NO PERIODIC GEODESICS IN JET SPACE

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Abstract. The J^k space of k-jets of a real function of one real variable x admits the structure of a subRiemannian manifold, which then has an associated Hamiltonian geodesic flow, and it is integrable. As in any Hamiltonian flow, a natural question is the existence of periodic solutions. Does J^k have periodic geodesics? This study will find the action-angle coordinates in T^*J^k for the geodesic flow and demonstrate that geodesics in J^k are never periodic.

1. Introduction

This paper is the first attempt to prove that Carnot groups do not have periodic subRiemannian geodesics; Enrico Le Donne made this conjecture. Here, we will establish the first case we found, which also has a simple and elegant proof.

This work is the continuation of [1, 2], in [1] J^k was presented as subRiemannian manifold, the subRiemannian geodesic flow was defined, and its integrability was verified. In [2], the subRiemannian geodesics in J^k were classified, and some of their minimizing properties were studied. The main goal of this paper is to prove:

Theorem A. *J^k does not have periodic geodesics.*

Following the classification of geodesics from [2] (see pg. 5), the only candidates to be periodic are the ones called x-periodic (the other geodesics are not periodic on the x-coordinate); so we are focusing on the x-periodic geodesics.

An essential tool during this work is the bijection made by Monroy-Perez and Anzaldo-Meneses [3, 4, 5], also described in [2] (see pg. 4), between geodesics on J^k and the pair (F, I) (module translation $F(x) \to F(x - x_0)$), where $F(x)$ is a polynomial of degree bounded by k and I is a closed interval called Hill interval. Let us formalize its definition.

Key words and phrases. Carnot group, Jet space, integrable system, Goursat distribution, sub Riemannian geometry, Hamilton-Jacobi, periodic geodesics.

Definition 1. *A closed interval I is called Hill interval of $F(x)$, if for each x inside I then $F^2(x) < 1$ and $F^2(x) = 1$ if x is in the boundary of I.*

By definition, the Hill interval I of a constant polynomial $F^2(x) = c^2 < 1$ is $\mathbb{R}$, while the Hill interval I of the constant polynomial $F(x) = \pm 1$ is a single point. Also, I is compact, if and only if, $F(x)$ is not a constant polynomial; in this case, if I is in the form $[x_0, x_1]$, then $F^2(x_1) = F^2(x_0) = 1$. This terminology comes from celestial mechanics, and I is the region where the dynamics governed by the fundamental equation (3.5) take place.

Geodesics corresponding to constant polynomials are called horizontal lines since their projection to (x, θ_0) planes are lines. In particular, geodesic corresponding to $F(x) = \pm 1$ are abnormal geodesics (see [6], [7] or [8]). Then this work will be restricted to geodesics associated with non-constant polynomials. x-periodic geodesics correspond to the pair $(F, [x_0, x_1])$, where x_0 and x_1 are regular points of $F(x)$, which implies they are simple roots of $1 - F^2(x)$.

Outline of the paper. In Section 2, Proposition 1 is introduced and Theorem A is proved. The main purpose of Section 3 is to prove Proposition 1. In sub-Section 3.1, the subRiemannian structure and the subRiemannian Hamiltonian geodesic function are introduced. In sub-Section 3.2, a generating function is presented and a canonical transformation from traditional coordinates in T^*J^k to action-angle coordinates (μ, ϕ) for the Hamiltonian systems are shown. In sub-Section 3.3, Proposition 1 is proved.

Acknowledgments. I want to express my gratitude to Enrico Le Donne for asking us about the existence of periodic geodesics and thus posing the problem. I want to thank my advisor Richard Montgomery for his invaluable help. This paper was developed with the support of the scholarship (CVU 619610) from "Consejo de Ciencia y Tecnologia" (CONACYT).

2. Proof of theorem A

Throughout the work the alternate coordinates $(x, \theta_0, \cdots, \theta_k)$ will be used, the meaning of which meaning is introduced in the Section 3 and described in more detail in [3, 4] or [2]. x-periodic geodesics have the property that the change undergone by the coordinates θ_i after one x-period is finite and does not depend on the initial point. We summarize the above discussion with the following proposition.

