METRIC LINES IN THE JET SPACE.

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Abstract. In the realm of sub-Riemannian Manifolds, a relevant question is: what are the metric lines (isometric embedding of the real line)? The space of k-jets of a real function of one real variable x, denoted by $J^k(\mathbb{R},\mathbb{R})$, admits the structure of a Carnot group. Every Carnot group is sub-Riemannian manifold, so is $J^k(\mathbb{R},\mathbb{R})$. This study aims to present a partial result about the classification of the metric lines within $J^k(\mathbb{R},\mathbb{R})$. The method is to use an intermediate 3-dimensional sub-Riemannian space $\mathbb{R}^3_F$ lying between the group $J^k(\mathbb{R},\mathbb{R})$ and the Euclidean space $\mathbb{R}^2$.

1. Introduction

The space of k-jets of a real function of one real variable x, denoted by $J^k(\mathbb{R},\mathbb{R})$, admits the structure of a Carnot group. Every Carnot group is sub-Riemannian manifold, so is $J^k(\mathbb{R},\mathbb{R})$. This work is the second in a series of papers whose final goal is to comprehensively classify the metric lines within $J^k(\mathbb{R},\mathbb{R})$. Let us commence by defining a metric line within the realm of sub-Riemannian geometry.

Definition 1. *Let M be a sub-Riemannian manifold with sub-Riemannian distance $dist_M(\cdot,\cdot)$, and $|\cdot|:\mathbb{R}\to[0,\infty)$ be the absolute value. We say that a curve $\gamma:\mathbb{R}\to M$ is a metric line if it is a globally minimizing geodesic, i.e.,*

$$|a-b| = dist_M(\gamma(a),\gamma(b)) \quad \textit{for all compact set} \quad [a,b] \subset \mathbb{R}.$$

For the precise definition of a sub-Riemannian geodesic, refer to Definition [1, sub-sub-Chapter 4.7.2] or [18, sub-Chapter 1.4]. Alternative terms for "metric line" are "globally minimizing geodesic" or "infinite geodesic".

In [3, 4, 18], A. Anzaldo-Meneses and F. Monroy-Perez showed a bijection between the set of pairs (F,I) and the set of geodesics in $J^k(\mathbb{R},\mathbb{R})$, where F is a polynomial of degree k or less, and I is a closed interval called the hill interval, see Definition 3 below. Subsequently, we presented an alternative proof in [12, Background Theorem]. Through the symplectic reduction of the sub-Riemannian geodesic flow on $J^k(\mathbb{R},\mathbb{R})$, we obtain a reduced Hamiltonian system H_F depending on the polynomial F (as shown in equation (2.2) below). In addition, we classified the geodesic within $J^k(\mathbb{R},\mathbb{R})$ according to

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their reduced dynamics, distinguishing between line, x-periodic, homoclinic, heteroclinic of the direct-type or heteroclinic of the turn-back, we elaborate in sub-Section 2.1.2 and present some examples in Figure 1.1. The following is the conjecture concerning metric lines within $J^k(\mathbb{R},\mathbb{R})$.

Conjecture 2. *The metric lines within $J^k(\mathbb{R},\mathbb{R})$ are precisely geodesics of the type: line, homoclinic and the heteroclinic of the direct-type.*

It is well know that the line geodesics are metric lines, see Corollary 11. In [12, Theorem 1], we proved that geodesics of type x-periodic and heteroclinic turn-back do not qualify as metric lines. Theorem A is the first main result of this work and proves Conjecture 2 for the case of heteroclinic of the direct-type geodesics.

Theorem A. *Heteroclinic of the direct-type geodesics are metric lines in $J^k(\mathbb{R},\mathbb{R})$.*

Conjecture 2 remains open for homoclinic geodesics. Theorem B is the second principal result of this work and provides a family of homoclinic geodesics that indeed qualify as metric lines.

Theorem B. *Let $F(x)$ be the polynomial $\pm(1-bx^{2n})$ and I be the hill interval $[0, \sqrt[2n]{\frac{2}{b}}]$. Then, the homoclinic-geodesic corresponding to the pair (F,I) is a metric line within $J^k(\mathbb{R},\mathbb{R})$ for all $k \geq 2n$ and $b>0$.*

Previous Results. In [6, 5, 7, 8], A. Andertov and Y. Sachkov proved Conjecture 2 for the case $k=1$ and $k=2$ using optimal synthesis. In [12, Theorem 2], we showed that a family of heteroclinic of the direct-type geodesics are metric lines.

The case $k=1$ corresponds to $J^1(\mathbb{R},\mathbb{R})$ being the Heisenberg group where the geodesics are x-periodic or geodesic lines. The case $k=2$ corresponds to $J^2(\mathbb{R},\mathbb{R})$ being Engel's group, denoted by Eng. Besides geodesic lines, up to a Carnot translation and dilation Eng has a unique metric line such that its projection to the plane $\mathbb{R}^2 \simeq \mathrm{Eng}/[\mathrm{Eng},\mathrm{Eng}]$ is the Euler-soliton. The family of metric lines defined by Theorem B is the generalization of A. Andertov and Y. Sachkov's result from [6, 5, 7, 8]. More specifically, when $n=1$ then the geodesic defined by the polynomial $F(x)=\pm(1-bx^2)$ is the one whose projection to the plane $\mathbb{R}^2$ is the Euler-soliton. For an exploration of the Euler-Elastica problem from the perspective of Calculus of Variations, refer to [21, sub-Chapter 5.5]. For further insights into Euler-Elastica and geodesics in Eng, consult [10, Section 4]. Review [1, sub-sub-Chapter 7.8.3] or [15, Chapter 14] for the relation between the Euler-Elastica and sub-Riemannian geodesics within the rolling problem and the Euclidean group.

Our Method. Two classical methods for demonstrating that a geodesic constitutes a metric line are optimal synthesis and weak KAM theory. For an introduction to optimal synthesis, refer to [15, sub-Chapter 9.4], [1, sub-Chapter 13.4] or [2, Chapter 13]. Further insights into weak KAM theory

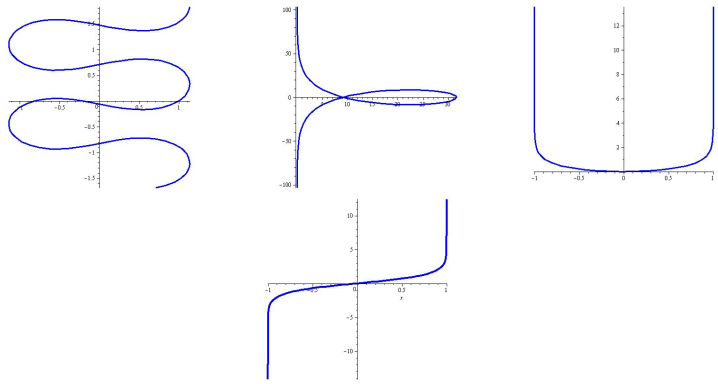

FIGURE 1.1. The images show the plane (x, θ_0) with the projections of geodesics in $J^k(\mathbb{R}, \mathbb{R})$. Successive panels showcase the projection of an x-periodic geodesic, a homoclinic geodesic, a turn-back geodesic, and a heteroclinic of the direct-type geodesic. The second panel explicitly illustrates the Euler-soliton solution to the Euler-Elastica problem, this geodesic corresponds to the scenario where $n = 1$ according to Theorem B.

in the Riemannian context can be found in [14], while details specific to the sub-Riemannian context are available in [19, sub-sub-chapter 1.9.2] or [12, Section 5]. Optimal synthesis necessitates the explicit integration of geodesic equations, while weak KAM theory relies on a global Calibration function. In both cases, the integrability of flows is a crucial requirement. Despite the integrability of the sub-Riemannian geodesic flow in $J^k(\mathbb{R}, \mathbb{R})$ as demonstrated in [10, Theorem 1.1], these methods alone cannot prove Conjecture 2. Explicitly integrating the equation of motion is infeasible in the general case, and the local Calibration functions found in [12, Section 5] or [11, sub-Section 3.2] lack a global extension.

Besides Theorem A and B, the main contribution of this work is the formalization of the method used in [12]. We will consider a sub-Riemannian manifold $\mathbb{R}^3_F$, called the magnetic space, and a sub-Riemannian submersion $\pi_F : J^k(\mathbb{R}, \mathbb{R}) \to \mathbb{R}^3_F$. Thanks to the fact that lift of a metric line is a metric line (Proposition 10), it is enough to prove that if $\gamma(t)$ is a sub-Riemannian geodesic corresponding to a polynomial F and satisfying the conditions of Theorems A or B, then the projection $\pi_F(\gamma(t))$ is a metric line in $\mathbb{R}^3_F$. In other words, we reduce the problem of studying metric lines in $J^k(\mathbb{R}, \mathbb{R})$ to studying metric lines in the magnetic space $\mathbb{R}^3_F$. Theorems 32 and 42 show that the curve $c(t) := \pi_F(\gamma(t))$ is a metric line, where $\gamma(t)$ is sub-Riemannian geodesic given by Theorems A and B, respectively. To prove Theorems 32 and 42, we consider a sequence of minimizing sub-Riemannian geodesics $c_n(t)$ joining every time farther away points on the geodesic $c(t)$, see Figures 3.1 and 4.1. We show that the sequence has a convergent subsequence $c_{n_j}(t)$ converging to a minimizing geodesic $c_\infty(t)$ corresponding to the polynomial F, since every two sub-Riemannian geodesics corresponding to the polynomial F are related by sub-Riemannian isometry we conclude that $c(t)$ is a metric line.

Outline. Section 2 introduces the preliminary results necessary to prove Theorem A and B. Sub-Section 2.1 briefly describes $J^k(\mathbb{R},\mathbb{R})$ as a sub-Riemannian manifold and summarizes some previous results from [12]. Between them, the most important are: the **Background Theorem** establishing the correspondence between sub-Riemannian geodesics and the pairs (F, I), the classification of sub-Riemannian geodesic, the formal definition of a sub-Riemannian submersion and Proposition 10. Sub-Section 2.2 presents the magnetic space $\mathbb{R}^3_F$ and some previous results from [12]: The correspondence between sub-Riemannian geodesics in the magnetic space $\mathbb{R}^3_F$ and the pairs (G, I) where G is a polynomial in a two-dimensional space Pen_F, the period map $\Theta(G, I)$, and an upper bound for the cut time. In addition, sub-Section 2.2 provides the relation between the sub-Riemannian geodesic in $\mathbb{R}^3_F$ and $J^k(\mathbb{R},\mathbb{R})$, the cost function definition, and some sub-Riemannian geodesics' properties.

The main goal of Section 3 is to prove Theorem 32. Sub-Section 3.1 presents a particular magnetic space for each heteroclinic geodesic of the direct-type and some essential properties of this space. In particular, Lemma 33 says that the Period map $\Theta(G, I)$ is one-to-one when restricted to space of heteroclinic geodesics of direct-type. Sub-sub-Section 3.2.1 sets up the proof, sub-sub-Section 3.2.2 presents the proof of Theorem 32 and sub-sub-Section 3.2.3 provides the formal proof of Theorem A.

The main goal of Section 4 is to prove Theorem 42 and has a similar structure to Section 3. In addition, Section 4 introduces Theorem 44, which says that the sub-Riemannian geodesic corresponding to the polynomial $F(x) = 1 - 2x^{2n+1}$ is not a metric line in the magnetic space $\mathbb{R}^3_F$.

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2. Preliminary

Here we will introduce the necessary results to prove Theorems A and B.

2.1. $J^k(\mathbb{R},\mathbb{R})$ **as a sub-Riemannian manifold.** Let $f(x)$ and $g(x)$ be smooth real-valued functions. We define an equivalent relation on the set of germs of smooth functions at x_0 that $f(x)$ and $g(x)$ by the following relation

$$f(x) \sim g(x) \qquad \text{if and only if} \qquad f(x) - g(x) = O(|x - x_0|^{k+1}).$$

Using the list of k derivatives of a function $f(x)$ at x_0, we identify the k-jet space of $f(x)$ at x_0 with its kth order Taylor expansion of f at x_0:

$$u_0 = f(x_0), \text{ and } u_j = \frac{d^j f}{dx^j}(x_0) \text{ for } j = 1, \dots, k.$$

We sweep out the k-jet space $J^k(\mathbb{R},\mathbb{R})$, by letting the base point x_0 and function $f(x)$ vary. Then, $J^k(\mathbb{R},\mathbb{R})$ is a $(k+2)$-dimensional manifold with global coordinates $(x, u_0, \dots, u_k)$. For more details about $J^k(\mathbb{R},\mathbb{R})$ as a Carnot group and sub-Riemannian manifolds, consult [1, sub-Chapter 10.2], [12, sub-Section 2.1], [10, Section 2] or [22, Section 3].

