathfrak{g}_1))) = \dim(\pi_1(\mathrm{Z}(\mathfrak{g}_0))) = \ell.$$

Moreover, since π_2 is surjective, we have $\pi_2(\mathrm{Z}(\mathfrak{g}_1)) \subset \mathrm{Z}(\mathfrak{g}_2)$ and, arguing as above, one can show that $\mathfrak{g}_2$ is metabelian and is $\mathfrak{a}_2 := \pi_2(\mathfrak{a}_1)$-simple. At this point, we are again faced with a dichotomy in which one of the following holds:

1. $\pi_2(\mathrm{Z}(\mathfrak{g}_1)) = \mathrm{Z}(\mathfrak{g}_2)$.
2. $\pi_2(\mathrm{Z}(\mathfrak{g}_1)) \subsetneq \mathrm{Z}(\mathfrak{g}_2)$.

In case 1., one may apply the induction hypothesis to $\mathrm{Z}(\mathfrak{g}_2)$ (since $\dim(\mathrm{Z}(\mathfrak{g}_2)) = \ell$) and there exists a Lie algebra $\tilde{\mathfrak{g}}$, with $\dim(\mathrm{Z}(\tilde{\mathfrak{g}})) = 1$ and a surjective Lie algebra homomorphism $\tilde{\pi} : \mathfrak{g}_2 \to \tilde{\mathfrak{g}}$ satisfying items (a), (b) in the statement of the Lemma and such that $\tilde{\mathfrak{g}}$ is $\tilde{\mathfrak{a}} := \tilde{\pi}(\mathfrak{a}_2)$-simple. In analogy with the above, the proof of the proposition follows by taking $\pi := \tilde{\pi} \circ \pi_2 \circ \pi_1 : \mathfrak{g}_0 \to \tilde{\mathfrak{g}}$ (see below).

In case 2., we repeat the above construction and consider a nontrivial subspace $\mathfrak{z}_2 \subset \mathrm{Z}(\mathfrak{g}_2)$ such that

$$\mathrm{Z}(\mathfrak{g}_2) = \pi_2(\mathrm{Z}(\mathfrak{g}_1)) \oplus \mathfrak{z}_2.$$

Then $\mathfrak{z}_2$ is an ideal in $\mathfrak{g}_2$ and we consider the metabelian Lie algebra $\mathfrak{g}_3 := \mathfrak{g}_2/\mathfrak{z}_2$ and the canonical projection $\pi_3 : \mathfrak{g}_2 \to \mathfrak{g}_3$ which is a surjective Lie algebra homomorphism satisfying,

$$\ker \pi_3 = \mathfrak{z}_2 \subset \mathrm{Z}(\mathfrak{g}_2) \subset \mathfrak{a}_2, \qquad \dim(\pi_3(\mathrm{Z}(\mathfrak{g}_2))) = \dim(\pi_2(\mathrm{Z}(\mathfrak{g}_1))) = \ell.$$

Again, using $\pi_3(\mathrm{Z}(\mathfrak{g}_2)) \subset \mathrm{Z}(\mathfrak{g}_3)$ and, arguing as above, one can show that $\mathfrak{g}_3$ is $\mathfrak{a}_3 := \pi_3(\mathfrak{a}_2)$-simple, and we may repeat the full argument.

Considering that $\mathfrak{g}_0$ is finite dimensional the above procedure ends at some point and may be summarized as follows. There exists a finite sequence of Lie algebras $\mathfrak{g}_0, \dots, \mathfrak{g}_R$, for some $R \geq 1$, and surjective Lie algebra homomorphisms $\{\pi_{r+1} : \mathfrak{g}_r \to \mathfrak{g}_{r+1}\}_{r=0}^{R-1}$ with the property that $\mathfrak{g}_r$ is $\mathfrak{a}_r$ simple with $\mathfrak{a}_{r+1} := \pi_{r+1}(\mathfrak{a}_r)$ for all $r = 0, \dots, R-1$. Furthermore, by construction,

$$\begin{aligned} &\ker \pi_{r+1} \subset \mathfrak{a}_r, \qquad r = 0, \dots, R-1, \\ &\mathrm{Z}(\mathfrak{g}_{r+1}) = \pi_{r+1}(\mathrm{Z}(\mathfrak{g}_r)) \oplus \ker \pi_{r+2}, \qquad r = 0, \dots, R-2, \\ &\pi_R(\mathrm{Z}(\mathfrak{g}_{R-1})) = \mathrm{Z}(\mathfrak{g}_R), \\ &\dim(\mathrm{Z}(\mathfrak{g}_R))) = \ell. \end{aligned} \tag{4.6}$$

Now, applying the induction hypothesis to $\mathfrak{g}_R$, there exists $\tilde{\pi} : \mathfrak{g}_R \to \tilde{\mathfrak{g}}$, surjective Lie algebra homomorphism, with $\dim \mathrm{Z}(\tilde{\mathfrak{g}}) = 1$, which satisfies $\ker \tilde{\pi} \subset \mathfrak{a}_R$, $\tilde{\pi}(\mathrm{Z}(\mathfrak{g}_R)) = \mathrm{Z}(\tilde{\mathfrak{g}})$, and such that $\tilde{\mathfrak{g}}$ is $\tilde{\mathfrak{a}} := \tilde{\pi}(\mathfrak{a}_R)$-simple. Using (4.6), it is not difficult to verify (as we did in the case $R = 1$ above) that the Lie algebra homomorphism $\pi := \tilde{\pi} \circ \pi_R \circ \cdots \circ \pi_1 : \mathfrak{g}_0 \to \tilde{\mathfrak{g}}$ satisfies the properties (a) and (b) in the statement of the Lemma and also $\tilde{\mathfrak{a}} = \pi(\mathfrak{a}_0)$.

