NO PERIODIC GEODESICS IN $J^k(\mathbb{R}, \mathbb{R}^n)$

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ABSTRACT. The space of k-jets of n real function of one real variable x admits the structure of a Carnot group, which then has an associated Hamiltonian geodesic flow. As in any Hamiltonian flow, a natural question is the existence of periodic solutions. Does the space of k-jets have periodic geodesics? This study will demonstrate the integrability of subRiemannian geodesic flow, characterize and classify the subRiemannian geodesics in the space of k-jets, and show that they are never periodic.

1. Introduction

This paper is the generalization of [10, 8, 9]: In [10], the space of k-jets of real function of a single variable was presented as a subRiemannian manifold, the subRiemannian geodesic flow was defined and its integablity was verified. In [8], the subRiemannian geodesics were classified and some of their minimizing properties were studied. In [9], the non-existence of periodic geodesics on the space of k-jets of a real function of a single variable was proved.

The k-jets space of n real functions of a single real variable, denoted here by $J^k(\mathbb{R}, \mathbb{R}^n)$ or J^k for short, is a $(n(k+1)+1)$-dimensional manifold endowed with a canonical rank $n+1$ distribution, i.e., a linear sub-bundle of its tangent bundle. This distribution is globally framed by n vector fields, denoted by $X_1, \cdots, X_{n+1}$ in Section 2, whose iterated Lie brackets give $J^k(\mathbb{R}, \mathbb{R}^n)$ the structure of a stratified group. Declaring $X_1, \cdots, X_{n+1}$ to be orthonormal endows $J^k(\mathbb{R}, \mathbb{R}^n)$ with the structure of a subRiemannian manifold, which is left-invariant under the group multiplication. Like any subRiemannian structure, the geodesics are projection of the solution to a Hamiltonian system defined on T^*J^k, called the geodesic flow on $J^k(\mathbb{R}, \mathbb{R}^n)$.

This paper has four main goals, the following theorem is the first.

Theorem A. *The subRiemannian geodesic flow on $J^k(\mathbb{R}, \mathbb{R}^n)$ is integrable.*

The bijection between geodesics on $J^k(\mathbb{R}, \mathbb{R})$ and the pairs (F, I) will be generalized, module translation $F(x) \to F(x - x_0)$, where $F(x)$ is a polynomial of degree k or less and I is a closed interval associated to $F(x)$, made by Monroy-Perez and Anzaldo-Meneses [2, 3, 4], also described in [8] (see pg. 4). In the present paper it will be a bijection between the geodesic

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in $J^k(\mathbb{R},\mathbb{R}^n)$ and the pairs (F,I), module translation $F(x)\to F(x-x_0)$, where $F(x)=(F^1(x),\cdots,F^n(x))$ is a polynomial vector of degree k or less and I is a closed interval associated to $F(x)$, see Definition 3.1 for more detail of I.

In Section 3, it will be described how to build a geodesic in $J^k(\mathbb{R},\mathbb{R}^n)$ given a pair (F,I) and prove the following main result.

Theorem B. *The prescription described in Section 3 yields a geodesic in $J(\mathbb{R},\mathbb{R}^n)$ parameterized by arclength. Conversely, any arc-length parameterized geodesic in $J^k(\mathbb{R},\mathbb{R}^n)$ can be achieved by this prescription applied to some polynomial vector $F(x)$ of degree k or less.*

$J^k(\mathbb{R},\mathbb{R}^n)$ comes with a projection $\Pi: J^k(\mathbb{R},\mathbb{R}^n)\to\mathbb{R}^{n+1}$ onto the Euclidean plane, which projects the frame $X_1,\cdots,X_{n+1}$ onto the standard coordinate frame $\{\partial/\partial x,\partial/\partial\theta_0^1,\cdots,\partial/\partial\theta_0^1\}$ of $\mathbb{R}^{n+1}$, see Section 2 for the meaning of the coordinates.