The jet space $J^k(\mathbb{R},\mathbb{R})$ is conventionally defined using the coordinates $(x, u_0, \dots, u_k)$. However, these coordinates do not readily reveal the symmetries inherent in the sub-Riemannian geodesic flow. The alternate coordinates $(x, \theta_0, \dots, \theta_k)$, also called exponential coordinates of second kind, elucidates these symmetries. For a formal definition and properties of these new coordinates, consult [3, 4]. The change of coordinates is given by

$$\theta_0 = u_k, \text{ and } \theta_j = \sum_{i=0}^{j} (-1)^i \frac{x^i}{(j-i)!} u_{k-i} \text{ for } j = 1, \dots, k.$$

In establishing $J^k(\mathbb{R},\mathbb{R})$ as a sub-Riemannian manifold, we observe that $J^k(\mathbb{R},\mathbb{R})$ possesses a natural rank 2 distribution, denoted by $\mathcal{D} \subset TJ^k(\mathbb{R},\mathbb{R})$, which is characterized by the k-Pfaffian equations

$$0 = d\theta_i - \frac{1}{i!} x^i d\theta_0, \qquad i = 1, \dots, k.$$

Thus $\mathcal{D}$ is globally framed by two vector fields

$$X = \frac{\partial}{\partial x} \text{ and } Y = \sum_{i=0}^{k} \frac{x^i}{i!} \frac{\partial}{\partial \theta_i}. \tag{2.1}$$

We declare these two vector fields to be orthonormal to define the sub-Riemannian structure on $J^k(\mathbb{R},\mathbb{R})$. Thus the sub-Riemannian metric is given by restricting $ds^2 = dx^2 + d\theta_0^2$ to $\mathcal{D}$. We endow the space $J^k(\mathbb{R},\mathbb{R})$ with a canonical projection $\pi : J^k(\mathbb{R},\mathbb{R}) \to \mathbb{R}^2 \simeq J^k(\mathbb{R},\mathbb{R})/[J^k(\mathbb{R},\mathbb{R}), J^k(\mathbb{R},\mathbb{R})]$, which in coordinates is given by $\pi(x, \theta_0, \dots, \theta_k) = (x, \theta_0)$.

2.1.1. *Reduced System.* A geodesic within $J^k(\mathbb{R},\mathbb{R})$ is determined by the pair (F, I). Let us explain: let F be a polynomial F of degree k or less, then the reduced Hamiltonian function H_F is given by

$$H_F = \frac{1}{2}(p_x^2 + F^2(x)), \tag{2.2}$$

and the reduced Hamiltonian equations are given by

$$\dot{x} = p_x \qquad \text{and} \qquad \dot{p}_x = -F(x)F'(x). \tag{2.3}$$

The condition $\frac{1}{2} = H_F$ implies that the reduced dynamics occur within a closed interval I, called hill interval, and the sub-Riemannian geodesic is parametrized by arc length. Let us formalize the hill interval definition.

Definition 3. *We say that a closed interval I is a hill interval associated to $F(x)$, if $|F(x)| < 1$ for every x in the interior of I and $|F(x)| = 1$ for every x in the boundary of I. If I is of the form $[x_0, x_1]$, then we call x_0 and x_1 the endpoints of the hill interval. We say that hill(F) is the hill region of F if hill(F) is union of all the hill intervals of F.*

By definition, if $F(x)$ is not a constant polynomial then I is compact. In contrast, if $F(x) = c$ is a constant polynomial where $|c| < 1$ then the hill interval is $I = \mathbb{R}$, and if $F(x)$ is constantly equal to ± 1 then every singleton is a hill interval.

Here, we prescribe the method to build a sub-Riemannian geodesic for a pair (F, I): first, find a solution to the reduced system (2.3) with initial condition $x(t_0)$ in I and energy $\frac{1}{2} = H_F$. Second, having found the solution $(x(t), p_x(t))$, we define a curve $\gamma(t)$ in $J^k(\mathbb{R}, \mathbb{R})$ using the following equation

$$\dot{\gamma}(t) = \dot{x}(t)X(\gamma(t)) + F(x(t))Y(\gamma(t)),$$

where $\dot{x}(t) = p_x(t)$ by Hamilton equations (2.3). The curve $\gamma(t)$ is defined for all time t in $\mathbb{R}$ by completeness of H_F.

The **Background Theorem** establishes the correspondence between the pair (F, I) and the sub-Riemannian geodesics in $J^k(\mathbb{R}, \mathbb{R})$.

Background Theorem. *The above prescription yields a geodesic in J^k parameterized by arc length. Conversely, every arc length parameterized geodesic in J^k can be achieved by this prescription applied to some polynomial $F(x)$ of degree k or less.*

The **Background Theorem** was proved first in [3, 4, 18], later we gave an alternative proof in [12, Appendix A].

Remark 4. *We make the following remark about the* ***Background Theorem***

(1) With the initial condition $x(t_0)$ located within the interior of I, we have the freedom to select the sign of the initial condition $p(t_0) = \pm\sqrt{1 - F(x(t_0))}$. Opting for a positive sign generates a solution where $x(t)$ progresses forward within the hill interval, i.e., $x(t)$ a has positive derivative in a neighborhood of t_0. Conversely, choosing a negative sign yields a solution where $x(t)$ moves backward within the hill interval, i.e., $x(t)$ has a positive derivative in a neighborhood of t_0.

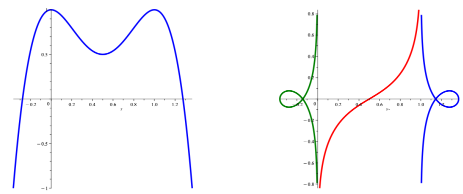

FIGURE 2.1. The first panel presents the graph of the polynomial $F(x) = 1 - 8x^2(1-x)^2$, which possesses three hill intervals, namely $[\frac{1}{2}(1-\sqrt{3}), 0]$, $[0, 1]$ and $[1, \frac{1}{2}(1+\sqrt{3})]$. The second panel displays the plane (x, θ_0) and the projection of three geodesics corresponding to the polynomial $F(x) = 1 - 8x^2(1-x)^2$. The hill intervals $[\frac{1}{2}(1-\sqrt{3}), 0]$ and $[1, \frac{1}{2}(1+\sqrt{3})]$ generate a homoclinic geodesic, and the hill interval $[0, 1]$ corresponds to a direct-type geodesic.

(2) *Given a solution $(x(t), p_x(t))$ with a initial condition $(x(t_0), p_x(t_0))$, the geodesic $\gamma(t)$ is unique up to constant of integrations $\theta_0(t_0)$, ..., $\theta_{k+1}(t_0)$.*
(3) *The choice of a different hill interval generates a different geodesics, as illustrated in Figure 2.1.*

2.1.2. *Classification Of Geodesic In Jet Space.* We classify the sub-Riemannian geodesics according to their reduced dynamics. Let $\gamma(t)$ be a geodesic in $J^k(\mathbb{R}, \mathbb{R})$ corresponding to the pair (F, I). Then, the x-component of $\gamma(t)$ is a solution of the reduced system 2.3 with a initial condition $x(t_0)$ in I. By the theory of a one-degree of freedom system, $x(t)$ is: trivial, uniform rectilinear motion, periodic, homoclinic, or heteroclinic. Then, $\gamma(t)$ is only one of the following options:

- We say that a geodesic $\gamma(t)$ is a line if the projected curve $\pi(\gamma(t))$ is a line in $\mathbb{R}^2$. Line geodesics correspond to constant polynomials or trivial solutions of the reduced dynamics.

Let $I = [x_0, x_1]$ be a hill interval of a non-constant polynomial $F(x)$.

- We say $\gamma(t)$ is x-periodic if its reduced dynamics is periodic. The reduced dynamics is periodic if and only if x_0 and x_1 are regular points of $F(x)$.
- We say $\gamma(t)$ is homoclinic if its reduced dynamics is a homoclinic orbit. The reduced dynamics has a homoclinic orbit if and only if one of the points x_0 and x_1 is regular and the other is a critical point of $F(x)$. Then a homoclinic geodesic has the property that $x(t)$ converges to the boundary point of I that is critical for $F(x)$, as t goes to $\pm\infty$.
- We say $\gamma(t)$ is heteroclinic if its reduced dynamics is a heteroclinic orbit. The reduced dynamics has a heteroclinic orbit if and only if both points x_0 and x_1 are critical of $F(x)$.

- We say a heteroclinic geodesic $\gamma(t)$ is turn-back if $F(x_0)F(x_1) = -1$.
- We say a heteroclinic geodesic $\gamma(t)$ is direct-type if $F(x_0)F(x_1) = 1$.

See figure 1.1 for a better undertanding of these names.

2.1.3. *Unitary Geodesics.* To prove Theorem A and B, we will introduce the concept of a unitary geodesic.

Definition 5. *We say a geodesic $\gamma(t)$ in $J^k(\mathbb{R},\mathbb{R})$ corresponding to the pair (F, I) is unitary if $I = [0,1]$. We say a heteroclinic of the direct-type geodesic (or homoclinic) $\gamma(t)$ is unitary, if in addition $F(x(t)) \to 1$ when $t \to \pm\infty$.*

The reflection $R_Y(x, \theta_0, \theta_1, \dots, \theta_k) := (x, -\theta_0, -\theta_1, \dots, -\theta_k)$ preserves the distribution $\mathcal{D}$, since $(R_Y)_*X = X$ and $(R_Y)_*Y = -Y$, where $(R_Y)_*$ is the push-forward of R_Y. Therefore, if we denote by $Iso(J^k(\mathbb{R},\mathbb{R}))$ the isometry group of $J^k(\mathbb{R},\mathbb{R})$, then the reflection R_Y is in $Iso(J^k(\mathbb{R},\mathbb{R}))$. We conclude that to classify metric lines it is enough to study unitary geodesics, since if $\gamma(t)$ is a heteroclinic of the direct-type or homoclinic geodesic corresponding to the polynomial $F(x)$, then $R_Y(\gamma(t))$ is a geodesic generated by $-F(x)$.

Lemma 6. *Let $\gamma(t)$ be a unitary heteroclinic of the direct-type geodesic for a polynomial $F(x)$ of degree k, then there exist natural numbers k_1 and k_2, and $q(x)$ is polynomial of degree $k - k_1 - k_2$ such that*

$$F(x) = 1 - x^{k_1}(1-x)^{k_2}q(x), \quad \textit{where} \;\; 1 < k_1, \;\; 1 < k_2,$$

and $0 < x^{k_1}(1-x)^{k_2}q(x) < 2$ if x is in $(0,1)$.

Proof. By construction, $F(x)$ is such that

$$F(0) = F(1) = 1, \;\; F'(0) = F'(1) = 0 \;\text{ and }\; |F(x)| < 1 \text{ if } \; x \; \text{ is in } \; (0,1),$$

then using the Euclidean algorithm we find the desired result. □

The Figure 2.1 display an example of polynomial $F(x)$ from Lemma 6, and its unitary heteroclinic of the direct-type geodesic.

2.1.4. *Carnot Dilatation.* Carnot groups have the property of admitting dilatations. The dilatation is a one-parameter group of automorphism of $\mathbb{G}$, denote by $\delta_u : \mathbb{G} \to \mathbb{G}$ and with u in $\mathbb{R} \setminus \{0\}$. The dilatation is compatible with the metric, that is $dist_{\mathbb{G}}(\delta_u g_1, \delta_u g_2) = |u| dis_{\mathbb{G}}(g_1, g_2)$. If $u \neq 0$ and $\gamma(t)$ is a sub-Riemannian geodesic parametrized by arc length, so is $\gamma_u(t)$, where

$$\gamma_u(t) := \delta_u \gamma(\frac{t}{u}).$$

For more details about the Carnot dilatation see [19, sub-Chapter 8.2].

Lemma 7. *If $\gamma(t)$ is a metric line in a Carnot group $\mathbb{G}$. Then, $\gamma_u(t)$ is a metric line in $\mathbb{G}$.*

Proof. Let us assume $\gamma(t)$ is a metric line in a Carnot group $\mathbb{G}$. Let $[a,b]$ be an arbitrary compact interval, then

$$dis_{\mathbb{G}}(\gamma_u(a),\gamma_u(b)) = dis_{\mathbb{G}}(\ \delta_u\gamma(\frac{a}{u}),\delta_u\gamma(\frac{b}{u})) = |u|dis_{\mathbb{G}}(\gamma(\frac{a}{u}),\gamma(\frac{b}{u}))$$
$$= |u||\frac{a}{u}-\frac{b}{u}| = |a-b|.$$

Since $[a,b]$ is arbitrary, we concluded $\gamma_u(t)$ is a metric line in $\mathbb{G}$. □

In the case of the jet space, the dilatations is given by

$$\delta_u(x,\theta_0,\theta_1,\dots,\theta_k) = (ux,u\theta_0,u^2\theta_1,\dots,u^{k+1}\theta_k).$$

The following Proposition tells us that every geodesic in $J^k(\mathbb{R},\mathbb{R})$ is related to unitary geodesic by a Carnot dilatation and translation.

Proposition 8. *Let $\gamma(t)$ be a geodesic in $J^k(\mathbb{R},\mathbb{R})$ associated to the pair (F,I) and let $h(\tilde{x}) = x_0 + u\tilde{x}$ be the affine map taking $[0,1]$ to $I = [x_0,x_1]$ with $u := x_1 - x_0$. If $\tilde{F}(\tilde{x}) := F(h(\tilde{x}))$ and $\tilde{\gamma}(t)$ is the geodesic in $J^k(\mathbb{R},\mathbb{R})$ corresponding to the pair $(\tilde{F},[0,1])$, then $\gamma(t)$ is related to $\tilde{\gamma}(t)$ by Carnot dilatation and translation, that is*

$$\gamma(t) = L_{(x_0,0\dots,0)}\big(\tilde{\gamma}_u(t)\big),$$

where L is the left translation.