$\square$

We are now ready to give the proof of Proposition 7.

Proof of Proposition 7. If $\dim \mathrm{Z}(\mathfrak{g}) = 1$ the result follows automatically from Proposition 8.

For the case $\dim \mathrm{Z}(\mathfrak{g}) > 1$, we apply Proposition 9 which guarantees the existence of a Lie algebra $\tilde{\mathfrak{g}}$, with $\dim \mathrm{Z}(\tilde{\mathfrak{g}}) = 1$ and a surjective Lie algebra homomorphism $\pi : \mathfrak{g} \to \tilde{\mathfrak{g}}$, satisfying $\ker \pi \subset \mathfrak{a}$, $\mathrm{Z}(\tilde{\mathfrak{g}}) = \pi(\mathrm{Z}(\mathfrak{g}))$, and such that $\tilde{\mathfrak{g}}$ is $\tilde{\mathfrak{a}} := \pi(\mathfrak{a})$-simple.

Now, given that $\dim \mathrm{Z}(\tilde{\mathfrak{g}}) = 1$, we may apply Proposition 8 to obtain a basis $\{\tilde{Z}_0, \tilde{Y}_j, \tilde{Y}_{\tilde{a}}, \tilde{X}_i\}$ of $\tilde{\mathfrak{g}}$ such that

$$\mathrm{Z}(\tilde{\mathfrak{g}}) = \langle \tilde{Z}_0 \rangle, \qquad \tilde{\mathfrak{a}} = \langle \tilde{Z}_0, \tilde{Y}_j, \tilde{Y}_{\tilde{a}} \rangle, \qquad \tilde{\mathfrak{X}} = \langle \tilde{X}_i \rangle$$

and

$$[\tilde{X}_i, \tilde{Y}_j] = \delta_{ij} \tilde{Z}_0. \tag{4.7}$$

We note that the indices i, j run from 1 to $n = \dim \mathfrak{X}$ since $\dim \mathfrak{X} = \dim \tilde{\mathfrak{X}}$ given that $\ker \pi \subset \mathfrak{a}$, $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{X}$ and $\tilde{\mathfrak{a}}$ was defined as $\pi(\mathfrak{a})$. On the other hand, the index $\tilde{a}$ runs from 1 to

$$\dim \tilde{\mathfrak{g}} - \dim(\mathrm{Z}(\tilde{\mathfrak{g}})) - 2n = \dim \mathfrak{g} - \dim \ker \pi - 1 - 2n. \tag{4.8}$$

Now, the condition that $\mathrm{Z}(\tilde{\mathfrak{g}}) = \pi(\mathrm{Z}(\mathfrak{g}))$ ensures the existence of $Z_0 \in \mathrm{Z}(\mathfrak{g})$ such that $\pi(Z_0) = \tilde{Z}_0$. On the other hand, given that $\pi : \mathfrak{g} \to \tilde{\mathfrak{g}}$ is surjective, there exists an injective linear map $\varphi : \tilde{\mathfrak{g}} \to \mathfrak{g}$ such that $\pi \circ \varphi = \mathrm{id}_{\tilde{\mathfrak{g}}}$. Let

$$Y_j := \varphi(\tilde{Y}_j), \qquad Y_{\tilde{a}} := \varphi(\tilde{Y}_{\tilde{a}}), \qquad X_i := \varphi(\tilde{X}_i).$$

Then $\{Z_0, Y_j, Y_{\tilde{a}}, X_i\}$ is a linearly independent subset in $\mathfrak{g}$, which satisfies

$$\pi(Z_0) = \tilde{Z}_0, \qquad \pi(Y_j) = \tilde{Y}_j, \qquad \pi(Y_{\tilde{a}}) = \tilde{Y}_{\tilde{a}}, \qquad \pi(X_i) = \tilde{X}_i. \tag{4.9}$$

Moreover, using that π is a Lie algebra homomorphism, and $\tilde{Y}_j, \tilde{Y}_{\tilde{a}}, \tilde{X}_i \notin \mathrm{Z}(\tilde{\mathfrak{g}})$, we have

$$\langle Z_0, Y_j, Y_{\tilde{a}}, X_i \rangle \cap \mathrm{Z}(\mathfrak{g}) = \langle Z_0 \rangle.$$