In subRiemannian geometry, a curve tangent to D is called C^1-rigid or singular if it is a critical point of the endpoint map, see [14] chapter 3, or [1, 11], in other words, given an initial and end points, a curve is C^1-rigid if it is the only curve tangent to D joining the given points. This property does not depend of the subRiemannian metric only on the distribution D, and it is said that the C^1-curve is minimizing regardless of how we measure length, that is, it is geodesic by virtue of its singular nature alone. Sometimes, a C^1-curve is not a solution to the geodesic equations and it is called abnormal geodesics, while, if it is also a solution to the geodesic equations it is called binormal geodesics.

It is well know that $J^1(\mathbb{R},\mathbb{R})$ does not have singular curves. In the case of $J^k(\mathbb{R},\mathbb{R})$, with $k>1$, C^1 curves are tangent to X_2 (see [15]), that is, C^1-curves correspond the the polynomial $F(x)=1$ and they are binormal geodesics. The third main result characterizes the C^1-curves in $J^k(\mathbb{R},\mathbb{R}^n)$ and ensures that they can be achieved by Theorem B.

Theorem C. *C^1-curves in $J^k(\mathbb{R},\mathbb{R}^n)$ are binormal and they correspond to constant vector polynomials such that $||F(x)||^2=1$.*

Using Theorem B, the geodesic in $J^k(\mathbb{R},\mathbb{R}^n)$ will be classified into two main families: line-geodesics and non-line-geodesics: We say that a geodesic $\gamma(t)$ is a line-geodesics if $\gamma(t)$ corresponds to a constant polynomial vector and its projection to $\mathbb{R}^{n+1}$ is a line, in particular, binormal geodesics are line-geodesics. We say that a geodesic $\gamma(t)$ is a non-line-geodesic if $\gamma(t)$ corresponds to a non-constant polynomial and its Hill interval is compact. Moreover, if $I=[x_0,x_1]$, we say that a non-line-geodesic $\gamma(t)$ is x-periodic (or regular), if x_0 and x_1 are regular points of $||F(x)||^2$, that is, exist $L(F,I)$ such that $x(t+L(F,I))=x(t)$. While, $\gamma(t)$ is critical if one point or both are critical points of $||F(x)||^2$; in this case the x-coordinate has an asymptotic behavior to the critical point and then the x-coordinate has an infinite period.

The fourth main result is the answer to a question by Enrico Le Donne: Does $J^k(\mathbb{R},\mathbb{R}^n)$ have periodic geodesics?

Theorem D. $J^k(\mathbb{R},\mathbb{R}^n)$ *does not have periodic geodesics.*

Following this classification, the only candidates to be periodic are x-periodic geodesics; so the focus is on non-constant vectors correspondig to x-periodic geodesics.

Remark 1: Viewing $J^k(\mathbb{R},\mathbb{R}^n)$ as a Carnot group, Theorem D is a particular case of the conjecture made by Enrico Le Donne.

Conjecture 1. *Carnot groups do not have periodic geodesics.*

Remark 2: In control theory a "chained normal form" is a control system that is locally diffeomorphic to the canonical distribution for $J^k(\mathbb{R},\mathbb{R}^n)$, see [16].

1.1. **Outline of paper.** The outline of the paper is as follows. In Section 2, the k-th jet space $J^k(\mathbb{R},\mathbb{R}^n)$ is presented as a subRiemannian manifold, as well as, the notation that will be followed throughout the work. The subRiemannian geodesic flow is defined and the proof of Theorem A is given. Finally, the Carnot structure of $J^k(\mathbb{R},\mathbb{R}^n)$ is presented. In Section 3, the prescription for constructing geodesic in $J^k(\mathbb{R},\mathbb{R}^n)$ given the pair (F,I) is described, the Hamilton equation are computed and Theorem B is proved. In Section 4, the abnormal equation is computed to show Theorem C. In Section 5, the proof of Theorem D is given.

