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METRIC LINES IN METABELIAN CARNOT GROUPS

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Abstract

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This work is devoted to metric lines (isometric embedding of the real line) in metabelian Carnot groups $\mathbb{G}$: we say that a group $\mathbb{G}$ is metabelian if $[\mathbb{G}, \mathbb{G}]$ is abelian. Theorems A and B provide a partial result about the classification of the metric lines in the jet-space of functions from $\mathbb{R}$ to $\mathbb{R}$, denoted by $J^k(\mathbb{R}, \mathbb{R})$. Theorem C is a complete classification of the metric lines in the Engel type Carnot groups, denoted by $\mathrm{Eng}(n)$. Both groups, $J^k(\mathbb{R}, \mathbb{R})$ and $\mathrm{Eng}(n)$ are examples of metabelian Carnot groups. The main tool to classify subRiemannian geodesic on $\mathbb{G}$ is a correspondence between the regular subRiemannian geodesics in a metabelian Carnot group $\mathbb{G}$ and the space of solutions to a family of classical electromechanical systems on Euclidean space. The method to prove Theorems A, B and C is to use an intermediate $(n+2)$-dimensional subRiemannian space $\mathbb{R}^{n+2}_F$ lying between the group $\mathbb{G}$ and the Euclidean space $\mathbb{R}^{d_1} \simeq \mathbb{G}/[\mathbb{G}, \mathbb{G}]$.

To Heather for be the love of my life.

To my parents for be the support of my life.

To my siblings for be my fellowships of life.

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Chapter 1

Introduction

This work is a report of the research done, from July 2019 to April 2023, under the advice of Richard Montgomery. My main goal was to characterize the metric lines on the jet space $J^k(R, R)$, an example of metabelian Carnot group.

§ 1.1 Metric lines in Carnot groups

A Carnot group $\mathbb{G}$ is a simple connected Lie group whose Lie algebra $\mathfrak{g}$ is graded, nilpotent, and its first layer $\mathfrak{g}_1$ generates the Lie algebra $\mathfrak{g}$. Every Carnot group $\mathbb{G}$ has a canonical protection $\pi : \mathbb{G} \to \mathbb{R}^{d_1}$, where $\mathbb{G}/[\mathbb{G}, \mathbb{G}] \simeq \mathfrak{g}_1 \simeq \mathbb{R}^{d_1}$ and d_1 is the dimension of the layer $\mathfrak{g}_1$, see (2.1) for more details. To give a subRiemannian structure to $\mathbb{G}$, we define the non-integrable distribution $\mathcal{D}_g := (L_g)_* \mathfrak{g}_1$, and we consider the inner product on $\mathcal{D}$ as the one who makes π a subRiemannian submersion where $\mathbb{R}^{d_1}$ is equipped with the Euclidean product. Let us formalize the subRiemannian submersion.

Definition 1. *Let $(M, \mathcal{D}_M, g_M)$ and $(N, \mathcal{D}_N, g_N)$ be two subRiemannian manifolds and let $\phi : M \to N$ a submersion, we consider the case $dim(M) \geq dim(N)$. We say that ϕ*

is a subRiemannian submersion if $\phi_* \mathcal{D}_M = \mathcal{D}_N$ *and* $\phi^* g_N = g_M$.

Here we will introduce the definition of a metric line in the context of subRiemannian geometry.

Definition 2. *Let* M *be a subRiemannian manifold, we denote by* $dist_M(\cdot, \cdot)$ *the subRiemannian distance on* M. *Let* $|\cdot| : \mathbb{R} \longrightarrow [0, \infty)$ *be the absolute value. We say that a geodesic* $\gamma : \mathbb{R} \longrightarrow M$ *is a metric line if* $|a - b| = dist_M(\gamma(a), \gamma(b))$ *for all compact set* $[a, b] \subseteq \mathbb{R}$.

See Definition 15 for the formal definition of a subRiemannian geodesic. An alternative term for 'metric line' is 'globally minimizing geodesic'. A classic result on metric lines is the following.

Proposition 3. *Let* $\phi : M \longrightarrow N$ *be a subRiemannian submersion and let* $c(t)$ *be a metric line in* N*, then the horizontal lift of* $c(t)$ *is a metric line in* M.

The proof of Proposition 3 is given in [12, p. 154].