Proposition 1. *Let $\gamma(t) = (x(t), \theta_0(t), \cdots, \theta_k(t))$ in J^k be an x-periodic geodesic corresponding to the pair (F, I). Then the x-period is*

$$L(F, I) = 2 \int_I \frac{dx}{\sqrt{1 - F^2(x)}}, \tag{2.1}$$

Moreover, it is twice the time it takes for the x-curve to cross its Hill interval exactly once. After one period, the changes $\Delta\theta_i := \theta_i(t_0 + L) - \theta_i(t_0)$ for $i = 0, 1, \ldots, k$ undergone by θ_i are given by

$$\Delta\theta_i(F, I) = \frac{2}{i!} \int_I \frac{x^i F(x) dx}{\sqrt{1 - F^2(x)}}. \tag{2.2}$$

In [2], a subRiemannian manifold $\mathbb{R}^3_F$, called magnetic space, was introduced and a similar statement like Proposition 1 was proved, see Proposition 4.1 from [2] (pg. 13), with an argument of classical mechanics, see [9] page 25 equation (11.5).

1 implies that a x-periodic geodesic $\gamma(t)$ corresponding to the pair (F, I) is periodic if and only if $\Delta\theta_i(F, I) = 0$ for all i.

Because that period L from equation (2.1) is finite, we can define an inner product in the space of polynomials of degree bounded by k in the following way;

$$< P_1(x), P_2(x) >_F := \int_I \frac{P_1(x) P_2(x) dx}{\sqrt{1 - F^2(x)}}. \tag{2.3}$$

This inner product is non-degenerate and will be the key to the proof of theorem A.

2.1. **Proof of Theorem A.**

Proof. We will proceed by contradiction. Let us assume $\gamma(t)$ is a periodic geodesic on J^k corresponding to the pair (F, I), where $F(x)$ is not constant, then $\Delta\theta_i(F, I) = 0$ for all i in $0, \cdots, k$.

In the context of the space of polynomials of degree bounded by k with inner product $<, >_F$, the condition $\Delta\theta_i(F, I) = 0$ is equivalent to $F(x)$ being perpendicular to x^i ($0 = \Delta\theta_i(F, I) =< x^i, F(x) >_F$), so $F(x)$ being perpendicular to x^i for all i in $0, 1, \cdots, k$. However, the set $\{x^i\}$ with $0 \leq i \leq k$ is a base for the space of polynomials with degree bounded by k, then $F(x)$ is perpendicular to any vector, so $F(x)$ is zero since the inner product is non-degenerate. Being $F(x)$ equals 0 contradicts the assumption that $F(x)$ is not a constant polynomial. □

Coming work: The proof of the conjecture in the meta-abelian group $\mathbb{G}$, that is, $\mathbb{G}$ is such that $0 = [[\mathbb{G}, \mathbb{G}], [\mathbb{G}, \mathbb{G}]]$.

3. Proof of proposition 1

3.1. **J^k as a subRiemannian manifold.** The subRiemannian structure on J^k will be here briefly described. For more details, see [1, 2]. We see J^k as $\mathbb{R}^{k+2}$, using $(x, \theta_0, \cdots, \theta_k)$ as global coordinates, then J^k is endowed with a natural rank 2 distribution $D \subset TJ^k$ characterized by the k Pafaffian equations

$$0 = d\theta_i - \frac{1}{i!}x^i d\theta_0, \qquad i = 1, \cdots, k. \tag{3.1}$$

D is globally framed by two vector fields

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = \sum_{i=0}^{k} \frac{x^i}{i!}\frac{\partial}{\partial \theta_i}. \tag{3.2}$$

A subRiemannian structure on $\mathcal{J}^k$ is defined by declaring these two vector fields to be orthonormal . In these coordinates the subRiemannian metric is given by restricting $ds^2 = dx^2 + d\theta_0^2$ to D.

3.1.1. *Sub-Riemannian geodesic flow.* Here it is emphasized that the projections of the solution curves for the Hamiltonian geodesic flow are geodesics, that is, if $(p(t), \gamma(t))$ is a solution for the Hamiltonian geodesic flow then $\gamma(t)$ is a geodesic on J^k.