We construct the desired basis of $\mathfrak{g}$ by adjoining a convenient basis of $\ker \pi$. Specifically, considering that $\ker \pi \subset \mathfrak{a}$ and $\mathrm{Z}(\mathfrak{g}) \subset \mathfrak{a}$, we construct a basis $\{Z_I, Y_{\hat{a}}\}$ of $\ker \pi$ such that $\ker \pi \cap \mathrm{Z}(\mathfrak{g}) = \langle Z_I \rangle$. Then

$$\{Z_0, Z_I, Y_j, Y_{\hat{a}}, Y_{\tilde{a}}, X_i\}$$

is a basis of $\mathfrak{g}$ which we claim has the desired properties. First note that the index I runs from 1 to $\dim(\mathrm{Z}(\mathfrak{g})) - 1$ as it should. Indeed, the condition $\pi(\mathrm{Z}(\mathfrak{g})) = \mathrm{Z}(\tilde{\mathfrak{g}}) = \langle \tilde{Z}_0 \rangle$ implies that $\dim(\ker \pi \cap \mathrm{Z}(\mathfrak{g})) = \dim(\mathrm{Z}(\mathfrak{g})) - 1$, as desired. As a consequence, $\mathrm{Z}(\mathfrak{g}) = \langle Z_0, Z_I \rangle$. Moreover, the index $\hat{a}$ runs on the (possibly empty) range from 1 to

$$\dim \ker \pi - (\dim(\mathrm{Z}(\mathfrak{g})) - 1).$$

In view of (4.8), this means that the combined range of the indices $\tilde{a}$ and $\hat{a}$ is from 1 to $\dim \mathfrak{g} - \dim(\mathrm{Z}(\mathfrak{g})) - 2n$ which is the desired range of the index a in the statement of the proposition.

It only remains to show that the commutation relations (4.1) hold. Using (4.7), (4.9) and the fact that π is a Lie algebra homomorphism, we get

$$\pi\left([X_i, Y_j]\right) = [\tilde{X}_i, \tilde{Y}_j] = \delta_{ij}\tilde{Z}_0 = \pi(\delta_{ij} Z_0).$$

Hence,

$$[X_i, Y_j] - \delta_{ij} Z_0 \in \ker \pi = \langle Z_I, Y_{\hat{a}} \rangle.$$

Therefore, there exist scalars $C_{ij}^I, C_{ij}^{\hat{a}} \in \mathbb{R}$ such that

$$[X_i, Y_j] = \delta_{ij} Z_0 + \sum_i C_{ij}^I Z_I + \sum_{\hat{a}} C_{ij}^{\hat{a}} Y_{\hat{a}},$$

as required. □

4.3. **Examples of $\mathbb{A}$-simple groups.** Even though Definition 5 seems technical and difficult to verify in examples, we show below that there is an abundant number of metabelian nilpotent groups which are $\mathbb{A}$-simple. We point out that a necessary condition for $\mathbb{G}$ to be $\mathbb{A}$-simple is that

$$\dim \mathfrak{X} = \dim(\mathfrak{g}/\mathfrak{a}) \leq \dim(\mathfrak{a}/\,\mathrm{Z}(\mathfrak{g})).$$

Indeed, this conclusion follows since the vectors $Y_1, \ldots, Y_n \in \mathfrak{a}$ in Definition 5 satisfy $Y_i \notin \mathrm{Z}(\mathfrak{g})$. The above inequality can be equivalently written as

$$\dim \mathfrak{g} + \dim(\mathrm{Z}(\mathfrak{g})) \leq 2 \dim \mathfrak{a}. \tag{4.10}$$

In Table 1 in the introduction, we present the low-dimensional Carnot groups from [29], indicating which ones are $\mathbb{A}$-simple, and which of those that fail to be $\mathbb{A}$-simple are not metabelian. The table shows that in low dimensions, most metabelian nilpotent groups are $\mathbb{A}$-simple. It may be verified that all examples of metabelian groups which are not $\mathbb{A}$-simple appearing in the table violate (4.10). On the other hand, to find examples of metabelian nilpotent groups which are not $\mathbb{A}$-simple and satisfy (4.10), one has to consider $\mathbb{G}$ of dimension ≥ 9.

4.3.1. *The Heisenberg group.* The most well-known example of a metabelian $\mathbb{A}$-simple group is the Heisenberg group.

The Heisenberg group $\mathbb{G} = \mathbb{H}^{2n+1}$ is a Carnot group of step 2 and dimension $(2n+1)$ whose Lie algebra is given by

$$[X_i, Y_i] = Z, \quad \text{for} \;\; i = 1, \ldots, n,$$

with all other brackets equal to zero. In this example $\mathrm{Z}(\mathfrak{g}) = \langle Z \rangle$ and $\mathfrak{a} = \langle Z, Y_1, \cdots, Y_n \rangle$ is a maximal abelian ideal. Therefore, the above relations show that $\mathbb{H}^{2n+1}$ satisfies Definition 5 with $\mathfrak{X} = \langle X_1, \dots, X_n\rangle$ and is therefore $\mathbb{A}$-simple.

4.3.2. *A Carnot group which is not $\mathbb{A}$-simple.* On the other hand, the best-known example of a metabelian Carnot group that is not $\mathbb{A}$-simple is the Cartan group $\mathbb{F}_{2,3}$ (whose Lie algebra is indicated as $N_{5,2,3}$ in Table 1) which is a Carnot group of step 3 and dimension 5 whose non-trivial brackets are the following

$$[X_1, X_2] = Y, \;\; [X_1, Y] = Z_1, \;\; \text{and} \;\; [X_2, Y] = Z_2.$$

This group has rank 2 and may be alternatively defined as the free Carnot group with 2 generators and step 3. In this example, $\mathrm{Z}(\mathfrak{g}) = \langle Z_1, Z_2\rangle$, $\mathfrak{a} = \langle Z_1, Z_2, Y\rangle$, and inequality (4.10) does not hold. Therefore, $\mathbb{F}_{2,3}$ is not $\mathbb{A}$-simple.