Definition 4. *Let* $\mathbb{G}$ *be a Carnot group. We say that a geodesic* $\gamma(t)$ *is a line if the projected curve* $\pi(\gamma(t))$ *in* $\mathbb{R}^{d_1}$ *is a line.*

As an immediate corollary to the Proposition 3, we get:

Corollary 5. *Geodesic lines are metric lines in every Carnot group.*

Metric lines in Jet Space

In [12], we showed a bijection between the set of pairs (F_μ, I) and the set of geodesics in $J^k(\mathbb{R}, \mathbb{R})$, where F_μ is a polynomial defined by (3.1.1) and I is a hill interval given

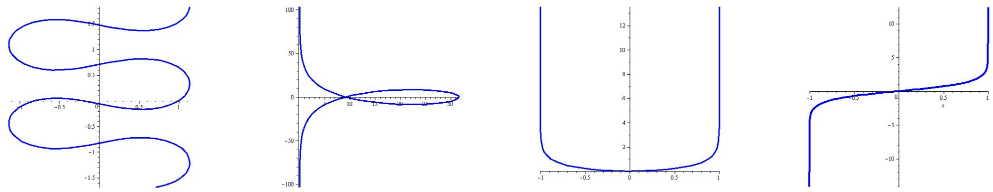

Figure 1.1: The images show the projection to $\mathbb{R}^2 \simeq J^k(\mathbb{R},\mathbb{R})/[J^k(\mathbb{R},\mathbb{R}), J^k(\mathbb{R},\mathbb{R})]$, with coordinates (x,θ_0), of geodesics in $J^k(\mathbb{R},\mathbb{R})$. The first panel presents a generic x-periodic geodesic, the second panel displays the Euler-soliton solution to the Euler-Elastica problem, see Theorem B. The third panel presents the projection of a turn-back geodesic, the forth panel displays the projection of a direct-type geodesic, see Theorem A.

by Definition 29. In addition, we classified the geodesic in $J^k(\mathbb{R},\mathbb{R})$ according to their reduced dynamics, that is, the geodesics in $J^k(\mathbb{R},\mathbb{R})$ are line, x-periodic, homoclinic, heteroclinic direct-type or heteroclinic turn-back, see sub-Section 3.1.2 or Figure 1.1. The Conjecture concerning metric lines in $J^k(\mathbb{R},\mathbb{R})$ is the following.

Conjecture 6. *The metric lines in $J^k(\mathbb{R},\mathbb{R})$ are precisely geodesics of the type: line, homoclinic and the heteroclinic direct-type.*

In [12], we proved geodesics of type x-periodic and heteroclinic turn-back are not metric lines. Theorem A is the first main result and proves Conjecture 6 for the case of direct-type geodesics.

Theorem A. *Heteroclinic direct-type geodesics are metric lines in $J^k(\mathbb{R},\mathbb{R})$.*

Conjecture 6 remains open for homoclinic geodesics. Theorem B is the second principal result and provides a family of homoclinic geodesics that are metric lines.

Theorem B. *The homoclinic-geodesic defined by the polynomial $F(x) = \pm(1 - bx^{2n})$ and hill interval $[0, \sqrt[2n]{\frac{2}{b}}]$ is a metric line in $J^k(\mathbb{R},\mathbb{R})$ for all $k \geq 2n$ and $b > 0$.*

Previous results: Conjecture 6 was proved by A. Andertov and Y. Sachkov in the case $k = 1$ and $k = 2$, see [22–25]. In [12], we showed that a family of direct-type geodesics are metric lines.

The case $k = 1$ corresponds to $\mathbb{G}$ being the Heisenberg group where the geodesics are x-periodic or metric lines. The case $k = 2$ corresponds to $\mathbb{G}$ being Engel's group, denoted by Eng; up to a Carnot translation and dilation Eng has a unique metric line such that its projection to the plane $\mathbb{R}^2 \simeq \mathrm{Eng}/[\mathrm{Eng}, \mathrm{Eng}]$ is the Euler-soliton, besides geodesic lines. The family of metric lines defined by Theorem B is the generalization of A. Andertov and Y. Sachkov's result from [22–25]. More specific, when $n = 1$ then the geodesic defined by the polynomial $F(x) = \pm(1 - bx^2)$ is the one whose projection to the plane $\mathbb{R}^2$ is the Euler-soliton. See [3] for more details about the Euler-Elastica and goedesics in Eng or see [1, p.240] for the relation of Euler-Elastica and the rolling problem.