Let $(p_x, p_{\theta_0}, \cdots, p_{\theta_k}, x, \theta_0, \cdots, \theta_k)$ be the traditional coordinates on T^*J^k, or in short way as (p, q). Let $P_1, P_2 : T^*J^k \to \mathbb{R}$ be the momentum functions of the vector fields X_1, X_2, see [6] 8 pg or see [10], in terms of the coordinates (p, q) are given by

$$P_1(p, q) := p_x, \qquad P_2(p, q) := \sum_{i=0}^{k} p_{\theta_i}\frac{x^i}{i!}. \tag{3.3}$$

Then the Hamiltonian governing the geodesic on J^k is

$$H_{sR}(p, q) := \frac{1}{2}(P_1^2 + P_2^2) = \frac{1}{2}p_x^2 + \frac{1}{2}(\sum_{i=0}^{k} p_{\theta_i}\frac{x^i}{i!})^2. \tag{3.4}$$

It is noteworthy that $h = 1/2$ implies that the geodesic is parameterized by arc-length. It can be noticed that H does not depend on θ_i for all i, then p_θ's define a $k + 1$ constants of motion.

Lemma 1. *The subRiemannian geodesic flow in J^k is integrable, if $(p(t), \gamma(t))$ is a solution then*

$$\dot{\gamma}(t) = P_1(t)X_1 + P_2(t)X_2, \qquad (P_1(t), P_2(t)) = (p_x(t), F(x(t))),$$

where $p_{\theta_i} = i!a_i$ and $F(x) = \sum_{i=0}^{k} a_i x^i$.

Proof. H does not depend on t and θ_i for all i, so $h := H_{sR}$ and p_{θ_i} are constants of motion, thus the Hamiltonian system is integrable. First equation form the Lemma 1 is consequence that P_1 and P_2 are linear in p_x and p_θ's. We denote by $(a_0, \cdots, a_k)$ the level set $i!a_i = p_{\theta_i}$, then by definition of P_1 and P_2 given by equation 3.3. □

3.1.2. *Fundamental equation.* The level set $(a_0, \cdots, a_k)$ defines a fundamental equation

$$H_F(p_x, x) := \frac{1}{2}p_x^2 + \frac{1}{2}F^2(x) = H|_{(a_0,\cdots,a_k)}(p, q) = \frac{1}{2}. \tag{3.5}$$

Here $H_F(p_x, x)$ is a Hamiltonian function in the phase plane (p_x, x), where the dynamic of $x(s)$ takes place in the Hill region $I = [x_0, x_1]$ and its solution $(p_x(t), x(t))$ with energy $h = 1/2$ lies in an algebraic curve or loop given by

$$\alpha_{(F,I)} := \{(p_x, x) : \frac{1}{2} = \frac{1}{2}p_x^2 + \frac{1}{2}F^2(x) \text{ and } x_0 \leq x \leq x_1\}, \tag{3.6}$$

and $\alpha_{(F,I)}$ is close and simple.

Lemma 2. *$\alpha(F, I)$ is smooth if and only if x_0 and x_1 are regular points of $F(x)$, in other words, $\alpha(F, I)$ is smooth if and only if the corresponding geodesic $\gamma(t)$ is x-periodic.*

Proof. A point $\alpha = (p_x, x)$ in $\alpha(F, I)$ is smooth if and only

$$0 \neq \nabla H_F(p_x, x)|_{\alpha(F,I)} = (p_x, F(x)F'(x)),$$

then α is smooth for all $p_x \neq 0$, the points $\alpha(F, I)$ such that $p_x = 0$ correspond to endpoints of the Hill interval I, since the condition $p_x = 0$ implies $F^2(x) = 1$, the point $\alpha = (0, x_0)$ is smooth if $F'(x_0) \neq 0$, as well as, the point $\alpha = (0, x_1)$ is smooth if $F'(x_1) \neq 0$. Then $\alpha(F, I)$ is smooth if and only x_0 and x_1 are regular points of $F(x)$. Also, $\alpha(F, I)$ is smooth is equivalent to $H_F(p_x, x)|_{\alpha(F,I)}$ is never zero, which is equivalent to the Hamiltonian vector field is never zero on $\alpha(F, I)$. □