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2. $J^k(\mathbb{R},\mathbb{R}^n)$ as a subRiemannian manifold

The k-jet of a smooth function $f:\mathbb{R}\to\mathbb{R}^n$ at a point $x_0\in\mathbb{R}$ is its k-th order Taylor expansion at x_0. We will this encode this k-jet as a $(k+2)$-tuple of real numbers as follows:

$$(j^k f) = (x_0, f^k(x_0), \cdots, f^1(x_0), f(x_0)) \in \mathbb{R}^{n(k+1)+1}.$$

As f varies over smooth functions and x_0 over $\mathbb{R}$, these k-jets sweep out the k-jet space. $J^k(\mathbb{R},\mathbb{R}^n)$ is diffeomorphic to $\mathbb{R}^{n(k+1)+1}$ and we will use the global coordinates

$$(x, u_k, \cdots, u_1, u_0) \in \mathbb{R}^{n(k+1)+1}.$$

Where, $u_i = (u_i^1, \cdots, u_i^n)$ and, if $f = u_0$, then $u_1 = du_0/dx$, and more general, $u_{i+1} = du_i/dx$, $j \geq 1$. These equations are rewritten into $du_0 = u_1 dx$, and in general, $du_i = u_{i+1}dx$, we see that $J^k(\mathbb{R},\mathbb{R}^n)$ is endowed with a natural rank $(n+1)$ distribution $D \subset TJ^k$ characterized by the nk Pfaffian equations

$$\begin{aligned} 0 &= du_0 - u_1 dx \\ 0 &= du_1 - u_2 dx \\ \vdots &= \vdots \\ 0 &= du_k - u_{k-1} dx. \end{aligned}$$

$J^k(\mathbb{R}, \mathbb{R}^n)$ has a natural definition using the coordinates u_i, but they do not reflect the symmetries of the dynamics, see the proof of Theorem A in Section 3. We will introduce the alternate coordinates θ_i for $J^k(\mathbb{R}, \mathbb{R}^n)$ describes in [2, 3] and also introduced in [8, 9], they are exponential coordinates of the second type, see [6] Section 6.2.;

$$\begin{aligned} \theta_0 &= u_k \\ \theta_1 &= xu_k - u_{k-1} \\ \vdots &= \vdots \\ \theta_k &= \frac{x^k}{k!} u_k - \frac{x^k}{k!} u_{k-1} dx + \cdots + (-1)^k u_0. \end{aligned}$$

D is globally framed by $(n+1)$ vector fields:

$$X_0 = \frac{\partial}{\partial x}, \qquad X_0^j = \sum_{i=0}^{k} \frac{x^i}{i!} \frac{\partial}{\partial \theta_i^j} \text{ for } 1 \leq j \leq n. \tag{2.1}$$

A subRiemannian structure on $J^k(\mathbb{R}, \mathbb{R}^n)$ is defined by declaring these $(n+1)$ vector fields to be orthonormal. In these coordinates the subRiemannian metric is defined by restricting $ds^2 = dx^2 + (d\theta_0^1)^2 + \cdots + (d\theta_0^n)^2$ to D.

During this work we will use the convention θ_i^j, where $i = 0, \cdots k$ and $j = 1, \cdots, n$, that is, i is used to denotes the vector θ_i and j denote the j-th entry of the vector θ_i

2.1. **Hamiltonian.** Let $(p_x, p_{\theta_0}, \cdots, p_{\theta_k}, x, \theta_0, \cdots, \theta_k)$ be the traditional coordinates for the cotangent bundle T^*J^k, or abbreviated as (p, q). Also, let $P_{X_0}, P_{X_0^1}, \cdots, P_{X_0^n} : T^*J^k \to \mathbb{R}$ be the momentum functions of the vector fields $X_0, X_0^1, \cdots, X_0^n$, in the coordinates (p, q); the momentum functions are given by

$$P_{X_0} = p_x, \qquad P_{X_0^j} = \sum_{i=0}^{k} \frac{x^i}{i!} p_{\theta_i^j} \text{ for } 0 \leq j \leq k. \tag{2.2}$$

Then the Hamiltonian governing the subRiemannian geodesic flow on $J^k(\mathbb{R}, \mathbb{R}^n)$ is

$$H = \frac{1}{2}(P_{X_0}^2 + P_{X_0^1}^2 + \cdots + P_{X_0^k}^2) \tag{2.3}$$

(see [14], pg 8). We will see in Section 3 that the condition $H = 1/2$ implies that the geodesics are parameterized by arc-length.