Proof. To show that both curves are the same it is enough to show that they satisfy the same differential equation and share identical initial conditions:

By construction $(\tilde{x}(t),\tilde{p}_x(t))$ satisfies the Hamiltonian system given by $H_{\tilde{F}}$, and $\dot{\tilde{\theta}}_i(t) = \frac{\tilde{x}^i(t)}{i!}\tilde{F}(\tilde{x}(t))$. Let us compute $\frac{d}{dt}L_{(x_0,0\dots,0)}\big(\tilde{\gamma}_u(t)\big)$;

$$\begin{aligned}
\frac{d}{dt}L_{(x_0,0\dots,0)}\big(\tilde{\gamma}_u(t)\big) &= (L_{(x_0,0\dots,0)})_*(\delta_u)_*\big(\dot{\tilde{\gamma}}(\frac{t}{u})\big)\frac{d}{dt}(\frac{t}{u}) \\
&= \frac{1}{u}(L_{(x_0,0\dots,0)})_*(\delta_u)_*\big(\dot{\tilde{x}}(\frac{t}{u})\frac{\partial}{\partial x} + \tilde{F}(\tilde{x}(\frac{t}{u}))\sum_{i=0}^{k}\frac{\tilde{x}^i(\frac{t}{u})}{i!}\frac{\partial}{\partial\theta_i}\big) \\
&= (L_{(x_0,0\dots,0)})_*\big(\dot{\tilde{x}}(\frac{t}{u})\frac{\partial}{\partial x} + \tilde{F}(u\tilde{x}(\frac{t}{u}))\sum_{i=0}^{k}\frac{(u\tilde{x}(\frac{t}{u}))^i}{i!}\frac{\partial}{\partial\theta_i}\big) \\
&= \dot{\tilde{x}}(\frac{t}{u})\frac{\partial}{\partial x} + \tilde{F}(x_0+u\tilde{x}(\frac{t}{u}))\sum_{i=0}^{k}\frac{(x_0+u\tilde{x}(\frac{t}{u}))^i}{i!}\frac{\partial}{\partial\theta_i}
\end{aligned}$$

If we set up $x(t) = x_0 + u\tilde{x}(\frac{t}{u})$, then

$$\frac{d}{dt}L_{(x_0,0\dots,0)}\big(\tilde{\gamma}_u(t)\big) = \dot{x}(t)\frac{\partial}{\partial x} + F(x(t))\sum_{i=0}^{k}\frac{x^i(t)}{i!}\frac{\partial}{\partial\theta_i} = \dot{\gamma}(t)$$

Without loss of generality, let us assume that $\gamma(0) = (x_0,0,\dots,0)$ and $\tilde{\gamma}(0) = (0,\dots,0)$, then

$$L_{(x_0,0\dots,0)}\big(\tilde{\gamma}_u(0)\big) = L_{(x_0,0\dots,0)}\big(\delta_u(0,\dots,0)\big) = L_{(x_0,0\dots,0)}\big((0,\dots,0)\big) = \gamma(0)$$

□

Lemma 7, Proposition 8 and the reflection R_Y imply that proving Theorems A and B for the unitary case is sufficient.

2.1.5. *Sub-Riemannian Submersion.* Let us formalize the definition of sub-Riemannian submersion and present Proposition 10.

Definition 9. *Let $(M, \mathcal{D}_M, g_M)$ and $(N, \mathcal{D}_N, g_N)$ be two sub-Riemannian manifolds and let $\phi : M \to N$ a submersion ($dim(M) \geq dim(N)$). We say that ϕ is a sub-Riemannian submersion if $\phi_* \mathcal{D}_M = \mathcal{D}_N$ and $\phi^* g_N = g_M$.*

We remark that the projection $\pi : J^k(\mathbb{R}, \mathbb{R}) \to \mathbb{R}^2$, defined in sub-Section 2.1, is a sub-Riemannian submersion. As a consequence, a curve $\gamma(t)$ in $J^k(\mathbb{R}, \mathbb{R})$ and its projection $\pi(\gamma(t))$ have the same arc length.

A classic result on metric lines is the following.

Proposition 10. *Let $\phi : M \to N$ be a sub-Riemannian submersion and let $c(t)$ be a metric line in N, then the horizontal lift of $c(t)$ is a metric line in M.*

The proof of Proposition 10 is given in [12, p. 154]. The following corollary is an immediate result to the Proposition 10.

Corollary 11. *Geodesic lines are metric lines in $J^k(\mathbb{R}, \mathbb{R})$.*

2.2. **The 3-Dimensional Magnetic Space.** In [12], we introduced the 3-dimensional sub-Riemannian manifold, denoted by $\mathbb{R}^3_F$ and called "magnetic sub-Riemannian structure" or "magnetic space", whose geometry depends on the choice of a polynomial $F(x)$. To endow $\mathbb{R}^3_F$ with the sub-Riemannian structure we use global coordinates (x, y, z) and consider $\omega := dz - F(x)dy$. Then, we define the two rank non-integrable distribution $\mathcal{D}_F$ and the sub-Riemannian metric on the distribution $\mathcal{D}_F$ by the Pfaffian equation $\omega = 0$ and $ds^2_{\mathbb{R}^3_F} = (dx^2 + dy^2)|_{\mathcal{D}_F}$, respectively. We provided a sub-Riemannian submersion π_F factoring the sub-Riemannian submersion $\pi : J^k(\mathbb{R}, \mathbb{R}) \to \mathbb{R}^2$, that is, $\pi = pr \circ \pi_F$, where the target of π_F is $\mathbb{R}^3_F$ and the target of pr is $\mathbb{R}^2$. If $F(x) = \sum_{i=0}^k \frac{a_i x^i}{i!}$, then the projections π_F and pr are given in coordinates by

$$\pi_F(x, \theta) = (x, \theta_0, \sum_{\ell=0}^k a_\ell \theta_\ell) = (x, y, z), \quad \text{and} \quad pr(x, y, z) := (x, y). \tag{2.4}$$

It follows that π_F maps the frame $\{X, Y\}$ defined in (2.1) into the frame $\{\tilde{X}, \tilde{Y}\}$, that is,

$$\tilde{X} := \frac{\partial}{\partial x} = (\pi_F)_* X \quad \text{and} \quad \tilde{Y} := \frac{\partial}{\partial y} + F(x)\frac{\partial}{\partial z} = (\pi_F)_* Y.$$

We conclude $\mathcal{D}_F$ is globally framed by the orthonormal vector fields $\{\tilde{X}, \tilde{Y}\}$.

When $F(x) = \frac{1}{2}x^2$, the magnetic space $\mathbb{R}^3_F$ is the Martinet Manifold, for a more in-depth understanding of the Martinet Manifold consult [19, sub-Chapter 3.2] or [1, Example 10.56]. For an explanation of the names "magnetic sub-Riemannian structure" or "magnetic space" refer to [12, sub-Section 4.1].

2.2.1. *Geodesics In The Magnetic Space.* The Hamiltonian function governing the sub-Riemannian geodesic flow in $\mathbb{R}^3_F$ is

$$H_{\pi^3_F}(p_x, p_y, p_z, x, y, z) = \frac{1}{2}p_x^2 + \frac{1}{2}(p_y + F(x)p_z)^2.$$

We say a curve $c(t) = (x(t), y(t), z(t))$ is a $\mathbb{R}^3_F$-geodesic parametrized by arc length in $\mathbb{R}^3_F$, if it is the projection of the sub-Riemannian geodesic flow with the condition $H_{\pi^3_F} = \frac{1}{2}$. Since $H_{\pi^3_F}$ does not depend on the coordinates y and z, they are cycle coordinates, so the momentum p_y and p_z are constant of motion. Refer to [16, p. 162] or [9, p. 67] for the definition of cycle coordinates and their properties. Since y and z are cycle coordinates, the translation $\varphi_{(y_0,z_0)}(x, y, z) = (x, y + y_0, z + z_0)$ represents an isometry.

Definition 12. *We denote by* $dist_{\mathbb{R}^3_F}(\ ,\ )$ *and* $Iso(\mathbb{R}^3_F)$*, the sub-Riemannian distance and the isometry group in* $\mathbb{R}^3_F$*.*

For more details about the definition of sub-Riemannian distance the sub-Riemannian group of isometries, consult [19, Chapter 1.4] or [1, sub-Chapter 3.2]. Consequently, the translation $\varphi_{(y_0,z_0)}$ belongs to $Iso(\mathbb{R}^3_F)$.

Definition 13. *We say that the two-dimensional linear space* Pen_F *is the pencil of* $F(x)$*, if* $Pen_F := \{G(x) = a + bF(x) : (a, b) \in \mathbb{R}^2\}$*.*

We define the lift of a curve in $\mathbb{R}^3_F$ to a curve in $J^k(\mathbb{R}, \mathbb{R})$.

Definition 14. *Let* $c(t)$ *be a curve in* $\mathbb{R}^3_F$*. We say that a curve* $\gamma(t)$ *in* $J^k(\mathbb{R}, \mathbb{R})$ *is the lift of* $c(t) = (x(t), y(t), z(t))$ *if* $\gamma(t)$ *solves*

$$\dot{\gamma}(t) = \dot{x}(t)X(\gamma(t)) + G(x(t))Y(\gamma(t)).$$

Now, we elucidate the $\mathbb{R}^3_F$-geodesics, their lifts, and their connection with the sub-Riemannian geodesics in $J^k(\mathbb{R}, \mathbb{R})$.

Proposition 15. *Let* $c(t)$ *be a* $\mathbb{R}^3_F$*-geodesic for the pair* (G, I)*, where* $G(x)$ *in* Pen_F *and* I *is a hill interval of* $G(x)$*, then the component* $x(t)$ *satisfies the* 1*-degree of freedom Hamiltonian system associated to the Hamiltonian function*

$$H_{(a,b)}(p_x, x) := \frac{1}{2}p_x^2 + \frac{1}{2}(a + bF(x))^2 = \frac{1}{2}p_x^2 + \frac{1}{2}G^2(x). \tag{2.5}$$

Having found a solution $(x(t), p_x(t))$ *with energy* $H_{(a,b)} = \frac{1}{2}$*, the coordinates* $y(t)$ *and* $z(t)$ *satisfy*

$$\dot{y} = G(x(t)) \ \ \text{and} \ \ \dot{z} = G(x(t))F(x(t)). \tag{2.6}$$

Moreover, every $\mathbb{R}^3_F$-geodesic is the π_F-projection of a geodesic in $J^k(\mathbb{R},\mathbb{R})$ corresponding to $G(x)$ in Pen_F. Conversely, the lifts of a $\mathbb{R}^3_F$-geodesic are precisely those geodesics corresponding to polynomials in Pen_F.

The proof was presented in [12, sub-Section 4.1].

Corollary 16. *Let $\gamma(t)$ be a sub-Riemannian geodesic in $J^k(\mathbb{R},\mathbb{R})$ corresponding to the pair (F,I), where $F(x)$ is polynomial of degree k or less and I is a hill interval for $F(x)$, and let $c(t)$ be the curve given by $\pi_F(\gamma(t))$, then $c(t)$ is a $\mathbb{R}^3_F$-geodesic corresponding to the pencil $(a,b)=(0,1)$.*

Proof. By construction, the pencil $(a,b)=(0,1)$ correspond to the polynomial $F(x)$. $\square$

We classify the sub-Riemannian geodesics in $\mathbb{R}^3_F$ according to their reduced dynamics, given by the reduced Hamiltonian $H_{(a,b)}$ from equation (2.5), in the same way as we did in sub-sub-Section 2.1.2.

Remark 17. *The solution $(x(t),p_x(t))$, with energy $H_{(a,b)}=\frac{1}{2}$, lays in a simple closed algebraic curve given by*

$$\alpha(G,I) := \{(x,p_x) : \frac{1}{2} = \frac{1}{2}p_x^2 + \frac{1}{2}G^2(x) \text{ and } x \in I\}. \tag{2.7}$$

Lemma 18. *The algebraic curve $\alpha(G,I)$ is smooth if and only if the pair (G,I) corresponds to a x-periodic $\mathbb{R}^3_F$-geodesic.*

Proof. According to the regular value theorem, the algebraic curve $\alpha(G,I)$ at a point (x,p_x) is smooth if $\nabla H(x,p_x)=(G(x)G'(x),p_x)\neq 0$. If $p_x\neq 0$ then $\nabla H(x,p_x)\neq 0$. The points $(x,0)$ in $\alpha(G,I)$ correspond to when x lies on the boundary of the hill interval I. Therefore, if $G(x)=\pm 1$, then $\nabla H(x,p_x)\neq 0$ if and only if $G'(x)\neq 0$, which is the condition for the geodesic to be x-periodic. $\square$

The sub-Riemannian geometry encompasses two type of geodesics: normal and abnormal. The sub-Riemannian geodesic flow governs the normal geodesics, while the endpoint map delineates the abnormal geodesics. For further insights into the endpoint map and abnormal geodesics, refer to [19, Chapter 3], [1, sub-sub-Chapter 4.3.2] or [13]. The following proposition characterizes the abnormal geodesics in $\mathbb{R}^3_F$.

Lemma 19. *A curve $c(t)$ in $\mathbb{R}^3_F$ is an abnormal geodesic if and only if $c(t)$ is tangent to the vector field $\tilde{Y}$ and $x(t)=x^*$ is a constant point in $\mathbb{R}$ such that $F'(x^*)=0$.*

Proof. Let us introduce the following result: Let M be three-dimensional sub-Riemannian manifold such that non-integrable distribution $\mathcal{D}$ is defined by $ker\ \omega$. Then, all the non-trivial abnormal geodesics are contained in the set

$$\mathfrak{M} := \{(x,y,z)\in\mathbb{R}^3_F : \big(\omega\wedge d\omega\big)|_{(x,y,z)} = 0\}.$$

For more details about $\mathfrak{M}$ and the above result consult [1, Proposition 4.38] or [19, sub-Chapter 5.2].