In contrast, the free Carnot group of step 4 and rank 2, denoted $\mathbb{F}_{2,4}$, is $\mathbb{A}$-simple. Indeed, this group has dimension 8 and the following non-trivial brackets

$$[X_1, X_2] = Y_3, \;\; [X_1, Y_3] = Y_2, \;\; [X_2, Y_3] = Y_1,$$
$$[X_1, Y_2] = Z_1, \;\; [X_1, Y_1] = [X_2, Y_2] = Z_2, \; \text{and} \; [X_2, Y_1] = Z_3.$$

We have $\mathrm{Z}(\mathfrak{g}) = \langle Z_1, Z_2, Z_3\rangle$, $\mathfrak{a} = \langle Z_1, Z_2, Z_3, Y_1, Y_2, Y_3\rangle$ as a maximal abelian ideal and the bracket relations $[X_1, Y_1] = [X_2, Y_2] = Z_2$ imply that $\mathbb{F}_{2,4}$ is an $\mathbb{A}$-simple group (with $\mathfrak{X} = \langle X_1, X_2\rangle$).

This is an example of a common phenomenon: it is often the case that adding an extra step to a non-$\mathbb{A}$-simple nilpotent group makes it $\mathbb{A}$-simple.

4.3.3. *Metabelian Carnot groups with a one-dimensional center.* Nilpotent Lie algebras with a one-dimensional center are important because they provide a foundational building block for understanding more complex Lie algebras and groups [18, 23, 12, 25]. Here we will show that every nilpotent, stratifiable, metabelian Lie algebra $\mathfrak{g}$ with a one-dimensional center is $\mathfrak{a}$-simple, for any maximal abelian ideal $\mathfrak{a}$ containing $[\mathfrak{g}, \mathfrak{g}]$.

Theorem 10. *Let $\mathfrak{g}$ be a metabelian stratified Lie algebra and let $\mathfrak{a} \subset \mathfrak{g}$ be a maximal abelian ideal containing $[\mathfrak{g}, \mathfrak{g}]$. If $\dim \mathrm{Z}(\mathfrak{g}) = 1$, then $\mathfrak{g}$ is $\mathfrak{a}$-simple. Consequently, let $\mathbb{G}$ be a metabelian Carnot group and let $\mathbb{A} \subset \mathbb{G}$ be a maximal abelian normal subgroup containing $[\mathbb{G}, \mathbb{G}]$. If $\mathrm{Z}(\mathbb{G})$ is one-dimensional, then $\mathbb{G}$ is $\mathbb{A}$-simple.*

The proof of the theorem relies on two lemmas given below. In order to state these results, we recall that the ***second center*** of a Lie algebra $\mathfrak{g}$ and the ***centralizer*** of a subset $\mathfrak{h} \subset \mathfrak{g}$ are given by

$$\mathrm{Z}_2(\mathfrak{g}) := \{W \in \mathfrak{g} : [\mathfrak{g}, [\mathfrak{g}, W]] = 0\},$$
$$\mathrm{C}(\mathfrak{h}, \mathfrak{g}) := \{W \in \mathfrak{g} : [\mathfrak{h}, W] = 0\}.$$

It is easily checked that $\mathrm{Z}_2(\mathfrak{g})$ is a subalgebra. Moreover, $\mathrm{Z}_2(\mathfrak{g})$ is a 2-step nilpotent ideal of $\mathfrak{g}$ and $\mathrm{Z}(\mathfrak{g}) \subset \mathrm{Z}_2(\mathfrak{g})$ (see e.g. [14]).

An essential tool in representation theory is the famous result known as Kirillov's Lemma [18, Lemma 1.1.12]. More recently, I. Beltitia and D. Beltita made the following generalization.

Lemma 11 (Theorem 3.1, [12])**.** ***(Generalization of Kirillov's Lemma for Nilpotent algebras)*** *Let $\mathfrak{g}$ be a nilpotent Lie algebra with step larger than two, and $\dim \mathrm{Z}(\mathfrak{g}) = 1$. Pick the linear subspaces $\mathfrak{W}$ and $\mathfrak{D}$ such that*

$$\mathrm{Z}_2(\mathfrak{g}) = \mathfrak{D} \oplus \mathrm{Z}(\mathfrak{g}) \quad \textit{and} \qquad \mathfrak{g} = \mathfrak{W} \oplus \mathrm{C}(\mathrm{Z}_2(\mathfrak{g}), \mathfrak{g}). \tag{4.11}$$

Then, $\dim \mathfrak{W} = \dim \mathfrak{D}$. Moreover, for fixed $0 \neq Z \in \mathrm{Z}(\mathfrak{g})$, if $Y_1, \cdots, Y_n$ is a basis for $\mathfrak{D}$, then there exists a unique basis $X_1, \cdots, X_n$ of $\mathfrak{W}$ such that $[X_i, Y_j] = \delta_{ij} Z$.