2.2. Proof of Theorem A.

Proof. The Hamiltonian H does not depend on the coordinate θ_i^j because the Hamilton equations $p_{\theta_i^j}$ is a constant of motion. Then $\{H, p_{\theta_i^j}\}$ is a set of $n(k+1)+1$ constants of motion that Poisson commute and they are linearly independent. □

2.3. Carnot Group structure. The frame $\{X_0, X_0^1, \cdots, X_0^n\}$ generates $(n(k+1)+1)$-dimensional nilpotent Lie algebra, under the iterated bracket. That is,

$$X_0^1 = [X_0, X_0^j], \ \cdots, \ X_k^j = [X_0, X_j^{k-1}], \cdots 0 = [X_0, X_k^j],$$

all the other Lie brackets $[X_m^\ell, X_i^j]$ are zero. Then the frame $\{X_0, X_i^j\}$ with $0 \leq i \leq k$ and $1 \leq j \leq n$ forms a $n(k+1)+1$-dimensional graded nilpotent Lie algebra:

$$\mathfrak{g}_k = V_1 \oplus \cdots \oplus V_{k+1}, V_1 = \{X_0, X_0^j\}, V_i = \{X_{i-1}^j\}, \ 1 \leq i \leq k, \ 1 \leq j \leq n.$$

Like any graded nilpotent Lie algebra, this algebra has an associated Lie group which is a Carnot group G w.r.t the subRiemannian structure. We can identify G with $J^k(\mathbb{R}, \mathbb{R}^n)$, using the flows of $\{X_0, X_i^j\}$. For more detail on the jets space as a Carnot group see [7].

3. Geodesic in $J^k(\mathbb{R}, \mathbb{R}^n)$

This Section describes how to build a geodesic on $J^k(\mathbb{R}, \mathbb{R}^n)$: Let us formalize the definition of the interval I.

Definition 3.1. *We say that a closed interval I is a Hill interval, associated to $F(x)$, if $F^2(x) < 1$ for all x in the interior of I and $G^2(x) = 1$ for x in the boundary of I. Then, i is compact if and only if $F(x)$ is not a constant polynomial, if I is in the form $[x_0, x_1]$, x_0 and x_1 are called endpoints of the Hill interval.*

Consider the Hamiltonian system of one degree of freedom defined on the plane phase space (p_x, x) and with potential $1/2||F(x)||^2$, in other words, a Hamiltonian function given by

$$H_F(p_x, x) = \frac{1}{2}p_x^2 + \frac{1}{2}||F(x)||^2; \tag{3.1}$$

then, the Hamilton equations are give by

$$\dot{x} = p_x, \qquad \dot{p}_x = (\frac{dF}{dx}, F(x)), \tag{3.2}$$

where dF/dx is the derivative of the polynomial vector and (,) is the Euclidean dot product on $\mathbb{R}^n$. Since the Hamiltonian is autonomous, we choose $H_F = 1/2$; then the dynamic takes place in the point where $||F(x)||^2 \leq 1$.

If $F(x)$ is not the constant polynomial vector, and $I = [x_0, x_1]$ is the Hill interval, then $\dot{x} = 0$ if and only if $x = x_0, x_1$. Moreover, x_0 and x_1 are equilibrium points, if and only if, x_0 and x_1 are critical points of $||F(x)||^2$, in other words, $0 = (dF/dx, F(x))$.