By construction of the magnetic $\mathbb{R}^3_F$, we have $\omega = dz - F(x)dy$ and $\omega \wedge d\omega = F'(x)dx \wedge dy \wedge dz$. Then, $\omega \wedge d\omega = 0$ if and only if $F'(x) = 0$. $\square$

2.2.2. *Cost Map In Magnetic Space.* In [12, sub-Section 7.2], we defined the *Cost* map and used it to prove the main result. Here, we introduce *Cost*, an auxiliary function to show Theorems A and B.

Definition 20. *Let $(c, \mathcal{T})$ be a pair of an $\mathbb{R}^3_F$-geodesic $c(t)$ parametrized by arc-length, and a time interval $\mathcal{T} := [t_0, t_1]$. For every pair $(c, \mathcal{T})$, we denote by $\Delta(c, \mathcal{T})$ the change perform by the time t, and the coordinates y, and z after the geodesic $c(t)$ travel during the time interval $\mathcal{T}$. Then, $[0, \infty) \times \mathbb{R}^2$ is the codomain of function $\Delta(c, \mathcal{T})$ given by*

$$\begin{aligned}\Delta(c, \mathcal{T}) &:= (\Delta t(c, \mathcal{T}), \Delta y(c, \mathcal{T}), \Delta z(c, \mathcal{T})) \\ &:= (t_1 - t_0, y(t_1) - y(t_0), z(t_1) - z(t_0)).\end{aligned}$$

For every pair $(c, \mathcal{T})$ we define the function $Cost(c, \mathcal{T})$, with codomain $[0, \infty) \times \mathbb{R}$, given by

$$Cost(c, \mathcal{T}) := (Cost_t(c, \mathcal{T}), Cost_y(c, \mathcal{T}))$$

where

$$\begin{aligned}Cost_t(c, \mathcal{T}) =& \Delta t(c, \mathcal{T}) - \Delta y(c, \mathcal{T}) \\ Cost_y(c, \mathcal{T}) =& \Delta y(c, \mathcal{T}) - \Delta z(c, \mathcal{T}).\end{aligned}$$

Let us prove that $Cost(c, \mathcal{T})$ is well-defined:

Proof. By construction,

$$|\Delta y(c, \mathcal{T})| \leq \Delta t(c, \mathcal{T}), \quad \text{so} \quad 0 \leq Cost_t(c, \mathcal{T}).$$

$\square$

We interpret $Cost_t(c, \mathcal{T})$ as the cost that takes to the geodesic $c(t)$ travel through the y-component in the positive direction. To give more meaning to this interpretation, we present the following Lemma.

Lemma 21. *Let $c(t)$ and $\tilde{c}(t)$ be two $\mathbb{R}^3_F$-geodesics. Let us assume that they travel from a point A to a point B in a time interval $\mathcal{T} = [t_0, t_1]$ and $\tilde{\mathcal{T}} := [\tilde{t}_0, \tilde{t}_1]$, respectively. If $Cost_t(c_1, \mathcal{T}) < Cost_t(c_2, \tilde{\mathcal{T}})$, then the arc length of $c(t)$ is shorter that the arc length of $\tilde{c}(t)$.*

Proof. We need to show that $\Delta t(c_1, \mathcal{T}) < \Delta t(c_2, \tilde{\mathcal{T}})$. Since $A = c(t_0) = \tilde{c}(\tilde{t}_0)$ and $B = c(t_1) = \tilde{c}(\tilde{t}_1)$, it follows that

$$\Delta y(c_1, \mathcal{T}) = \Delta y(c_2, \tilde{\mathcal{T}})$$

which implies

$$\Delta t(c_1, \mathcal{T}) - Cost_t(c_1, \mathcal{T}) = \Delta t(c_2, \tilde{\mathcal{T}}) - Cost_t(c_2, \tilde{\mathcal{T}}),$$

so $0 < Cost_t(c_2, \tilde{\mathcal{T}}) - Cost_t(c_1, \mathcal{T}) = \Delta t(c_2, \tilde{\mathcal{T}}) - \Delta t(c_1, \mathcal{T})$. $\square$

Proposition 22. *Let $c(t)$ be an $\mathbb{R}^3_F$-geodesic parametrized by arc-length for the pair (G, I), where $G(x)$ is in Pen_F and I is a hill interval of $G(x)$. Let $\mathcal{T}$ be a time interval. Let us consider the pair $(c, \mathcal{T})$, then we can rewrite the function $\Delta(c, \mathcal{T})$ from Definition 20 in terms of polynomial $G(x)$ and the the curve $x(\mathcal{T}) \subset I$ as follows;*

$$\Delta(c,\mathcal{T}) = (\int_{x(\mathcal{T})} \frac{dx}{\sqrt{1-G^2(x)}}, \int_{x(\mathcal{T})} \frac{G(x)dx}{\sqrt{1-G^2(x)}}, \int_{x(\mathcal{T})} \frac{G(x)F(x)dx}{\sqrt{1-G^2(x)}}).$$

In the same way, the map $Cost(c, \mathcal{T})$ from Definition 20 can be rewritten as follows:

$$Cost(c,\mathcal{T}) = (\int_{x(\mathcal{T})} \frac{1-G(x)}{\sqrt{1-G^2(x)}}dx, \int_{x(\mathcal{T})} \frac{(1-F(x))G(x)}{\sqrt{1-G^2(x)}}dx)$$

Proof. The Hamiltonian function given by equation (2.5) defines a one degree of freedom system, by Hamilton equation we have $\dot{x} = \pm\sqrt{1-G^2(x)}$. Then, we reduce to quadrature the dynamics in the following way

$$\Delta t(c,\mathcal{T}) = \int_{x(t_0)}^{x(t_1)} \frac{dx}{\sqrt{1-G^2(x)}} = \int_{x(\mathcal{T})} \frac{dx}{\sqrt{1-G^2(x)}},$$

where $\mathcal{T} := [t_0, t_1]$. To compute $\Delta y(c, \mathcal{T})$ and $\Delta z(c, \mathcal{T})$, we integrate the coordinates y and z in the same way using equation (2.6). To calculate $Cost_y(c, \mathcal{T})$ and $Cost_y(c, \mathcal{T})$, we use integral expression $\Delta(c, \mathcal{T})$ and subtract $\Delta y(c, \mathcal{T})$ to $\Delta t(c, \mathcal{T})$, and $\Delta z(c, \mathcal{T})$ to $\Delta y(c, \mathcal{T})$, respectively. □

Remark 23. *Remarks about Proposition 22:*

1) There is no ambiguity regarding the sign of $\dot{x} = \pm\sqrt{1-G^2(x)}$. If $\dot{x}(t)$ is positive within an interval $(t-\epsilon, t+\epsilon)$ for some $\epsilon > 0$, then the interval of integration is positively oriented $[x(t-\epsilon), x(t+\epsilon)]$. Conversely, if $\dot{x}(t)$ is negative within an interval $(t-\epsilon, t+\epsilon)$, then the interval of integration is negatively oriented $[x(t-\epsilon), x(t+\epsilon)]$. Therefore, if $\dot{x}(t)$ is negative, we utilize the positive root, and integrate on the positively oriented interval $[x(t+\epsilon), x(t-\epsilon)]$. We make the convention to chose the positive root and integrate on positively oriented intervals. For more details about this integration, refer to [16, Section 11] *for a general mechanical system or to* [12, sub-Section 4.3] *in the context of the magnetic space $\mathbb{R}^3_F$.*

2) We must regard $\frac{dx}{\sqrt{1-G^2(x)}}$ as closed but not exact one-form defined on the algebraic curve $\alpha(G, I)$ from Remark 17. Consequently, the function $\Delta t(c, \mathcal{T})$ not only depends on the endpoints $x(t_0)$ and $x(t_1)$, but also depends on the path $x(\mathcal{T})$. For instance, if c is $\mathbb{R}^3_F$-geodesic for the pair (G, I) and its reduced dynamics has period L, then $x(t) = x(t+nL)$ for all natural number n and

$$\Delta(c,[t,t+nL]) = nL(G,I) \text{ where } L(G,I) := 2\int_I \frac{dx}{\sqrt{1-G^2(x)}}, \tag{2.8}$$

where, we call n the degree of the map $(x,p_x):[t,t+nL]\to\alpha(G,I)$. The expression $L(G,I)$ is the classical formula for the period of a one degree of freedom system, review [16, Section 11]. *For a more deep understanding of the closed one-forms defining the functions $\Delta(c,\mathcal{T})$ and $Cost(c,\mathcal{T})$, consult* [9, Section 50] *for a general mechanical system or* [11, Section 2] *in the context of $J^k(\mathbb{R},\mathbb{R})$.*

The following lemma is consequence of Proposition 22.

Lemma 24. *Let $c(t)$ be an $\mathbb{R}^3_F$-geodesic for the pair (G,I). Then;*

(1)Let $\mathcal{T}_n$ be a sequence of time interval such that $\lim_{n\to\infty}\mathcal{T}_n=[-\infty,\infty]$, then $\lim_{n\to\infty}Cost_t(c,\mathcal{T}_n)$ is finite if and only if $c(t)$ is a homoclinic or direct-type geodesic such that $\lim_{t\to\pm\infty}G(x(t))=1$.

(2) $Cost_t(c,\mathcal{T})=0$ if and only if $G(x)\equiv 1$ on I.

By construction, if $c(t)$ is a homoclinic or heteroclinic geodesic, then the period $L(G,I)$ is not finite. Let us define the normalized period for geodesic such that $\lim_{t\to\pm\infty}G(x(t))=1$.

Definition 25. *Let (G,I) be a pair of a polynomial $G(x)$ in Pen_F, and I be one of its hill intervals. For very pair (G,I), we define the normalized period map $\Theta:(G,I)\to[0,\infty]\times\mathbb{R}$ given by*

$$\begin{aligned}\Theta(G,I)&:=(\Theta_1(G,I),\Theta_2(G,I))\\&=2\Big(\int_I\sqrt{\frac{1-G(x)}{1+G(x)}}dx,\int_I G(x)\frac{1-F(x)}{\sqrt{1-G^2(x)}}dx\Big).\end{aligned}$$

The following lemma provides some properties of the functions $\Delta(c,\mathcal{T})$, $Cost(G,\mathcal{T})$, and $\Theta(G,I)$.

Lemma 26. *The functions $\Delta(c,\mathcal{T})$, $Cost(G,\mathcal{T})$, and $\Theta(G,I)$ has the following properties:*

1) The functions $\Delta t(c,\mathcal{T})$, $\Delta y(c,\mathcal{T})$ and $\Delta z(c,\mathcal{T})$ are smooth function with respect to the (a,b) whenever they are finite. In particular, the period $L(G,I)$ is a smooth functions with respect to the (a,b) if and only if $\alpha(G,I)$ is a smooth curve.

2) If $c(t)$ is an $\mathbb{R}^3_F$-geodesic for the pair (G,I), where $\lim_{t\to\pm\infty}G(x(t))=1$ and $\lim_{n\to\infty}\mathcal{T}_n=[-\infty,\infty]$. Then,

$$\begin{aligned}&\lim_{n\to\infty}Cost(c,\mathcal{T}_n)=\frac{1}{2}\Theta(G,I)\ \text{ if } c(t) \text{ is a direct-type geodesic},\\&\lim_{n\to\infty}Cost(c,\mathcal{T}_n)=\Theta(G,I)\ \text{ if } c(t) \text{ is a homoclinic geodesic}.\end{aligned}$$

3) The functions $Cost_t(c,\mathcal{T})$ and $Cost_y(c,\mathcal{T})$ are smooth function with respect to the (a,b) whenever they are finite. In particular, The normalized period $\Theta_1(G,I)$ is a smooth functions with respect to the (a,b) if and only if $\alpha(G,I)$ is a smooth curve or $\lim_{t\to\pm\infty}G(x(t))=1$.

Proof. 1) Without loss of generality, let us focus on the period $\Delta y(G,I)$. The function $\Delta y(G,I)$ depends not only explicitly on the parameters (a,b)

by the definition of $G(x)$, but also implicitly, since the limits of integration x_0 and x_1 vary with changes in (a, b). Utilizing the implicit function theorem, we compute the partial derivatives of x_0 and x_1 with respect to a and b. To calculate the partial derivatives of $\Delta y(G, I)$ with respect to a and b, we initially set up the change of variable $x = h(\tilde{x})$ and denote $\tilde{F}(\tilde{x}) = F(h(\tilde{x}))$, where $h(\tilde{x})$ is the affine map defined in Proposition 8, to find:

$$\Delta y(G, I) = (x_1 - x_0) \int_0^1 \frac{a + b\tilde{F}(\tilde{x})}{\sqrt{1 - (a + b\tilde{F}(\tilde{x}))^2}} d\tilde{x}.$$

Then we compute the partial derivative with respect of a:

$$\frac{F'(x_1) - F'(x_0)}{bF'(x_1)F'(x_0)} \int_0^1 \frac{a + b\tilde{F}(\tilde{x})}{\sqrt{1 - (a + b\tilde{F}(\tilde{x}))^2}} d\tilde{x}$$
$$+ \frac{x_1 - x_0}{F'(x_1)F'(x_0)} \int_0^1 \frac{F'(x_1)F'(x_0) - \tilde{F}'(\tilde{x})\big((F'(x_0) - F'(x_1))\tilde{x} + F'(x_1)\big)}{(1 - (a + b\tilde{F}(\tilde{x}))^2)^{\frac{3}{2}}} d\tilde{x}.$$

We notice the second integrand has the property that when $\tilde{x} = 0$ and $\tilde{x} = 1$, the numerator is zero making the integral finite. Similar computations hold for the partial derivative with respect of b. This method is the standard technique for computing the partial derivative of a period with respect of the Energy or external parameter as the length of a rod in a pendulum; for further details, refer to [16, Section 11]. Alternatively, consult [17, sub-Chapter 3.10] or insights into the derivatives of elliptic functions.