To apply the above result in our setting, we must first establish the relation between the second center $\mathrm{Z}_2(\mathfrak{g})$ and the maximal abelian ideal $\mathfrak{a}$ in the statement of Theorem 10. This is done in the following lemma which relies on our additional assumptions that, in addition to being to being metabelian and having 1-dimensional center, the Lie algebra $\mathfrak{g}$ is stratifiable and the maximal abelian ideal $\mathfrak{a}$ contains $[\mathfrak{g}, \mathfrak{g}]$.

Lemma 12. *Let $\mathfrak{g}$ be a metabelian stratified Lie algebra of step $s > 2$ and let $\mathfrak{a} \subset \mathfrak{g}$ be a maximal abelian ideal containing $[\mathfrak{g}, \mathfrak{g}]$. If $\dim \mathrm{Z}(\mathfrak{g}) = 1$, then*

(a) *$\mathrm{Z}_2(\mathfrak{g}) \subseteq \mathfrak{a}$,*
(b) *$\mathfrak{a} = \mathrm{C}(\mathrm{Z}_2(\mathfrak{g}), \mathfrak{g})$.*

Proof. (a) Consider the stratification

$$\mathfrak{g} = \mathrm{V}_1 \oplus \cdots \oplus \mathrm{V}_s,$$

and note that the condition that $\dim \mathrm{Z}(\mathfrak{g}) = 1$ implies that $\mathrm{V}_s = \mathrm{Z}(\mathfrak{g})$. Indeed, by the grading $\mathrm{V}_s \subseteq \mathrm{Z}(\mathfrak{g})$ and since V_s is non-trivial we must have $\mathrm{V}_s = \mathrm{Z}(\mathfrak{g})$.

Next, we claim that we also have $\mathrm{Z}_2(\mathfrak{g}) = \mathrm{V}_{s-1} \oplus \mathrm{V}_s$. To see this, note that for every stratified Lie algebra of step s one has $\mathrm{V}_{s-1} \oplus \mathrm{V}_s \subseteq \mathrm{Z}_2(\mathfrak{g})$. On the other hand, if $W \in \mathrm{Z}_2(\mathfrak{g})$, then, by definition of $\mathrm{Z}_2(\mathfrak{g})$, we have $[\mathfrak{g}, W] \subset \mathrm{Z}(\mathfrak{g}) = \mathrm{V}_s$. In particular, $[\mathrm{V}_1, W] \subset \mathrm{V}_s$ which by the grading implies that $W \in \mathrm{V}_{s-1} \oplus \mathrm{V}_s$. Therefore, $\mathrm{Z}_2(\mathfrak{g}) \subseteq \mathrm{V}_{s-1} \oplus \mathrm{V}_s$, proving that $\mathrm{Z}_2(\mathfrak{g}) = \mathrm{V}_{s-1} \oplus \mathrm{V}_s$ as claimed.

Finally, for any stratified Lie algebra we always have

$$[\mathfrak{g}, \mathfrak{g}] = \mathrm{V}_2 \oplus \cdots \oplus \mathrm{V}_s\,. \tag{4.12}$$

Considering that $s > 2$ we conclude that $\mathrm{Z}_2(\mathfrak{g}) = \mathrm{V}_{s-1} \oplus \mathrm{V}_s \subseteq [\mathfrak{g}, \mathfrak{g}]$. Combining this with the hypothesis that $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{a}$ implies that $\mathrm{Z}_2(\mathfrak{g}) \subseteq \mathfrak{a}$ as required.

(b) We begin by observing that the hypothesis that $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{a}$ together with Eq. (4.12) imply

$$\mathrm{V}_2 \oplus \cdots \oplus \mathrm{V}_s \subseteq \mathfrak{a}. \tag{4.13}$$

We now show that there exists a direct complement $\mathfrak{X}$ of $\mathfrak{a}$ in $\mathfrak{g}$ which is contained in V_1. In other words, $\mathfrak{X}$ is such that

$$\mathfrak{g} = \mathfrak{X} \oplus \mathfrak{a}, \qquad \mathfrak{X} \subseteq \mathrm{V}_1\,. \tag{4.14}$$

For this matter, simply define $\mathfrak{X}$ as any direct complement of $\mathrm{V}_1 \cap \mathfrak{a}$ in V_1. Namely, $\mathfrak{X}$ is chosen such that

$$\mathrm{V}_1 = (\mathrm{V}_1 \cap \mathfrak{a}) \oplus \mathfrak{X}.$$

It is clear that this definition of $\mathfrak{X}$ satisfies $\mathfrak{X} \cap \mathfrak{a} = \{0\}$ and $\mathfrak{X} \subseteq \mathrm{V}_1$ as required. Finally, we check that $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{X}$ by using Grassmann's formula to compute

$$\begin{aligned}\dim \mathfrak{X} &= \dim \mathrm{V}_1 - \dim(\mathrm{V}_1 \cap \mathfrak{a}) \\ &= \dim \mathrm{V}_1 - (\dim \mathrm{V}_1 + \dim \mathfrak{a} - \dim(\mathrm{V}_1 + \mathfrak{a})) \\ &= \dim \mathfrak{g} - \dim \mathfrak{a},\end{aligned}$$

where, in the last equality we have used $\mathrm{V}_1 + \mathfrak{a} = \mathfrak{g}$ which is an easy consequence of (4.13).