Having found the solution $x(t)$, next we solve

$$\dot{\theta}_0^j(t) = F^j(x(t)), \tag{3.3}$$

for θ_0^i. Then, $c(t) = (x(t), \theta_0(t))$ is a curve on $\mathbb{R}^{n+1}$ parameterized by arc-length. Finally, we solve the horizontal lift equation associated to the curve $c(t)$

$$\begin{aligned} \dot{\theta}_1^j &= x(t)F^j(x(t)), \\ \dot{\theta}_2^j &= \frac{x^2(t)}{2!}F^j(x(t)), \\ \vdots &= \vdots \\ \dot{\theta}_k^j &= \frac{x^k(t)}{k!}F^j(x(t)). \end{aligned} \tag{3.4}$$

3.1. **Hamilton equations.** To proof Theorem B, we need to write down the Hamilton equations for the geodesic flow. Since the Hamiltonian function 2.3 is a left invariant function on the cotangent bundle of the Lie group G, the 'Lie-Poisson bracket' structure can be used for such Hamiltonian flows to find the equations, see Appendix [5] or chapter 4 [13]. That is, if X and Y are left invariant vector fields then

$$\{P_X, P_Y\} = -P_{[X,Y]}. \tag{3.5}$$

In this context, the Hamilton equations are read as $\dot{f} = \{f, H\}$. With the Hamiltonian of this system, they expand to

$$\dot{f} = \{f, P_0\}P_0 + \{f, P_{X_0^1}\}P_{X_0^1} + \cdots \{f, P_{X_0^n}\}P_{X_0^n}.$$

Using $\{P_0, P_{X_0^j}\} = -P_{X_1^j}$, we see that P_0 and $P_{X_0^j}$ evolves according to the equations

$$\dot{P}_0 = -P_{X_0^1}P_{X_1^1} - \cdots - P_{X_0^2}P_{X_1^2} \qquad \dot{P}_{X_0^j} = P_0 P_{X_1^j} \text{ for } 1 \le j \le n. \tag{3.6}$$

For $1 < i < k$, we have $\{P_{X_i^j}, P_0\} = P_{X_{i+1}^j}$ and $\{P_{X_i^j}, P_{X_\ell^m}\} = 0$, so

$$\begin{aligned} \dot{P}_{X_3^j} &= P_0 P_{X_2^j} \\ \dot{P}_{X_4^j} &= P_0 P_{X_3^j} \\ \vdots &= \vdots \\ \dot{P}_{X_{k-1}^j} &= P_0 P_{X_k^j} \\ \dot{P}_{X_k^j} &= 0, \end{aligned} \tag{3.7}$$

for all $1 \leq j \leq n$. We also compute the Hamilton equations for the coordinates $(x, \theta_0, \cdots, \theta_n)$,

$$\dot{x} = P_0 \qquad \dot{\theta}_i^j = \frac{x^i}{i!} P_{X_0^j} \text{ for } 0 \leq i \leq k \text{ and } 1 \leq j \leq n. \tag{3.8}$$

3.2. **Proof of Theorem B.**

Proof. Let $\gamma(t)$ be a curve corresponding to the pair (F, I), that is, the coordinates $x, \theta_0^i, \theta_j^i$ are solutions to the equations (3.1), (3.3) and (3.4), we will associate to $\gamma(t)$ some momentum functions and show that they hold equations (3.6) and (3.7), respectively.

Let $(p_x(t), x(t))$ be the solution to the equation (3.2) with $x(t)$ laying in the I, comparing with the geodesic equation from (3.8), we define $P_0 := p_x$. In the same way, comparing the equations (3.3) and (3.4) with the Hamilton equations (3.8) and (3.7) for θ_0^j and θ_i^j, we define $P_{X_0^j}(t) := F^j(x(t))$ and $P_{X_i^j}(t) := \frac{d^i}{dx^i} F^j(x(t))$. Then using the change rule we have

$$\dot{P}_{X_0^j}(t) = \frac{d}{dt} F^j(x(t)) = \frac{dF^j}{dx} \dot{x} = P_{X_1^j} P_0,$$

which is the equation (3.6). In the same way

$$\dot{P}_{X_i^j}(t) = \frac{d}{dt} \frac{d^i F^j}{dx^i}(x(t)) = \frac{d^{i+1} F^j}{dx^{i+1}} \dot{x} = P_{X_1^j} P_0. \tag{3.9}$$

Since $F^i(x)$ is a polynomial of degree k or less, we obtain $\dot{P}_{X_k^j}(t) = 0$ for all $j = 1, \cdot, n$, and the equation (3.9) is the same as equation (3.7).