Metric lines in Engel type

For more details about the definition of $\mathrm{Eng}(n)$ as a Carnot group and a subRiemannian manifold see 4.1. In [8], we showed the subRiemannian geodesic flow on $\mathrm{Eng}(n)$ is integrable. We classify the normal geodesics in $\mathrm{Eng}(n)$ according to their reduced dynamics, see Definition 62. Theorem C is the third principal result of this work and makes a complete classification of metric lines in $\mathrm{Eng}(n)$.

Theorem C. *Up to a Carnot translation* $\mathrm{Eng}(n)$ *has one family of metric lines, besides geodesic lines. This family is generated by* $F_\mu(r) = \pm(1 - br^2)$ *with* $0 < b$.

We remark that given a metric line $\gamma(t)$ in the family described by Theorem C, there

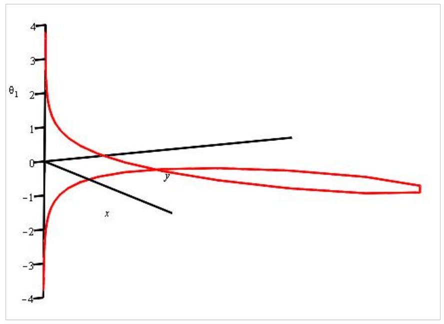

Figure 1.2: The images show the projection to $\mathbb{R}^3 \simeq \mathrm{Eng}(2)/[\mathrm{Eng}(2), \mathrm{Eng}(2)]$, with coordinates (x, y, θ_0), of one metric line in $\mathrm{Eng}(2)$ defined by Theorem C

exists a two-plane in $\mathbb{R}^{n+1} \simeq \mathrm{Eng}(n)/[\mathrm{Eng}(n), \mathrm{Eng}(n)]$ such that the projection of $\gamma(t)$ to this plane is the Euler-Elastica given by case $n = 1$ from Theorem B.

The method we will use to prove a geodesic is a metric line is the generalization of the one used in [12]. We will build a $(n+2)$-dimensional subRiemannian space $\mathbb{R}^{n+2}_F$ and a subRiemannian submersion $\pi_F : \mathbb{G} \to \mathbb{R}^{n+1}_F$, where $\mathbb{G}$ is a metabelian Carnot group, see 2.2. We will then show that the projection by π_F of the geodesics given by Theorems A, B and C are metric lines in $\mathbb{R}^{n+1}_F$.

Chapter 2

Preliminary

§ 2.1 A Carnot Group as a SubRiemannian Manifold

A Lie algebra $\mathfrak{g}$ is stratified if $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_s$ and call $\mathfrak{g}_r$ the layers of $\mathfrak{g}$. A stratified Lie algebra $\mathfrak{g}$ is graded if $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}$. A graded stratified Lie algebra $\mathfrak{g}$ is nilpotent if $\mathfrak{g}_{s+1} = 0$. We say a $\mathbb{G}$ is a Carnot group if $\mathbb{G}$ is a simply connected Lie group whose Lie algebra $\mathfrak{g}$ is graded stratified, nilpotent and bracket generated by $\mathfrak{g}_1$. We call s the step of $\mathbb{G}$ and denote by $[\mathbb{G}, \mathbb{G}]$ the commutator group of $\mathbb{G}$. Every Carnot group has a canonical projection $\pi : \mathbb{G} \to \mathbb{R}^{d_1}$ where $\mathbb{R}^{d_1} \simeq \mathfrak{g}_1 \simeq \mathbb{G}/[\mathbb{G}, \mathbb{G}]$. If g is in $\mathbb{G}$, then the formal definition is:

$$\pi(g) := g \;\; mod \; [\mathbb{G}, \mathbb{G}]. \tag{2.1.1}$$

The canonical injection of $[\mathbb{G}, \mathbb{G}]$ into $\mathbb{G}$ and the projection π define a short exact sequence: $0 \to [\mathbb{G}, \mathbb{G}] \hookrightarrow \mathbb{G} \xrightarrow{\pi} \mathbb{R}^{d_1} \to 0$, which tells that $\mathbb{G} \simeq \mathbb{R}^{d_1 + dim\,([\mathbb{G},\mathbb{G}])}$, topologically.