We now prove that our hypotheses on $\mathfrak{a}$ imply

$$\mathrm{C}(\mathfrak{a}, \mathfrak{g}) = \mathfrak{a}. \tag{4.15}$$

Given that $\mathfrak{a}$ is abelian, we obviously have $\mathfrak{a} \subseteq \mathrm{C}(\mathfrak{a}, \mathfrak{g})$. On the other hand, if $0 \neq X \in \mathfrak{X} \subset \mathrm{V}_1$, we claim that there exists $Y \in \mathfrak{a}$ such that $[X, Y] \neq 0$. Indeed, if this were not the case, then $\tilde{\mathfrak{a}} := \mathfrak{a} \oplus \langle X \rangle$ would be an abelian subalgebra. Moreover, since $X \in \mathrm{V}_1$ and $\mathfrak{g}$ is stratified, it is easy to check, using (4.13), that $\tilde{\mathfrak{a}}$ is an ideal, contradicting the maximality of $\mathfrak{a}$. This, together with the decomposition $\mathfrak{g} = \mathfrak{X} \oplus \mathfrak{a}$, allows us to conclude that if $W \in \mathfrak{g} \setminus \mathfrak{a}$, there exists $Y \in \mathfrak{a}$ such that $[W, Y] \neq 0$, which is equivalent to the statement $\mathfrak{a} \supseteq \mathrm{C}(\mathfrak{a}, \mathfrak{g})$ which proves (4.15).

Next, using that $\mathrm{Z}_2(\mathfrak{g}) \subseteq \mathfrak{a}$ (as established in item (a)), we immediately obtain $\mathrm{C}(\mathfrak{a}, \mathfrak{g}) \subseteq \mathrm{C}(\mathrm{Z}_2(\mathfrak{g}), \mathfrak{g})$, which in view of (4.15) is equivalent to

$$\mathfrak{a} \subseteq \mathrm{C}(\mathrm{Z}_2(\mathfrak{g}), \mathfrak{g}), \tag{4.16}$$

so we only need to prove that the opposite inclusion holds. We will show that

$$\mathfrak{X} \cap \mathrm{C}(\mathrm{Z}_2(\mathfrak{g}), \mathfrak{g}) = \{0\}, \tag{4.17}$$

which together with (4.14) and (4.16) implies $\mathrm{C}(\mathrm{Z}_2(\mathfrak{g}), \mathfrak{g}) \subseteq \mathfrak{a}$ as required. We will make use the identities

$$\mathrm{Z}(\mathfrak{g}) = \mathrm{V}_s, \qquad \mathrm{Z}_2(\mathfrak{g}) = \mathrm{V}_{s-1} \oplus \mathrm{V}_s,$$

established in the proof of item (a) above.

Proving (4.17) amounts to showing that for every $0 \neq X \in \mathfrak{X}$, there exists $Y \in \mathrm{Z}_2(\mathfrak{g})$ such that $[X, Y] \neq 0$. Fix then $0 \neq X \in \mathfrak{X}$ and let us construct such Y. In view of (4.15), the condition $X \notin \mathfrak{a}$ implies that there exists $Y_1 \in \mathfrak{a}$ such that $[X, Y_1] \neq 0$. Suppose $[X, Y_1] \in \mathrm{Z}(\mathfrak{g}) = \mathrm{V}_s$. Considering that $X \in \mathrm{V}_1$ (by (4.14)), the stratification of $\mathfrak{g}$ implies that

$Y_1 \in \mathrm{V}_{s-1} \subset \mathrm{Z}_2(\mathfrak{g})$, so we may take $Y := Y_1$. If instead, $[X, Y_1] \notin \mathrm{Z}(\mathfrak{g})$ then there exists $X_1 \in \mathfrak{g}$ such that $[X_1, [X, Y_1]] \neq 0$. Considering that $[X, Y_1] \in \mathfrak{a}$ and $\mathfrak{a}$ is abelian, we must have $X_1 \notin \mathfrak{a}$ and we may assume $X_1 \in \mathfrak{X}$. By the Jacobi identity, we have

$$\begin{aligned} 0 \neq [X_1, [X, Y_1]] &= [[X_1, X], Y_1] + [X, [X_1, Y_1]] \\ &= [X, [X_1, Y_1]], \end{aligned}$$

where we have used that $[[X_1, X], Y_1] = 0$ since $[X_1, X] \in [\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{a}$ and $\mathfrak{a}$ is abelian. Therefore, the vector $Y_2 := [X_1, Y_1] \in \mathfrak{a}$ is such that $[X, Y_2] \neq 0$. If $[X, Y_2] \in \mathrm{Z}(\mathfrak{g})$, then we argue as above to conclude that $Y_2 \in \mathrm{Z}_2(\mathfrak{g})$ and we take $Y := Y_2$. Otherwise, we repeat the construction to find $X_2 \in \mathfrak{X}$ such that $Y_3 := [X_2, [X_1, Y_1]] \in \mathfrak{a}$ satisfies $[X, Y_3] \neq 0$, and so on. In this way, we construct a list of vectors $X_1, \ldots, X_{a-1} \in \mathfrak{X}$ such that $Y_a := [X_{a-1}, [X_{a-2}, \cdots, [X_1, Y_1] \cdots]] \in \mathfrak{a}$ satisfies $[X, Y_a] \neq 0$. The process must finalize at some point due to the grading of the algebra. In other words, there exists a certain $r \geq 1$ for which $0 \neq [X, Y_r] \in \mathrm{V}_s = \mathrm{Z}(\mathfrak{g})$ and, repeating the argument above, we conclude that $Y := Y_r \in \mathrm{Z}_2(\mathfrak{g})$.