Conversely, let $\gamma(t)$ be a geodesic parameterized by arc-length with the initial condition $\gamma(0)$, that is, $\gamma(t)$ is the projection to the solution $(p(t), \gamma(t))$ of the Hamiltonian function (2.3), we will show that the coordinates $x, \theta_0^j, \theta_i^j$ of the geodesic $\gamma(t)$ hold the equations (3.1), (3.3) and (3.4), respectively.

Being $\gamma(t)$ a solution to the Hamilton equations $p_{\theta_j^i}(t)$ is constant, if $a_i^j := i! p_{\theta_i^j}$ and $F^j(x) := a_0^j + a_1^j x + \cdots + a_k^j x^k$ for all $1 \leq j \leq n$, then, using these expressions and $x_p = P_{X_0}$, the Hamiltonian function (2.3) became

$$H = \frac{1}{2}(P_{X_0}^2 + P_{X_0^1}^2 + \cdots + P_{X_0^k}^2) = \frac{1}{2}(p_x^2 + ||F(x)||^2) = H_F.$$

Thus the x-coordinate of the geodesic $\gamma(t)$ is a solution to the Hamiltonian system of one degree of freedom with potential $1/2||F(x)||$, defined by equation (3.1), where the initial condition $x(0)$ lays in a Hill interval I and, so does $x(t)$. In the same way, using the solution $x(t)$ and the Hamilton equation for θ_0^j, that is, $\dot{\theta}_0^j = \partial H / \partial p_{\theta_0^j} = F^j(x(t))$, thus the θ_0^j-coordinate of the geodesic $\gamma(t)$ is a solution to equation (3.3). Finally, the Hamilton equation for θ_i^j, that is, $\dot{\theta}_i^j = \partial H / \partial p_{\theta_i^j} = \frac{x^i}{i!} F^i(x(t))$ is equivalent to the horizontal equation (3.4). Thus, $\gamma(t)$ is a geodesic corresponding to the pair (F, I). $\square$

3.3. **Geodesics Classification in $J^k(\mathbb{R},\mathbb{R}^n)$.** Using the bijection between geodesics in $J^k(\mathbb{R},\mathbb{R}^n)$ and the pair (F,I), the geodesics are classified. Let $\gamma(t)$ be a geodesic corresponding to (F,I), as said before the first dichotomy is if the projected curve $\pi(\gamma(t)) = c(t)$ is a line or not.

- We say that $\gamma(t)$ is a line-geodesic if $F(x)$ is the constant polynomial vector, since equation (3.3) implies that the curve $c(t) = (x(t), \theta_0(t))$ in $\mathbb{R}^{n+1}$ is a line.
- We say that $\gamma(t)$ is a non-line-geodesic if $F(x)$ is not the constant polynomial vector with Hill interval $I = [x_0, x_1]$, since equation (3.2) implies that the x-dynamics takes place in I and curve $c(t) = (x(t), \theta_0(t))$ in $\mathbb{R}^{n+1}$ is not a line.

Let $\gamma(t)$ be a non-line-geodesic corresponding to (F,I), where $I = [x_0, x_1]$, the second dichotomy refers to the qualitative behavior of the $x(t)$ dynamic.