2) If $c(t)$ is a direct-type geodesic, then $x(t)$ is monotonically decreasing or decreasing, indicating that $x(t)$ travels the hill interval I once. If $c(t)$ is homoclinic geodesic, then $x(t)$ travels the hill interval I twice.

3) Without loss of generality let us focus on the normalized period $\Theta_1(G, I)$. The condition $\lim_{t\to\pm\infty} G(x(t)) = 1$ implies that -1 is a regular value of $G(x)$, ensuring that $\Theta_1(G, I)$ finite. Based on the above discussion, $\Theta_1(G, I)$ is a smooth function with respect the parameters (a, b).

□

2.2.3. *Sequence Of Geodesics On The Magnetic Space.* Let us present some classical results on metric spaces and definitions.

Lemma 27. *Let $c_n(t)$ be a sequence of minimizing geodesics on the compact interval $\mathcal{T}$ converging uniformly to a geodesic $c(t)$, then $c(t)$ is minimizing in the interval $\mathcal{T}$.*

Proof. Let $[t_0, t_1] \subset \mathcal{T}$, since $c_n(t)$ is sequence of minimizing geodesic then $dist_{\mathbb{R}^3_F}(c_n(t_0), c_n(t_1)) = |t_1 - t_0|$ for all n. By the uniform convergence, if $n \to \infty$ then $dist_{\mathbb{R}^3_F}(c(t_0), c(t_1)) = |t_1 - t_0|$. □

Lemma 27 implies the following proposition.

Proposition 28. *Let K be a compact subset of $\mathbb{R}^3_F$ and let $\mathcal{T}$ be a compact time interval. Let us define the following set of $\mathbb{R}^3_F$-geodesics*

$$Min(K,\mathcal{T}) := \left\{\mathbb{R}^3_F\text{-geodesics } c(t) : c(\mathcal{T}) \subset K \text{ and } c(t) \text{ is minimizing in } \mathcal{T}\right\}.$$

Then $Min(K,\mathcal{T})$ is a sequentially compact set with respect to the uniform topology.

Proof. Let $c_n(t)$ be an arbitrary sequence in $Min(K,\mathcal{T})$, we must prove $c_n(t)$ has a uniformly convergent subsequence converging to $c(t)$ in $Min(K,\mathcal{T})$. The space of geodesics $Min(K,\mathcal{T})$ is uniformly bounded and equi-Lipschitz because it consists of minimizing geodesics. By Arzela-Ascoli theorem, every sequence $c_n(t)$ in $Min(K,\mathcal{T})$ has a convergent subsequence $c_{n_s}(t)$ converging uniformly to a smooth curve $c(t)$. By Lemma 27 $c(t)$ is minimizing in $\mathcal{T}$. □

A useful tool for the proof of Theorem A and B is the following.

Lemma 29. *Let $c_1(t)$ be a $\mathbb{R}^3_F$-geodesic in $Min(K,\mathcal{T})$ and let $c_2(t)$ be a $\mathbb{R}^3_F$-geodesic. If $\varphi(x,y,z)$ is an isometry such that $c_2(\mathcal{T}') \subset \varphi(c_1(\mathcal{T}))$, then $c_2(t)$ is in $Min(\varphi(K),\mathcal{T}')$.*

Later on, it will be essential to ensure that a sequence of geodesics converges to a normal geodesic. Inspired by this necessity, we introduce the following definition.

Definition 30. *Let $\{c_n(t)\}_{n\in\mathbb{N}}$ be a sequence of geodesics in $\mathbb{R}^3_F$. We say $\{c_n(t)\}_{n\in\mathbb{N}}$ is strictly normal sequence if $c_n(t)$ is a normal geodesic for all n and every convergent sub-sequence converges to a normal geodesic.*

Let us introduce some standard notion from topology.

The following definition will ensure that a geodesic sequence has a convergent subsequence.

Definition 31. *Let K be a compact interval. Then, we define the following set*

$$Com(K) := \big\{(c(t),\mathcal{T}) :\ c(t) \text{ is a } \mathbb{R}^3_F\text{-geodesic, } \mathcal{T} \text{ is a compact time interval, and } x(t) \in K \text{ for all } t \in \partial\mathcal{T}\big\},$$

where ∂K is the boundary of K.

We say that region $K \times \mathbb{R}^2$ sub-set of $\mathbb{R}^3_F$ is geodesically compact if for every sequence $(c_n,\mathcal{T}_n)$ in $Com(K)$ holding the following conditions

(1) $lim_{n\to\infty}\mathcal{T}_n = [-\infty,\infty]$,

(2) $Cost(c_n,\mathcal{T}_n)$ is uniformly bounded with respect to the supremum norm.

Then there exist a compact subset K_x of $\mathbb{R}$ such that $x_n(\mathcal{T}_n) \subset K_x$ for all n.

3. Heteroclinic Of The Direct-Type Geodesic

This section is devoted to proving Theorem A. Let $\gamma_d(t)$ be an arbitrary heteroclinic of the direct-type geodesic in $J^k(\mathbb{R},\mathbb{R})$ for a polynomial $F_d(x)$.

We will consider the space $\mathbb{R}^3_{F_d}$ and the $\mathbb{R}^3_{F_d}$-geodesic $c_d(t) := \pi_{F_d}(\gamma_d(t))$. Then we will prove the following Theorem.

Theorem 32. *Let $\gamma_d(t)$ be an arbitrary heteroclinic of the direct-type geodesic in $J^k(\mathbb{R},\mathbb{R})$ for a polynomial $F_d(x)$. If $c_d(t) := \pi_{F_d}(\gamma_d(t))$, then $c_d(t)$ is a metric line $\mathbb{R}^3_{F_d}$.*

Without loss of generality, let us assume $\gamma_d(t)$ is a unitary geodesic and let $F_d(x)$ has the form given by Lemma 6. The goal is to show that for arbitrary T the geodesic is minimizing in the interval $[-T,T]$. The strategy to verify the goal is the following: For all $n > T$, we will take a sequence of geodesics $c_n(t)$ minimizing in the interval $[0,T_n]$ and joining the points $c_d(-n)$ and $c_d(n)$, see 3.1. Then, we will identify a convergent subsequence $c_{n_j}(t)$ converging to a $\mathbb{R}^3_{F_d}$-geodesic $c_\infty(t)$ in $Min(K,\mathcal{T})$ and isometry φ in $Iso(\mathbb{R}^3_{F_d})$ such that $c([-T,T]) \subseteq \varphi(c_\infty(\mathcal{T}))$, where K is a compact subset of $\mathbb{R}^3_{F_d}$ and $\mathcal{T}$ is a compact interval. By Lemma 29, $c_d(t)$ is minimizing in $[-T,T]$. Since T is arbitrary, $c_d(t)$ is a metric line.

Let $c_d(t) = (x(t),y(t),z(t))$. Without loss of generality, we can assume that $0 \leq \dot{x}(t)$, since heteroclinic geodesics are either strictly monotone increasing or decreasing, and the proof for the case $0 \geq \dot{x}(t)$ is similar. In addition, we can chose an initial condition $c_d(0) = (x(0),0,0)$ for some $x(0)$ in $(0,1)$, since we can utilize the t, y, and z translations.

3.1. **The Magnetic Space For Heteroclinic Geodesic.**

Lemma 33. *Let q_{max} be equal to $\max_{x\in[0,1]}\{x^{k_1}(1-x)^{k_2}q(x)\}$, where $q(x)$, k_1 and k_2 are given by Lemma 6. The set of all the heteroclinic geodesics of the direct-type $\mathbb{R}^3_{F_d}$-geodesic with hill interval $[0,1]$ is parametrized by*

$$Pen_d := Pen_d^+ \cup Pen_d^-,$$

where

$$Pen_d^{\pm} :=\{(a,b) = \pm(s,1-s): \quad s \in (1-\frac{2}{q_{max}},1)\}.$$

Moreover, the map $\Theta_2 : Pen_d^+ \to \mathbb{R}$ is one to one, and the map $Cost(c_d,\mathcal{T})$ is uniformly bounded in the sup norm by $\Theta_d := \Theta_1(F_d,[0,1])$ for all time interval $\mathcal{T}$.

The set Pen_d^+ corresponds to the set of unitary heteroclinic geodesics of the direct-type. While, the set Pen_d^- defines the set of heteroclinic geodesics of the direct-type such that $\lim_{t\to\pm\infty} G(x(t)) = -1$.

Proof. Let us prove Pen_d^+ parametrizes unitary heteroclinic geodesics of the direct-type. Let $G_s(x)$ be the polynomial defined by $(a,b) = (s,1-s)$. Then, it is enough to prove that $G_s(x)$ holds the conditions: being a non constant polynomial, $G_s(0) = G_s(1) = 1$, $G_s'(0) = G_s'(1) = 0$, and $-1 < G_s(x) < 1$ if x is in $(0,1)$. By Lemma 6, $F_d(x) = 1 - x^{k_1}(1-x)^{k_2}q(x)$, where k_1 and k_2 are bigger than 1, $0 < x^{k_1}(1-x)^{k_2}q(x) < 2$ if x is in $(0,1)$, and q_{max} is in

$(0,2)$. Then, $G_s(x) = 1-(1-s)x^{k_1}(1-x)^{k_2}q(x)$ implies $G_s(0) = G_s(1) = 1$, and $G'_s(0) = G'_s(1) = 0$. In addition, $-1 < 1 + (s-1)q_{max} < 1$ yields $1 - \frac{2}{q_{max}} < s < 1$. So $G_s(x)$ is a non-constant polynomial for all s in $(1 - \frac{2}{q_{max}}, 1)$ and satisfies the required conditions. The proof for the set Pen_d^- is the same. Let us proof that if $G(x) = a + bF_d(x)$ generates a direct-type geodesic with hill interval $[0,1]$, then $G(x)$ is in Pen_{F_d}. Indeed, $G(x)$ must satisfy $G(0) = G(1) = \pm 1$, since $F_d(0) = F_d(1) = 1$, we have $a + b = \pm 1$. So if $a = s$ and we chose the positive sign, we get $b = 1 - s$, and if $a = -s$ and we chose the negative sign, we get $b = s - 1$. The rest of the conditions follow the same way that the first part of the proof.

To prove that $\Theta_2 : Pen_d^+ \to \mathbb{R}$ is one to one, we consider the one-parameter family of polynomials $G_s(x) = 1 - (1-s)x^{k_1}(1-x)^{k_2}q(x)$. The identities $1-G_s^2(x) = (1-G_s(x))(1+G_s(x))$ and $(1-s)(1-F_d(x)) = 1-G_s(x)$ imply the following equality

$$\frac{1 - F_d(x)}{\sqrt{1 - G_s^2(x)}} = \frac{\sqrt{1 - G_s(x)}}{(1-s)\sqrt{1 + G_s(x)}}.$$

Then, the quotient is well defined in the interval $[0,1]$ and $\Theta_2(G_s, [0,1])$ is finite. Thus, $\Theta_2(G_s, [0,1])$ can be regarded as a function of s on Pen_d. Let us proceed to calculate its derivative:

$$\begin{aligned}
\frac{d}{ds}\Theta_2(G_s, [0,1]) &= \frac{d}{ds}\int_{[0,1]} \frac{\big(1 - F_d(x)\big)G_s(x)}{\sqrt{1 - G_s^2(x)}}dx \\
&= \int_{[0,1]} \big(1 - F_d(x)\big)\frac{d}{ds}\Big(\frac{G_s(x)}{\sqrt{1 - G_s^2(x)}}\Big)dx \\
&= \int_{[0,1]} \frac{\big(1 - F_d(x)\big)\frac{d}{ds}G_s(x)}{(1 - G_s^2(x))^{\frac{3}{2}}}dx.
\end{aligned}$$

We notice that $\frac{d}{ds}G_s(x) = 1 - F_d(x)$, so

$$\frac{d}{ds}\Theta_2(G_s, [0,1]) = \int_{[0,1]} \frac{\big(1 - F_d(x)\big)^2}{(1 - G_s^2(x))^{\frac{3}{2}}}dx = \int_{[0,1]} \frac{\sqrt{1 - G_s(x)}}{(1-s)^2(1 + G_s(x))^{\frac{3}{2}}}dx.$$

Then, $\frac{d}{ds}\Theta_2(G_s, [0,1])$ is finite and positive for all s in $(1 - \frac{2}{q_{max}}, 1)$.

Since $F_d(x) \neq -1$ if x is in $[0,1]$, the constant Θ_d is finite. Let us prove that maximum norm of $Cost(c_d, \mathcal{T})$ is bounded by Θ_d for all time interval $\mathcal{T}$. Using Proposition 22 and the condition $|F_d(x)| \leq 1$ for x in $[0,1]$, we find that:

$$\begin{aligned}
|Cost_y(c_d, \mathcal{T})| &< Cost_t(c_d, \mathcal{T}) \\
&< 2\int_{[0,1]} \sqrt{\frac{1 - F_d(x)}{1 + F_d(x)}}dx =: \Theta_1(F_d, [0,1]).
\end{aligned}$$

□

We remark that Pen_d^+ defines the heteroclinic geodesics of the direct type such that

$$\lim_{t\to\infty} y(t) = \infty \text{ and } \lim_{t\to-\infty} y(t) = -\infty,$$

while Pen_d^- defines the heteroclinic geodesics of the direct type such that

$$\lim_{t\to\infty} y(t) = -\infty \text{ and } \lim_{t\to-\infty} y(t) = \infty.$$

Lemma 34. *Let $\Omega(F_d)$ be the region $hill(F_d)\times\mathbb{R}^2$, then $c_d(t)$ is minimizing between the curves that lay in $\Omega(F_d)$.*

The proof is consequence of the calibration function defined on the region $\Omega(F_d)$ and provided in [12, Section 5].