□

We are now ready to give the proof of Theorem 10.

Proof of Theorem 10 . Every nilpotent Lie algebra of step 2 and 1-dimensional center is a Heisenberg algebra [12, Remark 2.4]. We showed that the Heisenberg algebra is $\mathfrak{a}$-simple in Section 4.3.1. Therefore, the statement is true for the step 2 case.

If the step $s \geq 3$, we apply Lemmas 11 and 12. Specifically, let $\mathfrak{X}$ be any complement of $\mathfrak{a}$ in $\mathfrak{g}$ and $\mathfrak{D}$ be any subspace such that $\mathrm{Z}_2(\mathfrak{g}) = \mathfrak{D} \oplus \mathrm{Z}(\mathfrak{g})$ (as in the statement of Lemma 11). Let $Y_1, \ldots, Y_n$ be a basis of $\mathfrak{D} \subset \mathrm{Z}_2(\mathfrak{g})$. Considering that $\mathfrak{a} = \mathrm{C}(\mathrm{Z}_2(\mathfrak{g}), \mathfrak{g})$ (by Lemma 12), and taking $\mathfrak{W} = \mathfrak{X}$, we conclude from Lemma 11 the existence of a basis $X_1, \ldots, X_n$ of $\mathfrak{X}$ such that $0 \neq [X_i, Y_i] \in \mathrm{Z}(\mathfrak{g})$. But the linearly independent vectors $Y_1, \ldots, Y_n \in \mathfrak{a}$ since $\mathrm{Z}_2(\mathfrak{g}) \subseteq \mathfrak{a}$ by Lemma 12. Hence, the Definition 5 of $\mathfrak{a}$-simple algebra is satisfied. □

4.3.4. *The jet space $\mathcal{J}^k(\mathbb{R}^n, \mathbb{R}^m)$.* It is well-known that the Heisenberg group $\mathbb{H}^3$ is diffeomorphic as a Carnot group to the jet-space $\mathcal{J}^1(\mathbb{R}, \mathbb{R})$ [19]. In this Section, we will briefly introduce the jet space $\mathcal{J}^k(\mathbb{R}^n, \mathbb{R}^m)$ as a Carnot group and show it is metabelian. We refer the reader to [50] for a more extensive explanation of the Carnot structure of the jet spaces. Our goal in this section prove that the jet space $\mathcal{J}^k(\mathbb{R}^n, \mathbb{R}^m)$ is $\mathbb{A}$-simple which is the content of the following.

Theorem 13. *The jet-space $\mathcal{J}^k(\mathbb{R}^n, \mathbb{R}^m)$ is an $\mathbb{A}$-simple group.*

Let us briefly recall the preliminaries to prove the theorem. If $U \subseteq \mathbb{R}^n$ is an open set and $x_0 \in U$, then we say that two functions $\mathbf{f}, \mathbf{g} \in C^k(U, \mathbb{R}^m)$

are equivalent at x_0, denoted $\mathbf{f} \sim_{x_0} \mathbf{g}$, if and only if their Taylor expansions of order k at x_0 are equal. The k***-jet space*** over U is given by

$$\mathcal{J}^k(U, \mathbb{R}^m) = \bigcup_{x_0 \in U} C^k(U, \mathbb{R}^m)/\sim_{x_0} .$$

We will denote elements in $\mathcal{J}^k(U, \mathbb{R}^m)$ by $j^k_{x_0}(\mathbf{f})$.

Let us make this construction in detail for the case $\mathcal{J}^k(\mathbb{R}^n, \mathbb{R})$ for simplicity. The number of partial derivatives of order k for a function $f : \mathbb{R}^n \to \mathbb{R}$ is

$$\binom{n+k-1}{k} = \frac{(n+k-1)!}{k!(n-1)!}.$$

If I is a k-index, i.e., $I = (i_1, \cdots, i_n)$ satisfies $|I| = i_1 = \cdots + i_n = k$, then we will use the following notation

$$\partial_I f(x_0) = \frac{\partial^k f}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}}(x_0).$$

We denote the set of k-indexes by $I(k)$ and let

$$\overline{I}(k) = I(0) \cup \cdots \cup I(k).$$

For $I \in \overline{I}(k)$ and $t \in \mathbb{R}^n$, we define

$$I! = i_1! i_2! \cdots i_n!, \text{ and } t^I = (t_1)^{i_1}(t_2)^{i_2} \cdots (t_n)^{i_n}.$$

The k-th oder taylor polynomial of f at x_0 is given by

$$T^k_{x_0}(f)(t) = \sum_{I \in \overline{I}(k)} \partial_I f(x_0) \frac{(t-x_0)^I}{I!}.$$

Therefore, two functions $f \sim_{x_0} g$ if and only if $T^k_{x_0}(f)(t) = T^k_{x_0}(g)(t)$.