A subRiemannian structure on a smooth manifold M is given by the pair $(\mathcal{D}, (\cdot, \cdot))$, where $\mathcal{D}$ is a non-integrable distribution and $(\cdot,\cdot)$ is an inner product on $\mathcal{D}$. Every Carnot group is a subRiemannian manifold with the Carnot-Carathéodory distance. Let us define the subRiemannian structure on a Carnot group.

Definition 7. *Chose a Euclidean inner product on $\mathfrak{g}_1$. The Corresponding subRiemannian structure $\mathbb{G}$ is given by $\mathcal{D}(g) := (L_g)_*\mathfrak{g}_1$ and inner product on $\mathcal{D}(g)$ is such that π is a subRiemannian submersion. The Carnot-Carathéodory distance on $\mathbb{G}$ is given by*

$$dist_{\mathbb{G}}(g_1,g_2) := \inf\Bigg\{ \int_a^b ||\dot{\gamma}(t)||_{\mathbb{G}} : \gamma(t) : [a,b] \to \mathbb{G} \ \ \textit{absolutely contiouons} \\ \gamma(a) = g_1 \ \ \gamma(b) = g_2 \ \ \textit{and} \ \ \dot{\gamma}(t) \in \mathcal{D}_{\gamma(t)} \ \ \textit{for a.e.}\ t \in [a,b] \Bigg\}.$$

The property of $\mathfrak{g}_1$ being bracket generating implies the distribution $\mathcal{D}$ is controllable, by Chow's theorem, see [11, p. 9], [20, p. 201] or [1, p. 80].

2.1.1 Metabelian Carnot groups

Let us introduce the formal definition of metabelian group.

Definition 8. *We say $\mathbb{G}$ is a metabelian group if $[\mathbb{G},\mathbb{G}]$ is abelian. Every metabelian group has a normal abelian subgroup $\mathbb{A}$ containing $[\mathbb{G},\mathbb{G}]$.*

See [26] for more algebraic details of the definition.

We consider the left action of $\mathbb{A}$ on $\mathbb{G}$, which is proper and free, so the quotient $\mathbb{G}/\mathbb{A}$ is well-defined. Let us denote by $\mathcal{H}$ the quotient $\mathbb{G}/\mathbb{A}$, and let $\pi_{\mathbb{A}} : \mathbb{G} \to \mathcal{H}$ be the canonical projection. Let g be in $\mathbb{G}$, then the canonical projection $\pi_{\mathbb{A}} : \mathbb{G} \to \mathcal{H}$ is given by

$$\pi_{\mathbb{A}}(g) := g \ \ mod \ \ \mathbb{A}. \tag{2.1.2}$$

Group	Dimension
$\mathbb{G}$	$n+m$
$\mathbb{A} \simeq \mathcal{V} \times [\mathbb{G},\mathbb{G}]$	$m = n_1 + dim([\mathbb{G},\mathbb{G}])$
$[\mathbb{G},\mathbb{G}]$	$m - n_1$
$\mathbb{R}^{d_1} = \mathcal{H} \oplus \mathcal{V} \simeq \mathfrak{g}_1 \simeq \mathbb{G}/[\mathbb{G},\mathbb{G}]$	$d_1 = n + n_1$
$\mathcal{H} := \mathbb{G}/\mathbb{A}$	n
$\mathcal{V} := \mathcal{H}^{\perp} \subseteq \mathbb{R}^{d_1}$	n_1

Table 2.1: Dimension of the groups.

The canonical injection of $\mathbb{A}$ into $\mathbb{G}$ and the projection $\pi_{\mathbb{A}}$ define a short exact sequence; $0 \to \mathbb{A} \hookrightarrow \mathbb{G} \xrightarrow{\pi_{\mathbb{A}}} \mathcal{H} \to 0$, which tells us $\mathbb{G} \simeq \mathbb{A} \times \mathcal{H}$, topologically. Thanks to the subRiemannian inner product, given the Lie algebra $\mathfrak{a}$ we can decompose $\mathfrak{g}_1$ as the direct sum of two sub-spaces.