Remark 35. *There exist $T_d^* > 0$ such that $y_d(t) > 0$ if $T_d^* < t$, and $y_d(t) < 0$ if $-T_d^* > t$.*

Proof. The intermediate value theorem implies the existence of T_d^*. Since by construction, $\lim_{t\to\infty} y_d(t) = \infty$ and $\lim_{t\to-\infty} y_d(t) = -\infty$. □

Lemma 36. *If F_d is a polynomial given by Lemma 6. Then, the region $[0,1]\times\mathbb{R}^2 \subset \mathbb{R}^3_{F_d}$ is geodesically compact.*

The proof is Appendix A.1.

3.2. Proof of Theorem 32.

3.2.1. *Set Up The Proof Of Theorem 32.* Let T be arbitrarily large and consider the sequence of points $c_d(-n)$ and $c_d(n)$ where $T < n$ and n is in $\mathbb{N}$. Let $c_n(t) = (x_n(t), y_n(t), z_n(t))$ be a sequence of minimizing $\mathbb{R}^3_{F_d}$-geodesics, in the interval $[0, T_n]$ such that:

$$c_n(0) = c_d(-n), \;\; c_n(T_n) = c_d(n) \text{ and } T_n \leq 2n. \tag{3.1}$$

We call the equations and inequality from (3.1) the endpoint conditions and the shorter condition, respectively. Since the endpoint condition holds for all n, then the sequences of endpoints $c_n(0)$ and $c_n(T_n)$ hold the asymptotic conditions:

$$\lim_{n\to\infty} c_n(0) = (0,-\infty,-\infty), \;\; \lim_{n\to\infty} c_n(T_n) = (1,\infty,\infty). \tag{3.2}$$

In addition, the endpoint condition implies the difference of endpoints are equal

$$\Delta y(c_d,[-n,n]) = \Delta y(c_n,[0,T_n]), \text{ and } \Delta z(c_d,[-n,n]) = \Delta z(c_n,[0,T_n]).$$

Therefore, $Cost_y(c_d,[-n,n]) = Cost_y(c_n,[0,T_n])$ for all n. Lemma 26 yields the asymptotic period condition:

$$\lim_{n\to\infty} Cost_y(c_n,[0,T_n]) = \frac{1}{2}\Theta_2(F_d,[0,1]). \tag{3.3}$$

We remark that a equation (3.3) does not tell that $c_n(n)$ converges to $c_d(t)$ neither $G_n(c)$ converges to $F_d(x)$. It only tells that that the sequence $Cost_y(c_n,[0,T_n])$ in $\mathbb{R}$ converges to the value $\frac{1}{2}\Theta_2(F_d,[0,1])$.

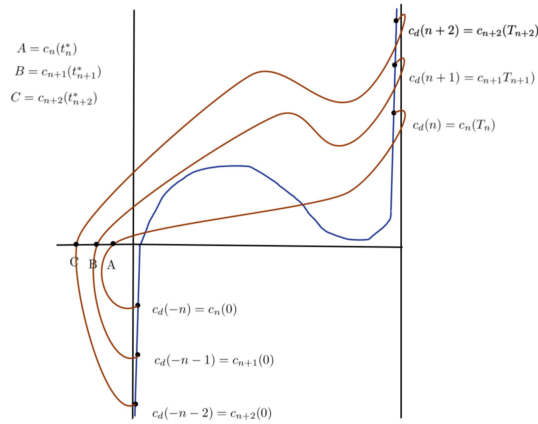

FIGURE 3.1. The images show the projection to $\mathbb{R}^2$, with coordinates (x, y), of a heteroclinic of the direct-type geodesic $c_d(t)$ and the sequence of geodesics $c_n(t)$.

Corollary 37. *The sequence of $\mathbb{R}^3_{F_d}$-geodesics $c_n(t)$ is not a sequence of geodesic lines and does not converge to a geodesic line. In particular, $c_n(t)$ is strictly normal sequence.*

Proof. Lemma 34 implies that if $c_n(t)$ is shorter than $c_d(t)$, then $c_n(t)$ must leave the region $\Omega(F_d)$ and come back. Thus $c_n(t)$ is a geodesic for non-constant polynomial $G_n(x)$, and $c_n(t)$ is not a geodesic line.

The sequence $c_n(t)$ cannot converge to a geodesic line; since $\lim_{n\to\infty} c_n(t)$ must satisfy the asymptotic condition given by equation (3.2), the only line in the plane (x, y) that travel from $y = -\infty$ to $y = \infty$ in a finite x-interval is the vertical line, but the x-interval of the vertical line is a point. In particular, Lemma 19 implies $c_n(t)$ is strictly normal sequence. □

The construction of the $\mathbb{R}^3_{F_d}$-geodesic $c_n(t)$ is such that the initial condition $c_n(0)$ is unbounded. The following Proposition provides a bounded initial condition.

Proposition 38. *Let n be a natural number larger than T_d^*, where T_d^* is given by Remark 35. Then, there exist a time t_n^* in $(0, T_n)$, and a compact set $K_0 \subset \mathbb{R}^3_{F_d}$ such that $c_n(t_n^*)$ is in K_0 for all $n > T_d^*$.*

Proof. Let n be a natural number larger than T_d^*. By construction, $y_n(0) < 0$ and $y_n(T_n) > 0$, the intermediate value theorem implies that exist a t_n^* in $(0, T_n)$ such that $y_n(t_n^*) = 0$. Lemma 33 tells that $Cost(c_n, [0, T_n])$ is uniformly bounded in the sup norm, and Lemma 36 says that the region $[0, 1] \times \mathbb{R}^2$ is geodesically compact, then there exists a compact set K_x such that $x_n(t)$ is in K_x for all t in $[0, T_n]$ and for all n.

Let us prove that $z_n(t_n^*)$ is bounded: the endpoint conditions imply $Cost_y(c_d, [-n, n]) = Cost_y(c_n, [0, T_n])$ and by definition of $Cost_y$, it follows that:

$$z_n(t_n^*) - z_n(0) = \Delta z(c_n, [0, t_n^*]) = \Delta y(c_n, [0, t_n^*]) - Cost_y(c_n, [0, t_n^*]),$$
$$z_d(0) - z_d(-n) = \Delta z(c_d, [-n, 0]) = \Delta y(c_d, [-n, 0]) - Cost_y(c_d, [-n, 0]).$$

By construction;

$$\Delta y(c_n, [0, t_n^*]) = \Delta y(c_d, [-n, 0]),\ z_d(0) = 0 \text{ and } z_n(0) = z_d(-n),$$

then

$$\begin{aligned} |z_n(t_n^*)| &= |\Delta z(c_n, [0, t_n^*]) - \Delta z(c_d, [-n, 0])| \\ &\leq |Cost_y(c_n, [0, t_n^*])| + |Cost_y(c_d, [-n, 0])|. \end{aligned} \tag{3.4}$$

Lemma 33 says $Cost_y(c_d, [-n, 0])$ is bounded by Θ_d. In addition, Lemma 53 implies $Cost_y(c_n, [0, t_n^*])$ is bounded by C_z^*, consult the Appendix B. So $|z_n(t_n^*)|$ is bounded by $\Theta_d + C_z$.

Let K_0 be the compact set given by $K_x \times [-1, 1] \times [-\Theta_d - C_z^*, \Theta_d + C_z^*]$. We just proved $c_n(t_n^*)$ is in K_0. □

Let us reparametrize the sequence of minimizing $\mathbb{R}^3_{F_d}$-geodesics $c_n(t)$. Let $\tilde{c}_n(t)$ be a minimizing $\mathbb{R}^3_{F_d}$-geodesic in the interval $\mathcal{T}_n := [-t_n^*, T_n - t_n^*]$ given by $\tilde{c}_n(t) := c_n(t + t_n^*)$. Then, $\tilde{c}_n(0)$ is bounded and $\tilde{c}_n(t)$ is a minimizing $\mathbb{R}^3_{F_d}$-geodesics in the interval $\mathcal{T}_n$.

Corollary 39. *There exists a subsequence $\mathcal{T}_{n_j}$ such that $\mathcal{T}_{n_j} \subset \mathcal{T}_{n_{j+1}}$.*

Proof. On one side $\tilde{c}_n(0)$ is bounded, on the other side the endpoints $\tilde{c}_n(-t_n^*)$ and $\tilde{c}_n(T_n - t_n^*)$ are unbounded. Then $[-t_n^*, T_n - t_n^*] \to [-\infty, \infty]$ when $n \to \infty$, and we can take a subsequence of intervals $\mathcal{T}_{n_j}$ such that $\mathcal{T}_{n_j} \subset \mathcal{T}_{n_{j+1}}$. □

For simplicity, we will use the notation $\mathcal{T}_n$ for the subsequence $\mathcal{T}_{n_j}$.

Lemma 40. *Let N be a fixed natural number larger than T_d^*. Then there exist compact set $K_N \subset \mathbb{R}^3_F$ such that $\tilde{c}_n(t)$ is in $Min(K_N, \mathcal{T}_N)$ if $n > N$.*

Proof. Since $\tilde{c}_n(t)$ is minimizing on the interval $\mathcal{T}_n$, it follows that $\tilde{c}_n(t)$ is minimizing on the interval $\mathcal{T}_N \subset \mathcal{T}_n$ if $n > N$. Moreover, $\Delta x(\tilde{c}_n, \mathcal{T}_N)$ and $\Delta y(\tilde{c}_n, \mathcal{T}_N)$ are bounded by T_N, since T_N is the length of the interval $\mathcal{T}_N$, and by construction $|\dot{\tilde{x}}_n| \leq 1$ and $|\dot{\tilde{y}}_n| \leq 1$. Using equation (2.6), we have

$$|\Delta z(\tilde{c}_n, \mathcal{T}_N)| \leq \int_{-t_N^*}^{T_n - t_N^*} |F(x(t))| dt \leq T_N \max_{x \in K_x + [-T_n, T_n]} |F(x)| =: C_z.$$

Let K_N be the compact set $K_0 + [-T_n, T_n] \times [-T_n, T_n] \times [-C_z, C_z]$. We just prove that $\tilde{c}_n(\mathcal{T}_N) \subset K_N$. □

Therefore, $\tilde{c}_n(t)$ has a convergent subsequence $\tilde{c}_{n_j}(t)$ converging to a $\mathbb{R}^3_{F_d}$-geodesic $c_\infty(t)$ in $Min(K_n, \mathcal{T}_N)$. Corollary 37 implies that $c_\infty(t)$ is a normal $\mathbb{R}^3_{F_d}$-geodesic, then we can associate $c_\infty(t)$ to a polynomial $G(x)$ in Pen_{F_d}. The following Lemma tells that $G(x) = F_d(x)$.

Lemma 41. *$G(x) = F_d(x)$ is the unique polynomial in the pencil of $F_d(x)$ satisfying the asymptotic conditions given by (3.2) and (3.3).*

Proof. By Proposition 28, $\tilde{c}_n(t)$ has a convergent subsequence $\tilde{c}_{n_s}(t)$ converging to a minimizing geodesic $c_\infty(t)$ on the interval $\mathcal{T}_N$. Being a $\mathbb{R}^3_{F_d}$-geodesic, $c_\infty(t)$ is associated to a polynomial $G(x) = a + bF_d(x)$. By Proposition 22 the coordinates y and z diverge when x approaches to 0 if and only if $G(0) = 1$ and $G'(0) = 0$, by construction $x = 0$ is a critical point of $G(x)$, then $G(0) = a + b$ must be equal 1, to satisfy the asymptotic conditions given by (3.2). Then (a, b) is in Pen^+_d, the set defined in Lemma 33. Since the map $\Theta_2 : Pen^+_d \to \mathbb{R}$ is one to one, the unique polynomial in Pen^+_d satisfying the condition (3.2) and (3.3) is $G(x) = F_d(x)$. □

3.2.2. *Proof of Theorem 32.*

Proof. Since $c_\infty(t)$ and $c_d(t)$ are $\mathbb{R}^3_{F_d}$-geodesics for $F_d(x)$ with the same hill interval, there exists a translation $\varphi_{(y_0,z_0)}$ in $Iso(\mathbb{R}^3_{F_d})$ sending $c_\infty(t)$ to $c_d(t)$. Using that N is arbitrary and $c_d([-T, T])$ is bounded, we can find compact sets $K := K_N$ and $\mathcal{T} := \mathcal{T}_N$ such that $c_d([-T, T]) \subset \varphi_{(y_0,z_0)}(c_\infty(\mathcal{T}))$ and c_∞ is in $Min(K, \mathcal{T})$. Lemma 29 implies that $c_d(t)$ is minimizing in $[-T, T]$ and T is arbitrary. Therefore, $c_d(t)$ is a metric line in $\mathbb{R}^3_{F_d}$. □

3.2.3. *Proof of Theorem A.*

Proof. Let $\gamma_d(t)$ be an arbitrary heteroclinic of the direct-type geodesic. By Theorem 32, $c_d(t) := \pi_{F_d}(\gamma_d(t))$ is a metric line in $\mathbb{R}^3_{F_d}$. Since π_{F_d} is a sub-Riemannian submersion and $\gamma_d(t)$ is the lift of $c_d(t)$, then Proposition 10 implies $\gamma_d(t)$ is a metric line in $J^k(\mathbb{R}, \mathbb{R})$. □

4. Homoclinic Geodesics In Jet Space

This chapter is devoted to proving Theorem B. Let $\gamma_h(t)$ be the homoclinic geodesic in $J^k(\mathbb{R}, \mathbb{R})$ for $F_h(x) := \pm(1 - bx^{2n})$. We will consider the space $\mathbb{R}^3_{F_h}$ and the geodesic $c_h(t) := \pi_{F_h}(\gamma_h(t))$, then we will prove the following Theorem.