If $\mathbf{f} = (f^1, \ldots, f^m)$ is a map $f : U \to \mathbb{R}^m$, then we apply the above construction to the coordinates $f^\ell : U \to \mathbb{R}$. We can endow $\mathcal{J}^k(\mathbb{R}^n, \mathbb{R}^m)$ with global coordinates as follows, we denote by $(x, u^{(k)})$ the coordinates of the point $T^k_{x_0}(\mathbf{f})(t)$, where

$$x(j^k_{x_0}(\mathbf{f})) = x_0, \text{ and } u^\ell_I(j^k_{x_0}(\mathbf{f})) = \partial_I f^\ell(x_0), \text{ for } I \in \overline{I}(k), \ \ \ell = 1, \ldots, m.$$

So the formal definition of $u^{(k)}$ is the following

$$u^{(k)} := \{u^\ell_I : I \in \overline{I}(k), \ \ \ell = 1, \ldots, m\}.$$

The jet space $\mathcal{J}^k(U, \mathbb{R}^m)$ has a natural distribution $\mathcal{D}^k_{x_0}$defined by the following set of Pfaffian equations

$$0 = du^\ell_I - \sum_{i=1}^n u^\ell_{I+e_i} dx^i, \text{ for all } I \in \overline{I}(k-1), \text{ and } \ell = 1, \cdots, m.$$

The distribution $\mathcal{D}^k_{x_0}$ has rank $n + m(\frac{(n+k-1)!}{k!(n-1)!})$, and is globally framed by the vector fields

$$
\begin{aligned}
X_i &:= \frac{\partial}{\partial x_i} + \sum_{\ell=1}^{m} \sum_{I \in \overline{I}(k-1)} u_{I+e_i} \frac{\partial}{\partial u^\ell_I}, \text{ where } i = 1, \cdots, n, \\
Y^\ell_I &:= \frac{\partial}{\partial u^\ell_I}, \text{ where } I \in \overline{I}(k), \text{ and } \ell = 1, \cdots, m.
\end{aligned}
\tag{4.18}
$$

The non-trivial commutators are

$$
[Y^\ell_{I+e_i}, X_i] = Y^\ell_I, \; I \in \overline{I}(k-1), \text{ and } \ell = 1, \cdots, m. \tag{4.19}
$$

Evaluating these vector fields at the origin $(x, u^{(k)}) = (0, 0)$, we define the Lie algebra $\mathfrak{g} := \mathfrak{j}^k(\mathbb{R}^n, \mathbb{R}^m)$ having the same commutation relations. It admits the stratification

$$
\mathfrak{g} = V_1 \oplus \cdots \oplus V_k,
$$

whose layers are given by

$$
\begin{aligned}
V_1 &= \operatorname{span}\left\{\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_m}\right\} \oplus \operatorname{span}\left\{\frac{\partial}{\partial u^\ell_I}\right\}_{I \in I(k)}, \\
V_j &= \left\{\frac{\partial}{\partial u^\ell_I} : \; I \in I(k-j+1) \text{ and } \ell = 1, \cdots, m\right\}, \text{ where } j = 2, \cdots, k.
\end{aligned}
$$

The Lie bracket relations show that $V_{j+1} = [V_1, V_j]$, where $j = 1, \cdots, k$, and $0 = [V_i, V_j]$ for all $i, j > 1$. It follows that $\mathfrak{g}$ is a k-step stratified metabelian nilpotent Lie algebra, so $\mathbb{G} := \mathcal{J}^k(\mathbb{R}^n, \mathbb{R}^m)$ is a metabelian Carnot group. Actually, the maximal abelian ideal $\mathfrak{a}$ and the center $\mathrm{Z}(\mathfrak{g})$ are given by

$$
\mathfrak{a} = \langle Y^\ell_I \rangle_{I \in \overline{I}(k), \ell=1,\ldots,m}, \qquad \mathrm{Z}(\mathfrak{g}) = \langle Y^\ell_I \rangle_{I \in I(0), \ell=1,\ldots,m},
$$

where it is understood that the vector fields Y^ℓ_I defined by (4.18) are evaluated at the origin. The group multiplication of $\mathcal{J}^k(\mathbb{R}^n, \mathbb{R}^m)$ may be determined via the Baker–Campbell–Hausdorff formula, see [50, Section 4].

We now present the proof of Theorem 13.

Proof of Theorem 13. If we define $\mathfrak{X} := \langle X_i \rangle_{i=1,\ldots,n}$ with the vector fields X_i defined by (4.18) evaluated at the origin, then we have $\mathfrak{g} = \mathfrak{X} \oplus \mathfrak{a}$ and the commutation relations (4.19) show that the definition of $\mathfrak{a}$-simple is satisfied if we take $Y_j := Y^1_{e_j} \in \mathfrak{a}$. Indeed, eq. (4.19) implies that

$$
[X_i, Y_j] = \delta_{ij} Y^1_0 \in \mathrm{Z}(\mathfrak{g}), \text{ for } i, j = 1, \cdots, n.
$$

Therefore, $\mathcal{J}^k(\mathbb{R}^n, \mathbb{R}^m)$ is $\mathbb{A}$-simple. $\square$

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