Definition 9. *Let $\mathbb{G}$ be a metabelian Carnot group, and let $\mathfrak{a}$ be the Lie algebra of a maximal abelian subgroup $\mathbb{A}$ containing $[\mathbb{G},\mathbb{G}]$. Then $\mathfrak{g}_1 = \mathfrak{h} \oplus \mathfrak{v}$, where $\mathfrak{v} := \mathfrak{a} \cap \mathfrak{g}_1$ and $\mathfrak{h}$ is the orthogonal complement of $\mathfrak{v}$ in $\mathfrak{g}_1$. In addition, $\mathcal{D}(g) = \mathcal{D}_{\mathfrak{h}}(g) \oplus \mathcal{D}_{\mathfrak{v}}(g)$ where $\mathcal{D}_{\mathfrak{v}}(g) := (L_g)_*\mathfrak{v}$ and $\mathcal{D}_{\mathfrak{h}}(g) := (L_g)_*\mathfrak{h}$ are left-invariant subspace.*

We will identify $\mathcal{H}$ as a subset of $\mathbb{R}^{d_1} \simeq \mathfrak{g}_1$. Then $\mathbb{R}^{d_1} = \mathcal{H} \oplus \mathcal{V}$, where $\mathcal{V}$ s the orthogonal complement of $\mathcal{H}$ with respect of the Euclidean product in $\mathbb{R}^{d_1}$. The map π is compatible with the splitting of $\mathcal{D}(g)$ and $\mathbb{R}^{d_1}$, that is,

$$d\pi_g(\mathcal{D}_{\mathfrak{h}}(g)) = T_{\pi_{\mathbb{A}}(g)}\mathcal{H} \quad \text{and} \quad d\pi_g(\mathcal{D}_{\mathfrak{v}}(g)) = T_{\pi_{\mathbb{A}(g)}}\mathcal{V}.$$

Let $\{E^i\}$, $\{E^k_{\mathfrak{a}}\}$ and $\{E^j_{\mathfrak{a}}\}$ be a base for $\mathfrak{h}$, $\mathfrak{v}$ and $[\mathfrak{g},\mathfrak{g}]$ with $1 \leq i \leq n$, $1 \leq k \leq n_1$ and $n_1 + 1 \leq j \leq m$. We denote by X^i, Y^k and Y^j the left extension of E^i, $E^j_{\mathfrak{a}}$ and $E^j_{\mathfrak{a}}$, that is, $X^i(g) := (L_g) * E^i$, $Y^k(g) := (L_g) * E^k_{\mathfrak{a}}$ and $Y^j(g) := (L_g) * E^j_{\mathfrak{a}}$. Then $\{X^i\}$ is a base for $\mathcal{D}_{\mathfrak{h}}$ with $1 \leq i \leq n$, and $\{Y^k\}$ is a base for $D_{\mathfrak{v}}$ with $1 \leq k \leq n_1$. We remark: The frame

$\{X^i, Y^k\}$ is an orthonormal base of $\mathcal{D}$.

2.1.2 The Left Action

Definition 10. *The left-action of* $\mathbb{A}$ *on* $\mathbb{G}$ *is a function* $\varphi : \mathbb{A} \times \mathbb{G} \to \mathbb{G}$ *given by* $\varphi(a, g) := a * g$ *such that* $\varphi(a_1 * a_2, g) := \varphi(a_1, \varphi(a_2, g))$*, where* $*$ *is the Carnot multiplication.*

By construction, $\varphi(a, g)$ is in $Iso(\mathbb{G})$ and since $\mathbb{A}$ is abelian, $\varphi(a_1 * a_2, g) = \varphi(a_2 * a_1, g)$. If ξ is in $\mathfrak{g}$ then the action of $\mathbb{A}$ on $\mathbb{G}$ defines the infinitesimal generator $\sigma : \mathfrak{a} \to \mathfrak{g}$ in the following way

$$\sigma_\xi(g) = \frac{d}{dt}\varphi(\exp(t\xi), g)|_{t=0} = \frac{d}{dt}\exp(t\xi) * g|_{t=0}. \tag{2.1.3}$$

The map σ sends a vector ξ in $\mathfrak{a}$ to a Killing vector field σ_ξ since φ is in $Iso(\mathbb{G})$. We say the vector field X and the map is σ are $\mathbb{A}$-invariant if $X(a * g) = (L_a)_* X(g)$ and $\sigma_\xi(a * g) = (L_a)_*\sigma(g)$. The infinitesimal generator σ_ξ is equinvariant in general, see [20, p. 108] or [11, p. 161] for more details. It is a general property of infinitesimal generators that $\mathbb{A}$ abelian implies that $\sigma_\xi(g)$ is $\mathbb{A}$-invariant.