Theorem 42. *Let $\gamma_h(t)$ be an arbitrary homoclinic geodesic in $J^k(\mathbb{R}, \mathbb{R})$ for the polynomial $F_h(x) := \pm(1 - bx^{2n})$. If $c_h(t) := \pi_{F_h}(\gamma_h(t))$, then $c_h(t)$ is a metric line $\mathbb{R}^3_{F_h}$.*

The following corollary is an immediate consequence of Theorem 42.

Corollary 43. *The Metric lines in the Martinet Manifold are precisely geodesic of the type line and homoclinic.*

Proof. Let us consider the polynomial $F_h(x) = 1 - \frac{x^2}{2}$, then $\mathbb{R}^3_{F_h}$ is the Martinet manifold, however it is not in the standard form. The change of variable $\phi(x, y, z) = (x, -y, y - z)$, which we can think it as a sub-Riemannian submersion, maps standard Martinet Manifold $(\mathbb{R}^3, \frac{x^2}{2}dy - dz)$ to the magnetic space $\mathbb{R}^3_{F_h}$. Since $\phi^*(F_h(x)dy - dz) = \frac{x^2}{2}dy - dz$.

The geodesic in the Martinet Manifold are of the type: line, x-periodic and homoclinic. The x-periodic are not metric lines, consult [12, sub-Section 6.2].

Proposition 10 and Theorem 42 imply geodesic of type line and homoclinic are metric lines. ◻

Without loss of generality, we will consider the polynomial $F_h(x) := 1 - 2x^{2n}$ with hill interval $[0,1]$. The strategy to prove Theorem 32 is the same as the one used for Theorem 42.

Before prove Theorem 42, we present the following.

Theorem 44. *Let $F_h(x)$ be the polynomial $1-2x^{2n+1}$, and $\mathbb{R}^3_{F_h}$ be the magnetic space. If $c(t)$ is the homoclinic $\mathbb{R}^3_{F_h}$-geodesic corresponding to $F_h(x)$. Then, $c(t)$ is not a metric line $\mathbb{R}^3_{F_h}$.*

Theorem 44 say that we cannot use the the magnetic space $\mathbb{R}^3_F$ to prove the Conjecture 2 for the general homoclinic case. Since the method used to prove Theorem 42 does not work for the odd case $F(x) := 1 - 2x^{2n+1}$. The proof of Theorem 44 is in Appendix C.

4.1. **The Magnetic Space For the Homoclinic Geodesics.** Without loss of generality, $c_h(0) = (1,0,0)$, by use of the t, y and z translations. By the time reversibility of the reduced Hamiltonian H_F given by (2.2), it follows that $x(-n) = x(n)$ and $\Delta x(c_h, [-n,n]) := x(n) - x(-n) = 0$ for all n.

Lemma 45. *Let $c_h(t)$ be the homoclinic $\mathbb{R}^3_{F_h}$-geodesic for $F_h(x) := 1-2x^{2n}$, then*

$$\Theta_2(F_h, [0,1]) < 0.$$

Proof. By construction, $-xF_h'(x) = 2n(1 - F_h(x))$. Using integration by parts it follows that

$$\begin{aligned}\Theta_2(F_h,[0,1]) &= \frac{-1}{n}\int_{[0,1]} \frac{xF_h'(x)F_h(x)dx}{\sqrt{1-F_h^2(x)}} \\ &= \frac{1}{n}x\sqrt{1-F_h^2(x)}\Big|_0^1 - \frac{1}{n}\int_{[0,1]}\sqrt{1-F_h^2(x)}dx.\end{aligned}$$

The desired result follows by $x\sqrt{1-F_h^2(x)}\Big|_0^1 = 0$. ◻

Corollary 46. *The set of all the homoclinic $\mathbb{R}^3_{F_h}$-geodesics is parametrized by*

$$Pen_h := Pen_h^+ \cup Pen_h^-,$$

where

$$Pen_h^{\pm} := \{(a,b) = \pm(s, 1-s) : \ \ s \in (-\infty, 1)\}.$$

Moreover, the map $\Theta_2 : Pen_h^+ \to \mathbb{R}$ is one to one, and the map $Cost(c_h, \mathcal{T})$ is uniformly bounded by $\Theta_1(F_h,[0,1]) := \Theta_h$ for all time interval $\mathcal{T}$.

Proof. Let us prove Pen_h^+ parametrizes homoclinic geodesics. Let $G_s(x)$ be the polynomial defined by $(a,b)=(s,1-s)$. So, $G_s(x)=1-2(1-s)x^{2n}$ and one of its hill interval is $[0,\ \sqrt[2n]{\frac{1}{1-s}}]$. Then, it follows: $G_s(x)$ is not a constant polynomial, $G(0)=-G(\sqrt[2n]{\frac{1}{1-s}})=1$, $G'(0)=0$, and $-1<G(x)<1$ if x is in $(0,\ \sqrt[2n]{\frac{1}{1-s}})$. The proof for the set Pen_d^- is the same. Let us proof that if $G(x)=a+bF_d(x)$ generates a homoclinic geodesic, then $G(x)$ is in Pen_h. Indeed, $G(x)$ has a unique critical point $x=0$, then $G(x)$ must satisfy $G(0)=\pm 1$, since $F_d(0)=1$, we have $a+b=\pm 1$. So if $a=s$ and we chose the positive sign, we get $b=1-s$, and if $a=-s$ and we chose the negative sign, we get $b=s-1$.

To prove that $\Theta_1(a,b): Pen_h^+\to\mathbb{R}$ is one to one, we consider the one-parameter family of homoclinic polynomial $G_s(x)$ with hill interval $[0,\ \sqrt[2n]{\frac{1}{1-s}}]$. Thus, $\Theta_1(G_s,[0,\ \sqrt[2n]{\frac{1}{1-s}}]):(0,\infty)\to\mathbb{R}$ is a one variable function and it is enough to show it is a monotone decreasing function. Let us set up the change of variable $x=\sqrt[2n]{\frac{1}{1-s}}\tilde{x}$, then $G_s(x)=F_h(\tilde{x})$ and

$$\begin{aligned}\Theta_2(G_s,[0,\ \sqrt[2n]{\frac{1}{1-s}}]) &= \int_{[0,\ \sqrt[2n]{\frac{1}{1-s}}]}\frac{(1-F_h(x))G_s(x)}{\sqrt{1-G_s^2(x)}}dx\\ &= (\sqrt[2n]{\frac{1}{1-s}})^{2n+1}\Theta_2(F_h,[0,1]).\end{aligned}$$

Since $\frac{1}{1-s}$ is monotone increasing and $\Theta_2(F_h,[0,1])$ is negative. Then, we conclude $\Theta_2(G_s,[0,\ \sqrt[2n]{\frac{1}{1-s}}])$ is a monotone decreasing function with respect to s. □

Corollary 47. *There exists $T_h^*>0$ such that $y_h(t)>0$ if $T_h^*<t$ and $y_h(t)<0$ if $-T_h^*>t$. Moreover, $Cost_y(c_h,[-t,t])<0$ if $T_h^*<t$.*

Proof. Since $Cost_y(c_h,[-t,t])\to\Theta_2(F_h,[0,1])$ as $t\to\infty$ and $\Theta_2(F_h,[0,1])<0$, we can find the desired T_h^*. The rest of the proof is equal to Remark 35. □

Lemma 48. *If $F_h=1-2x^{2n}$. Then, the region $[0,1]\times\mathbb{R}^2\subset\mathbb{R}^3_{F_h}$ is geodesically compact.*

The proof of Lemma 48 is the same as the one for Lemma 36.

4.2. **Set Up The Proof Of Theorem 42.** Let T be arbitrarily large and consider the sequence of points $c_h(-n)$ and $c_h(n)$ where $T<n$ and n is in $\mathbb{N}$. Let $c_n(t)=(x_n(t),y_n(t),z_n(t))$ be a sequence of minimizing $\mathbb{R}^3_{F_h}$-geodesics in the interval $[0,T_n]$ such that:

$$c_n(0)=c_h(-n),\ \ c_n(T_n)=c_h(n)\ \text{ and }\ T_n\le 2n. \tag{4.1}$$

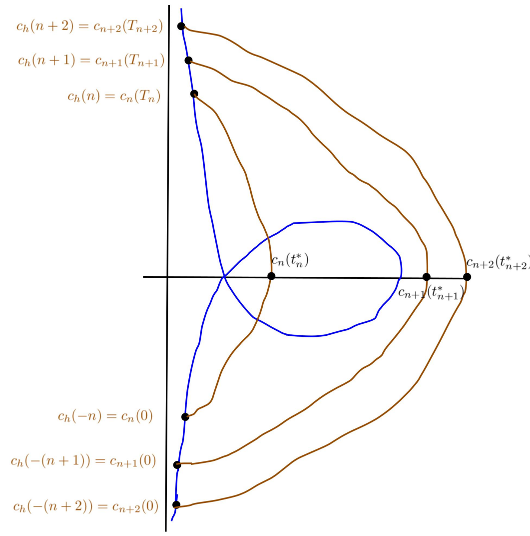

FIGURE 4.1. The images show the projection to $\mathbb{R}^2$, with coordinates (x, y), of a homoclinic-geodesic $c_h(t)$ and the sequence of minimizing geodesics $c_n(t)$.

See figure 4.1. We call the equations and inequality from (4.1) the end-point conditions and the shorter condition, respectively. Since the endpoint condition holds for all n, the sequence $c_n(t)$ has the asymptotic conditions:

$$\lim_{n\to\infty} c_n(0) = (0, -\infty, -\infty), \quad \lim_{n\to\infty} c_n(T_n) = (0, \infty, \infty), \tag{4.2}$$

and the asymptotic period condition

$$\lim_{n\to\infty} Cost_y(c_n, [0, T_n]) = \Theta_2(F_h, [0, 1]). \tag{4.3}$$

The following Corollary tells us $c_n(t)$ is not a sequence of line geodesics. We remark that applying the calibration function found in [12, Section 5] is only possible for every sub-interval of the time intervals $(-\infty, 0)$ or $(0, \infty)$, in other words the calibration method does not work on an interval containing the time $t = 0$, which correspond to the point when the x coordinate bounce on the point $x = 1$, for more details see [12, Section 5].

Corollary 49. *Let n be larger than T_h^*, where T_h^* is given by Corollary 47, then the sequence of geodesics $c_n(t)$ neither is a sequence of geodesic lines, nor converges to a geodesics line. In particular, $\{c_n(t)\}_{n\in\mathbb{N}}$ is a strictly normal sequencce geodesic.*

Proof. Let us assume that $c_n(t)$ is a sequence of geodesic lines. Since $\Delta x(c_h, [-t, t])) = 0$ for all n and $\Delta y(c_h, [-t, t])) > 0$ for all $n > T_h^*$, the unique geodesic line satisfying these conditions is the vertical line, which is generated by the polynomial $G_n(x) = 1$. Since $1 - F_h(x) > 0$ for all x, then

$(1 - F_h(x))G_n(x) > 0$ for all x and it follows that:

$$Cost_y(c_n, [0, T_n]) = \int_0^T (1 - F_h(x(t)))G_n(x(t))dt > 0.$$

This contradicts the endpoint conditions given by (4.1) since if $T_h^* < t$ then $Cost_y(c_h, [-t, t]) < 0$. The same proof follows if $c_n(t)$ converges to a geodesics line $c(t)$ generated by $G(x) = 1$, since there exists N big enough that $G_n(x) > \frac{1}{2}$ for $n > N$. □

Notice that this proof cannot be done in the case $F_h(x) = 1 - 2x^{2n+1}$. In Appendix C under the hypothesis $F_h(x) = 1 - 2x^{2n+1}$, we will find a sequence of curves $c_n(t)$ shorter than $c_h(t)$ that converges to the abnormal geodesic.

The proof that the sequence of geodesic $c_n(t)$ converge to a normal geodesic $c_\infty(t)$ for the polynomial $F_h(x)$ is the same as the one provided in sub-Section 3.2. As well as the proof of Theorems 42 and B are the same as the ones for Theorems 32 and A, respectively.

5. Conclusion

We formalized the method used in [12] to prove that a particular geodesic is a metric line. Theorem A proves the Conjecture 2 for the heteroclinic of the direct-type case, and the problem remains open for the homoclinic case. Theorem 44 says we cannot use the space $\mathbb{R}^3_F$ to prove the Conjecture for the homoclinic case. However, Theorem 44 does not imply that the Conjecture is false. The homoclinic case can be solved by showing the corresponding period map in $J^k(\mathbb{R}, \mathbb{R})$ restricted to the homoclinic geodesics is one-to-one.