3.1.3. *Arnold-Liouville manifold.* The Arnold-Liouville manifold $M|_F$ is given by

$$M_F := \{(p, q) \in T^*J^k : \frac{1}{2} = H_F(p_x, x), \quad p_{\theta_i} = i!a_i\}.$$

In the case $\gamma(t)$ is x-periodic, M_F is diffeomorphic to $\mathbb{S}^1 \times \mathbb{R}^{k+1}$, where $\mathbb{S}^1$ is the simple closed and smooth curve $\alpha(F, I)$.

$\alpha(F, I)$ has two natural charts using x as coordinates and given by solve the equation $H_F = 1/2$ with respect of p_x, namely, $(p_x, x) = (\pm\sqrt{1 - F^2(x)}, x)$. Having this in mind,

Lemma 3. *Let $d\phi_t$ be the close one-form on $M_F \subset T^*J^k$ give by*

$$d\phi_h := \frac{p_x}{\Pi(F,I)}|_{M_F} dx = \frac{\sqrt{1-F^2(x)}}{\Pi(F,I)} dx, \tag{3.7}$$

where $\Pi(F,I)$ is the area enclosed by $\alpha(F,I)$. Then,

$$\int_{\alpha(F,I)} d\phi_h = 1 \qquad \frac{\partial}{\partial h}\Pi(F,I) = L(F,I).$$

as a consequence exist the inverse function $h(\Pi)$.

Proof. Let $\Omega(F,I)$ be the closed region by $\alpha(F,I)$, then $d\phi_h$ can be extended to $\Omega(F,I)$ and Stokes' Theorem implies

$$\begin{aligned} \Pi(F,I) := \int_{\alpha(F,I)} p_x dx &= \int_{\Omega(F,I)} dp_x \wedge dx, \\ &= 2\int_I \sqrt{2h - F^2(x)}|_{h=\frac{1}{2}} dx. \end{aligned} \tag{3.8}$$

This tell that $\int_{\alpha(F,I)} d\phi_h = 1$, thus $d\phi_h$ is not exact.

$\Pi(F,I)$ is a function of h, so

$$\frac{\partial}{\partial h}\Pi(F,I) = \frac{\partial}{\partial h}\int_I d\phi_h = \int_I \frac{2dx}{\sqrt{1-F^2(x)}}. \tag{3.9}$$

□

$\Pi(F,I)$ is also called an adiabatic invariant see [11] pg 297. We will use Π when we use it as a variable and $\Pi(F,I)$ for the adiabatic invariant.

3.2. **Action-angle variables in T^*J^k.** We will consider the actions $\mu = (\Pi, a_0, \cdots, a_k)$ and find its angle coordinates $\phi = (\phi_h, \phi_0, \cdots, \phi_k)$, such the set (μ, ϕ) of coordinates are an action-angle coordinates in T^*J^k.

Lemma 4. *There exist a canonical transformation $\Phi(p,q) = (\mu,\phi)$, where ϕ_h is the local function defined by the close form $d\phi_h$ from Lemma 3 and*

$$\phi_i = -\int^x \frac{\tilde{x}^i F(\tilde{x}) d\tilde{x}}{\sqrt{1-F^2(\tilde{x})}} + i!\theta_i \qquad x \in I \text{ and } i = 0, \cdots, k.$$

To construct the canonical transformation $\Phi(p,q)$, we will look for its generating function $S(\mu,q)$, of the second type that satisfies the three following conditions.

$$p = \frac{\partial S}{\partial q}, \quad \phi = \frac{\partial S}{\partial \mu}, \quad H(\frac{\partial S}{\partial q}, q) = h(\Pi) = \frac{1}{2}, \tag{3.10}$$

where $h(\Pi)$ is the function defined in Lema 3. For more detail on the definition of $S(\mu, q)$, see [11] Section 50 or [9].