- We say that $\gamma(t)$ is x-periodic or regular, that is, exist $L(F,I)$ such that $x(t + L(F,I)) = x(t)$, if x_0 and x_1 are regular points of the potential $1/2||F(x)||^2$, if and only if, x_0 and x_1 are simple roots of $1 - ||F(x)||^2$, if and only if, $1 - ||F(x)||^2 = (x - x_0)(x_1 - x)q(x)$, where $q(x)$ is not zero if x is in I.
- We say that $\gamma(t)$ is critical, if one or both endpoints x_0 and x_1 are critical points of the potential $1/2||F(x)||^2$, if and only if, one or both endpoints x_0 and x_1 are not simple roots of $1 - ||F(x)||^2$. Then, by equation (3.1), the critical points are equilibrium points of a one degree of freedom system, and the solution $x(t)$ has an asymptotic behavior to the critical points.

3.3.1. *Periods.* x-periodic geodesics have the property that the change undergone by the coordinates θ^i_j after one x-period $L(F,I)$ is finite and does not depend on the initial point. This is summarized in the following proposition.

Proposition 3.1. *Let $\gamma(t) = (x(t), \theta_0(t), \cdots, \theta_k(t))$ in $J^k(\mathbb{R},\mathbb{R}^n)$ be an x-periodic geodesic corresponding to the pair (F,I). Then the x-period is*

$$L(F,I) = 2\int_I \frac{dx}{\sqrt{1 - ||F(x)||^2}}, \tag{3.10}$$

and is twice the time it takes for the x-curve to cross its Hill interval exactly once. After one period, the changes $\Delta\theta^j_i := \theta^j_i(t_0 + L) - \theta^j_i(t_0)$ for $i = 0, 1, \ldots, k$ and $j = 1, \cdots, n$ undergone by θ^j_i are given by

$$\Delta\theta^j_i(F,I) = \frac{2}{i!}\int_I \frac{x^i F^j(x)dx}{\sqrt{1 - ||F(x)||^2}}. \tag{3.11}$$

The proof of this Proposition is equivalent to the proofs of Proposition 4.1 from [8] (pg. 13) or Proposition 2.1 from [9] (pg. 2). In [8] an argument of classical mechanics was used, see [12] pg. 25 equation (11.5); while, in

[9], a generating function to find action-angle coordinates for Hamiltonian systems was constructed, see [5] Section 50.

Then a x-periodic geodesic $\gamma(t)$ corresponding to the pair (F,I) is periodic if and only if $\Delta\theta_i^j(F,I)=0$ for all for $i=0,1,\ldots,k$ and $j=1,\cdots,n$.

4. Abnormal Geodesics and proof of Theorem C

Proof. To prove Theorem C, we will compute the abnormal geodesics using the Pontryagin's maximun principle [1]. Let us look at the following optimal control problem:

$$\dot\gamma(t)=u_0X_0+u_1X_0^1+\cdots+v^nX_0^n, \tag{4.1}$$

with the boundary conditions $\gamma(0)$ and $\gamma(T)$ and performance the functional

$$\ell=\int_0^T\sqrt{u_0^2+u_1^2+\cdots+u_n^2}dt\to\ min, \tag{4.2}$$

with the condition $u_0^2+u_1^2+\cdots+u_n^2=1$. Then the Hamiltonian associated to the optimal control problem (4.1) and (4.2) for the abnormal case is

$$H(p,q,u)=u_0p_x+\sum_{j=1}^n\sum_{i=0}^k p_{\theta_i^j}u_j\frac{x^i}{i!}. \tag{4.3}$$

From Pontryagin's Maximum principle for this Hamiltonian we obtain a Hamiltonian system for the variables p:

$$\dot p_x=-\frac{\partial H}{\partial x}=-\sum_{j=1}^n\sum_{i=1}^k p_{\theta_i^j}u_j\frac{x^{i-1}}{(i-1)!},\ \ \dot p_{\theta_i^j}=-\frac{\partial H}{\partial\theta_i^j}=0, \tag{4.4}$$

for all $i=0,\cdots,k$ and $j=1,\cdots,n$, the maximun condition is

$$\max_{u\in\mathbb{R}^{n+1}}H(p(t),q(t),u),$$

where $u(t)$ and $q(t)$ is the optimal process, and the condition $p\neq 0$ of non-triviality. So, Theorem C is equivalent to showing that the optimal process corresponds to $u_0=0$ and u_j constant different than zero such that $u_1^2+\cdots+u_n^2=1$.