Appendix A. Proof Of Lemma 36

Definition 50. *Let us consider the vector space of polynomial on $\mathbb{R}$ of degree bounded by k, and let $||F||_\infty := \sup_{x\in[0,1]} |F(x)|$ be the uniform norm. We denote by $B(k)$ the closed ball of radius 1.*

Definition 51. *We say a polynomial F is unitary if F has a hill interval $[0, 1]$, and let $\mathcal{P}_N(k)$ be the set of unitary polynomials.*

Corollary 52. *If $G_n(x)$ is a sequence of non-constant polynomials in Pen_F with hill interval $I_n = [x_n, x'_n]$ such that $G_n(x_n) = G_n(x'_n) = 1$, $\lim_{n\to\infty} x_n = -\infty$ and $\lim_{n\to\infty} x'_n = \infty$, then $F(x)$ must be even degree.*

Proof. Let $G_n(x)$ be equal to $a_n + b_nF(x)$. There exists K_x a compact set containing all the roots of $F(x)$, and let n be large enough that $K_x \subset I_n$. Let us assume $F(x)$ is an odd degree. Without loss of generality, let us assume $F(x'_n) > 0$ and $F(x_n) < 0$, then $0 = G(x'_n) - G(x_n) = b_n(F(x'_n) - F(x_n))$ and $b_n = 0$ since $F(x'_n) - F(x_n) > 0$, which is a contradiction to the assumption that $G_n(x)$ is a sequence of non-constant polynomials. □

A.1. **Proof Of Lemma 36.**

Proof. Let $(c_n, \mathcal{T}_n)$ be a pair in $Com([0,1])$ where $\lim_{n\to\infty} \mathcal{T}_n = [-\infty, \infty]$. We will prove that if $x_n(\mathcal{T}_n)$ is unbounded, then $\Theta(c, \mathcal{T}_n)$ is unbounded.

The sequence of $c_n(t)$ of $\mathbb{R}^3_F$-geodesics, induces a sequence of pairs (G_n, I_n) where $G_n(x)$ is a polynomial in Pen_F and I_n is one of its hill interval. Here, we will use the notation $x_n(\mathcal{T}_n) = [(\hat{x}_0)_n, (\hat{x}_1)_n] \subseteq I_n$, where $(\hat{x}_0)_n$ and $(\hat{x}_1)_n$ are the endpoint of the image $x_n(\mathcal{T}_n)$, which may coincide with the endpoint of the hill interval I_n. We use the sequence $G_n(x)$ to define a sequence of unitary polynomials $\hat{G}_n(\tilde{x}) := G_n(h_n(\tilde{x}))$ where $h_n(\tilde{x}) = (\hat{x}_0)_n + u_n\tilde{x}$ with $u_n := (\hat{x}_0)_n - (\hat{x}_1)_n$. The condition of $x_n(\mathcal{T}_n)$ being unbounded implies $\lim_{n\to\infty} u_n = \infty$. Since $\hat{G}_n(\tilde{x})$ is in $B(k)$, then there exists a subsequence $\hat{G}_{n_j}(\tilde{x})$ converging to $\hat{G}(\tilde{x})$. Let us proceed by the following cases: case $\hat{G}(\tilde{x}) \neq 1$ or case $\hat{G}(\tilde{x}) = 1$.

Case $\hat{G}(\tilde{x}) \neq 1$: Setting the change of variable $x = h_{n_j}(\tilde{x})$, we have

$$Cost_t(c_{n_j}, \mathcal{T}_{n_j}) = u_{n_j} \int_{[0,1]} \sqrt{\frac{1-\hat{G}_{n_j}(\tilde{x})}{1+\hat{G}_{n_j}(\tilde{x})}} d\tilde{x},$$

the above expression may have some degree, see remark 23, but its irrelevant for the proof. Then, Fatou's lemma implies

$$0 < \int_{[0,1]} \sqrt{\frac{1-\hat{G}(\tilde{x})}{1+\hat{G}(\tilde{x})}} d\tilde{x} \leq \liminf_{n_j\to\infty} \int_{[0,1]} \sqrt{\frac{1-\hat{G}_{n_j}(\tilde{x})}{1+\hat{G}_{n_j}(\tilde{x})}} d\tilde{x}.$$

Then, $u_{n_j} \to \infty$ implies $Cost_t(c_{n_j}, \mathcal{T}_{n_j}) \to \infty$ when $n_j \to \infty$.

Case $\hat{G}(\tilde{x}) = 1$: Then, for big enough n_j there exists a positive constant $0 < A_{n_j} < 1$ such that $A_{n_j} < \hat{G}_{n_j}(\tilde{x})$ for all $\tilde{x}$ in $[0,1]$. If $1 - F(x) \neq 0$ for all x in $\mathbb{R}$, then we have

$$A_{n_j} |\mathcal{T}_{n_j}| \min_{x\in\mathbb{R}} |1 - F(x)| \leq |Cost_y(c, \mathcal{T}_{n_j})|,$$

where $|\mathcal{T}_{n_j}|$ is the Lebesgue measure of the set $\mathcal{T}_{n_j}$. We conclude that the left side of the above equation goes to infinite as $n_j \to \infty$, so does $|Cost_y(c, \mathcal{T}_{n_j})|$.

Let us assume $1 - F(x)$ changes sign, we proceed by cases: Case 1, $(\hat{x}_0)_{n_j}$ and $(\hat{x}_1)_{n_j}$ are both unbounded. Case 2, $(\hat{x}_0)_{n_j}$ is bounded and $(\hat{x}_1)_{n_j}$ is unbounded. Case 3, $(\hat{x}_0)_{n_j}$ is unbounded and $(\hat{x}_1)_{n_j}$ is bounded.

Case 1: by Corollary 52 implies $F(x)$ is even and without loss of generality, let us assume that there exist a compact set K'_x such that $1 - F(x) < 0$ for all x in K'_x. We split the integral for $Cost_y(c, \mathcal{T}_{n_j})$ given by Definition 25 and equation 2.6 in the following way

$$\begin{aligned} Cost_y(c, \mathcal{T}_{n_j}) = &\int_{x_{n_j}^{-1}(K'_x)} (1 - F(x(t)))G_{n_j}(x(t))dt \\ &+ \int_{\mathcal{T}_{n_j} \setminus x_{n_j}^{-1}(K'_x)} (1 - F(x(t)))G_{n_j}(x(t))dt, \end{aligned} \tag{A.1}$$

The second integral domain the first one. Since the set K'_x is compact and $|1-F(x)|$ is bounded, while $|1-F(x)|$ is not bounded in the complement of K'_x. A similar proof follows for Cases 2 and 3.

□

Appendix B. Proof of Lemma 53

Lemma 53. *Let t_n^* be the time in $(0,T_n)$ given by Proposition 38. Then, $Cost_y(c_n,[0,t_n^*])$ is bounded by a positive constant C_z^*.*

Let us denote by $\phi_z(G)$ the one-form $\frac{G(x)(1-F(x))}{\sqrt{1-G^2(x)}}dx$ on the algebraic curve $\alpha(G,I)$. Following the construction from sub-sub-Section 3.2.1, $c_n(t)$ is a $\mathbb{R}^3_{F_d}$-geodesic correspoding to the polynomial $G_n(x)$ in Pen_{F_d}, and $Cost_y(c_n,[0,T_n])$ is finite. If $\alpha(t)=(x(t),p_x(t))$ is a solution to the Hamiltonian system (2.3), then Lemma 26 implies that the one-form $\phi_z(G)$ restricted to $\alpha([0,T_n])\subset\alpha(G_n,I_n)$ is smooth. The proof of Lemma 53 relies in this fact and compactness of $B(k)$.

Proof. We will use the same notation from the proof of Lemma 36: the sequence of $c_n(t)$ of $\mathbb{R}^3_F$-geodesics, induces a sequence of pairs (G_n,I_n). We consider the path $x_n(\mathcal{T}_n)=[(\hat{x}_0)_n,(\hat{x}_1)_n]\subseteq I_n$, define a sequence of unitary polynomials $\hat{G}_n(\tilde{x}):=G_n(h_n(\tilde{x}))$, and set up the change of variable $x=h_n(\tilde{x})$. Then, we can see the function $Cost_y(c_n,[0,T_n])$ as a smooth map from the duple $(\hat{G}_n,u_n)$ to the real numbers, where $\hat{G}_n$ is in $B(k)$ and u_n is bounded by the length of the compact set K_x. That is the function $Cost_y$ is a map from a compact set to the real number $\mathbb{R}$. Then, the function $Cost_y$ is proper, so $K_{\Theta_d}:=Cost_y^{-1}[-\Theta_d,\Theta_d]$ is compact.

To find the bound C_z^*, we proceed in two steps: first, the smoothness of the one-form $\phi_z(G_n)$ implies $h_n^*\phi_z(G_n)$ is smooth, where h_n^* is the pull-back of $h_n(\tilde{x})$. We fix the lower bound of the integral $h_n^{-1}(x(0)$ and vary the upper bound $h_n^{-1}(x(t))$, where $t\in[0,T_n]$. Then, the integral of $h_n^*\phi_z(G_n)$ is smooth with respect the upper bound $h_n^{-1}(x(t))$ and we can determine its maximum. Second, we take the maximin with respect the pair $(\hat{G}_n,u_n)$ in K_{Θ_d}. That is,

$$C_z^*:=\max_{(\hat{G}_n,u_n)\in K_{\Theta_d}}\left\{\max_{h_n^{-1}(x(t))\in[0,1]}\left\{\left|\int_{h_n^{-1}(x([0,t]))}h_n^*\phi_z(G_n)\right|\right\}\right\}$$

□

Appendix C. Proof Of Theorem 44

For simplicity, we will prove Theorem 44 for the case $F(x)=1-2x^3$. Let $c(t)$ be a $\mathbb{R}^3_F$-geodesic for $F(x)=1-2x^3$ with initial point $c(0)=(1,0,0)$ and hill interval $[0,1]$. Let us consider the time interval $[-n,n]$, since the reduced system is reservable it follows $x_n:=x(n)=x(-n)$ and $x_n([-n,n])=[x_n,1]$.

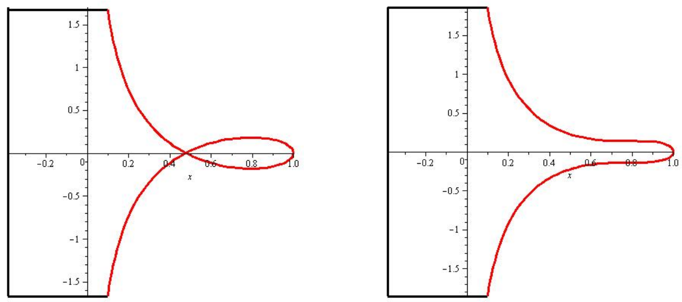

FIGURE C.1. Both images show the projection of the geodesic $c(t)$ for $F(x) = 1 - 2x^3$ and the curve $\tilde{c}(t)$ to the (x, y) and (x, z) planes, respectively.

By Proposition 22, the relation between the curve $x_n([-n, n])$ and n is given by

$$n = \int_{[x_n,1]} \frac{dx}{\sqrt{1 - F^2(x)}}.$$

In addition, the change in $\Delta y(c, n)$ and $\Delta z(c, n)$ are given by

$$\Delta y(c, n) := 2\int_{[x_n,1]} \frac{F(x)dx}{\sqrt{1 - F^2(x)}} \text{ and } \Delta z(c, n) = 2\int_{[x_n,1]} \frac{F^2(x)dx}{\sqrt{1 - F^2(x)}}.$$

Therefore

$$c(-n) = (x(-n), -\frac{\Delta y(F, n)}{2}, -\frac{\Delta z(F, n)}{2})$$

and

$$c(n) = (x(n), \frac{\Delta y(F, n)}{2}, \frac{\Delta z(F, n)}{2}).$$

Corollary 54. *If $F(x) = 1 - 2x^3$ and n is large enough, then*

$$\Delta y(F, n) < \Delta z(F, n) \text{ and } \lim_{n\to\infty} \frac{\Delta z(F, n)}{\Delta y(F, n)} = 1.$$

Proof. If $F(x) = 1 - 2x^3$, then the same integration by parts, used to prove Corollary 45, implies the inequality $\Delta y(F, n) - \Delta z(F, n) > 0$. L'Hopital rules shows $\lim_{n\to\infty} \frac{\Delta z(F,n)}{\Delta y(F,n)} = 1$. □

C.1. **Proof Of Theorem 44.**

Proof. For every large enough n we can find $0 < \epsilon_n$ and $0 < \delta_n$ such that

$$\Delta z(F, n) = (1 + \epsilon_n)\Delta y(F, n) \text{ and } F(-\delta_n) = 1 + \epsilon_n.$$

If $T_1 = x_n + \delta_n$, $T_2 = T_1 + \Delta y(F,n)$ and $T_3 = T_1 + T_2$, then for every n we define the following curve $\tilde{c}_n(t)$ in $\mathbb{R}^3_F$ in the interval $[0, T_3]$ as follows

$$\tilde{c}_n(t) = \begin{cases} c(-n) + (-t, 0, 0) \ \text{ where } \ t \in [0, T_1] \\ c(-n) + (-T_1, t - T_1, 0) \ \text{ where } \ t \in [T_1, T_2] \\ c(-n) + (-T_1 + t - T_2, \Delta y(F,n), \Delta z(F,n)) \text{ where } t \in [T_2, T_3]. \end{cases}$$

See figure C.1. By construction, $c(-n) = \tilde{c}_n(0)$ and $c(n) = \tilde{c}_n(T_3)$, the relation between the n and $\Delta y(F,n)$ is given by

$$2n = \Delta y(F,n) + Cost_t(F, [-n,n]),$$

while the relation between T_3 and $\Delta y(F,n)$ is given by

$$T_3 = \Delta y(F,n) + 2(\delta_n + x_n).$$

If $n \to \infty$, then $Cost_t(F, [-n,n]) \to \Theta_1(F, [0,1]) > 0$ and $2(\delta_n + x_n) \to 0$. Thus there exists an n such that $Cost_t(F, [-n,n]) > 2(\delta_n + x_n)$, in other words $T_3 < 2n$ and we conclude that $\tilde{c}_n(t)$ is shorter that $c(t)$.

$\square$

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