To find $S(\mu, q)$, we will solve the subRiemannian Hamilton-Jacobi equation associated with the subRiemannian geodesic flow. For more details about the definition of this equation in subRiemannian geometry and its relations with the Eikonal equation, see [6] 8 pg or [2].

Proof. The subRiemannian Hamilton-Jacobi equation is given by

$$h|_{1/2} = \frac{1}{2}(\frac{\partial S}{\partial x})^2 + \frac{1}{2}(\sum_{i=0}^{k} \frac{x^i}{i!}\frac{\partial S}{\partial \theta_i})^2. \tag{3.11}$$

Take the ansatz

$$S(\mu, q) := f(x) + \sum_{i=0}^{k} i! a_i \theta_i,$$

as a solution. The equation (3.11) becomes equation (3.5), then the generating function is given by

$$S(\mu, q) = \int_{x_0}^{x} \sqrt{2h(\Pi) - F^2(\tilde{x})}\, d\tilde{x} + \sum_{i=0}^{n} i! a_i \theta_i \tag{3.12}$$

Here, $h(\Pi) = 1/2$ and $S(\mu, q)$ is a local function, since x must lay in the Hill region I, that is, $S(\mu, q)$ is defined in the sub-set $\mu \times I \times \mathbb{R}^{k+1}$.

We can see that conditions 1 and 3 of equation (3.10) are satisfied: $p(\mu, q) = \partial S/\partial q$ and $H(p(\mu, q), q) = h$. To find the new coordinates ϕ, we use the condition 2:

$$\frac{\partial S}{\partial h} = \int^{x} \frac{d\tilde{x}}{\sqrt{1 - F^2(\tilde{x})}} = \phi_h,$$

$$\frac{\partial S}{\partial a_i} = -\int^{x} \frac{\tilde{x}^i F(\tilde{x}) d\tilde{x}}{\sqrt{1 - F^2(\tilde{x})}} + i!\theta_i = \phi_i.$$

□

Note: In [2] a projection $\pi_F : J^k \to \mathbb{R}^3_F$ was built and the solution to the subRiemannian Hamilton-Jacobi equation on the magnetic space $\mathbb{R}^3_F$ was found. The solution given by equation (3.12) is the pull-back by π_F of the solution previously found it in $\mathbb{R}_F$, where π_F is in fact, a subRiemannian submersion.

Corollary 1. *(μ, ϕ) are action-angle coordinates.*

Proof. Using the Hamilton equations for the new coordinates (μ, ϕ), we have $\phi_t = t$ and $\phi_i = const$. □

Note: that h and ϕ_t are action-angles coordinates for the Hamiltonian H_F.

3.2.1. *Horizontal derivative.* A horizontal derivative ∇_{hor} of a function $S : J^k \to \mathbb{R}$ is the unique horizontal vector field that satisfies; for every q in J^k,

$$< \nabla_{hor} S, v >_q = dS(v), \quad \text{for } \ v \ \in D_q, \tag{3.13}$$

where $<, >_q$ is th subRiemannian metric in D_q. For more detail see [6] pg 14-15 or [10].

Lemma 5. *Let $\gamma(t)$ be a geodesic parameterized by arc-length corresponding to the pair (F, I) and S_F the solution given by equation* (3.12)*, then*

$$dS_F(\dot{\gamma})(t) = 1.$$

Proof. Let us prove that $\dot{\gamma}(t) = (\nabla_{hor} S_F)_{\gamma(t)}$, which is just a consequence that S_F is solution to the Hamilton-Jacobi equation, that is,

$$X_1(S_F)|_{\gamma(t)} = \frac{\partial S}{\partial x}|_{\gamma(t)} = p_x(t),$$

but, Lemma 1 implies that $P_1(t) = p_x(t)$, so $P_1(t) = X_1(S_F)|_{\gamma(t)}$. As well,

$$X_2(S_F)|_{\gamma(t)} = \sum_{i=0}^{k} \frac{x^i(t)}{i!} \frac{\partial S}{\partial \theta_i}|_{\gamma(t)} = \sum_{i=0}^{k} a_i x^i(t) = F(x(t)),$$

also, Lemma 1 implies that $P_2(t) = F(x(t))$, so $P_2(t) = X_2(S_F)|_{\gamma(t)}$. As a consequence;

$$\nabla_{hor} S|_{\gamma(t)} := X_1(S_F)|_{\gamma(t)} X_1 + X_2(S_F)|_{\gamma(t)} X_2 = P_1(t) X_1 + P_2(t) X_2,$$