From the maximum condition, we obtain

$$0=\frac{\partial H}{\partial u_0}=p_x\qquad 0=\frac{\partial H}{\partial u_j}=\sum_{i=0}^k p_{\theta_i^j}\frac{x^i}{i!}; \tag{4.5}$$

differentiating these equations with respect to time, we get that

$$0=\sum_{j=1}^n\sum_{i=1}^k p_{\theta_i^j}u_j\frac{x^{i-1}}{(i-1)!},\qquad 0=u_0\sum_{i=1}^k p_{\theta_i^j}\frac{x^{i-1}}{(i-1)!} \tag{4.6}$$

with $j=1,\cdots,n$. The second equation in (4.6) implies that $u_0=0$ or $\sum_{i=1}^k p_{\theta_i^j}\frac{x^{i-1}}{(i-1)!}=0$. If $u_0=0$, then x is constant, since $\dot x=u_0$ according to equation (4.1). If $\sum_{i=1}^{k} p_{\theta_i^j}\frac{x^{i-1}}{(i-1)!} = 0$, we can differentiate again to get $u_0\sum_{i=2}^{k} p_{\theta_i^j}\frac{x^{i-2}}{(i-2)!} = 0$. This equation also implies that $\sum_{i=2}^{k} p_{\theta_i^j}\frac{x^{i-2}}{(i-2)!} = 0$ or $u_0 = 0$. Following this process, it is concluded that $u_0 = 0$, and therefore x is constant.

First equation in (4.6) is rewritten as interior product:

$$0 = (u_1, \cdots, u_n)\cdot(\sum_{i=1}^{k} p_{\theta_i^1}\frac{x^{i-1}}{(i-1)!}, \cdots, \sum_{i=1}^{k} p_{\theta_i^n}\frac{x^{i-1}}{(i-1)!}).$$

This is a lineal equation and it has $(n-1)$ lineal independent solutions, which can be written in terms of the constants $p_{\theta_i^j}$ and x for $j = 1, \cdots, n$. Hence, u_j is constant.

Then, to build the abnormal geodesic $\gamma(t)$ as in the prescription of Section 3, we define $F_j(x) = u_j$ and the condition $u_0^2 + \cdots + u_n^2 = 1$, which implies $||F(x)||^2 = 1$. Therefore, the abnormal geodesics are line-geodesics and a solution to the Hamiltonian flow from (2.3). $\square$

5. Proof of Theorem D

Because that period $L(F, I)$ in equation (3.10) is finite, we can define an inner product in the space of polynomials of degree k or less as follows

$$< P_1(x), P_2(x) >_F := \int_I \frac{P_1(x)P_2(x)dx}{\sqrt{1 - F^2(x)}}. \tag{5.1}$$

This inner product is not degenerated and will be the key to the proof of Theorem D.

5.1. **Proof of Theorem D.**

Proof. It will be proceeded by contradiction. Let us assume $\gamma(t)$ is a periodic geodesic on $J^k(\mathbb{R}, \mathbb{R}^n)$ corresponding to the pair (F, I), where $F(x)$ is not a constant polynomial vector; then $\Delta\theta_i^j(F, I) = 0$ for all $i = 0, \cdots, k$ and $j = 1, \cdots, n$.

In the context of the space of polynomials of degree k or less with inner product $< \ , \ >_F$, the condition $\Delta\theta_i^j(F, I) = 0$ for all i and j is equivalent to each $F^j(x)$ being perpendicular to x^i for all $i \in 0, 1, \cdots, k$ ($0 = \Delta\theta_i^j(F, I) =< x^i, F^j(x) >_F$). But $\{x^i\}$ is a basis for the space of polynomials of degree k or less, then each $F^j(x)$ is perpendicular to any vector, so each $F^j(x)$ is zero since the inner product is not degenerated. This is a contradiction to the assumption that $F(x)$ is not a constant polynomial. $\square$

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