Lemma 1 implies $P_1(t)X_1 + P_2(t)X_2 = \dot{\gamma}(t)$. Thus, $\nabla_{hor} S = \dot{\gamma}(t)$ and $dS_F(v)|_q =< \nabla_{hor} S_F, v >$ for all D_q. In particular,

$$dS_F(\dot{\gamma}) =< \dot{\gamma}(t), \dot{\gamma}(t) >= 1,$$

since t is arc-length parameter. □

3.3. **Proof of Proposition 1.**

Proof. It is well-known that the fundamental system system H_F with energy $1/2$ has period $L(F, I)$ given by equation (2.1) and the relation between $\Pi(F, I)$ and $L(F, I)$ is given by Lemma 3, see [11] pg 281. Let $\gamma(t)$ be a x-periodic corresponding to (F, I), we are interested in seeing the change suffered by the coordinates θ_i after one $L(I, F)$. For that, we consider the change in $S(\mu, q)$ after $\gamma(t)$ travel form t to $t+L(F,I)$, in other words,

$$L(F,I) = \int_t^{t+L(F,I)} dS(\dot{\gamma}(t))dt = \Pi(F,I) + \sum_{i=0}^{n} i!a_i\Delta\theta_i(F,I). \tag{3.14}$$

On the left side of the equation is a consequence of Lemma 5, and the right side is the integration term by term. The derivative of equation (3.14) with respect to a_i to find $-\frac{\partial}{\partial a_i}\Pi(F,I) = i!\Delta\theta_i$, which is equivalent to equation (2.2).

We differentiate $\Delta\theta_i := \theta_i(t+L) - \theta_i(t)$ respect to t, to see that $\Delta\theta_i(F,I)$ is independent of the initial point. The derivative is

$$\frac{x^i(t+L)F(x(t+L))}{\sqrt{1-F^2(x(t+L))}} - \frac{x^i(t)F(x(t))}{\sqrt{1-F^2(x(t))}},$$

but $x(t+L) = x(t)$. □

References

[1] Alejandro Bravo-Doddoli. Higher elastica: Geodesics in the jet space. *European Journal of Mathematics*.

[2] Alejandro Bravo-Doddoli and Richard Montgomery. Geodesics in Jet Space. *Regular and Chaotic Dynamics*, 2021.

[3] A. Anzaldo-Meneses and F. Monroy-Perez. Goursat distribution and subriemannian structures. *Journal of Mathematical Physics*, 44(12):6101–6111.

[4] A. Anzaldo-Meneses and F. Monroy-Perez. Integrability of nilpotent subriemannian structures.

[5] Felipe Monroy-Pérez and A. Anzaldo-Meneses. Optimal control on nilpotent lie groups. *Journal of Dynamical and Control Systems*, 8:487–504, 2002.

[6] Richard Montgomery. *A tour of subriemannian geometries, their geodesics and applications*. Number 91. American Mathematical Soc.

[7] Robert Bryant and Hsu Lucas L. Rigidity of integral curves of rank 2 distributions. *Inventiones mathematicae*, 114(2):435–462, 1993.

[8] Richard Montgomery and Michail Zhitomirskii. Geometric approach to goursat flags. volume 18, pages 459–493.

[9] LD Landau and EM Lifshitz. Mechanics third edition: Volume 1 of course of theoretical physics. *Elsevier Science*, 1976.

[10] Andrei Agrachev, Davide Barilari, and Ugo Boscain. *A Comprehensive Introduction to Sub-Riemannian Geometry*. Cambridge University Press, 2019.

[11] Vladimir Igorevich Arnol'd. *Mathematical methods of classical mechanics*. Springer Science.

[12] Ben Warhurst. Jet spaces as nonrigid carnot groups. *Journal of Lie Theory*, 15(1):341–356, 2005.

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