We remark that $(L_g)_*\mathfrak{a}$ and $\sigma(\mathfrak{a})$ are the same as abstract Lie algebras and as sub-vector spaces of $T_g\mathbb{G}$. However, they are different Lie algebras inside $T_g\mathbb{G}$. In general, only the left-invariant vector fields in $\sigma(\mathfrak{a})$ and $(L_g)_*\mathfrak{a}$ are the ones corresponding to the left translation of the last layer $\mathfrak{g}_s$.

2.1.3 Exponential Coordinates of the Second Kind

We use the frame X^i and Y^j to give coordinates to the Carnot group at a point g in the following way: define a map from the coordinates $(x, \theta) \in \mathbb{R}^{n+m}$ to $\mathbb{G}$ by

$$\Phi(x) := \prod_{i=0}^{n-1} \exp(x_{n-i} X^{n-i}) \quad \text{and} \quad \Phi(\theta) := \exp(\sum_{k=1}^{n_1} \theta_k Y^k + \sum_{j=n_1+1}^{m} \theta_j Y^j).$$

Definition 11. *The exponential coordinates (x, θ) are given by a unique chart $(\mathbb{R}^{n+m}, \Phi)$ where a point is given by $g := \Phi(x, \theta) := \Phi(\theta) * \Phi(x)$.*

Proposition 12. *Let $\mathbb{G}$ be a metabelian Carnot group and let $g = (x, \theta)$ be in $\mathbb{G}$. Then the left-invariant vector fields and the left-invariant one-forms on $\mathbb{G}$ are given by*

$$X^1(g) = \frac{\partial}{\partial x_1}, \quad X^i = \frac{\partial}{\partial x_i} + \sum_{j=n_1+1}^{m} \mathcal{A}_{ij}^M(x) \frac{\partial}{\partial \theta_j} \qquad 2 \leq i \leq n,$$

$$Y^k(g) = \frac{\partial}{\partial \theta_k} + \sum_{j=n_1+1}^{m} \mathcal{A}_{kj}^E(x) \frac{\partial}{\partial \theta_j} \qquad 1 \leq k \leq n_1,$$

$$\Theta_k(g) = d\theta_k \quad \textit{and} \quad \Theta_j(g) = d\theta_j - \sum_{i=1}^{n} \mathcal{A}_{ij}^M(x) dx_i - \sum_{k=1}^{n_1} \mathcal{A}_{kj}^E(x) d\theta_k,$$

where $\mathcal{A}_{ij}^M(x)$ and $\mathcal{A}_{kj}^E(x)$ are homogeneous polynomial functions on the horizontal coordinates.

The proof the Proposition 12 is in [8] or in [10] in the conetext of the Jet-Space.

2.1.4 The Metabelian Carnot Groups as Principal Bundle

We can think of $\pi_{\mathbb{A}} : \mathbb{G} \to \mathcal{H}$ as a principal $\mathbb{A}$-bundle. Where we identify $\mathcal{H}$ with a sub-vector space $\mathcal{D}_{\mathfrak{h}} \subseteq T_g\mathbb{G}$, which is complementary to $(L_g)_*\mathfrak{a} \subseteq T_g\mathbb{G}$, that is $\mathcal{D}_{\mathfrak{h}} \oplus (L_g)_*\mathfrak{a} = T_g\mathbb{G}$. This way, $\mathcal{H}$ defines a connection on our principal bundle $\pi_{\mathbb{A}}$.

Note: $(L_g)_*\mathfrak{a}$ represents the vertical space for $\pi_{\mathbb{A}}$, and $\mathcal{D}_{\mathfrak{h}}$ is an $\mathbb{A}$-invariant choice of horizontal space by left-translation, that is $d\pi_{\mathbb{A}}((L_g)_*\mathfrak{a}) = 0$ and $(L_a)_*\mathcal{D}_{\mathfrak{h}}(g) = \mathcal{D}_{\mathfrak{h}}(ag)$, as a connection on principal $\mathbb{A}$-bundle requires. For more bundles with connections, see [18, Chapter 8], [11, Chapter 12] , or [20, sub-Chapter 2.9].

$\sigma(g)$ sends the canonical base $\{E^\ell\}$ for $\mathbb{A}$ to the frame of Killing vector fields $\{\sigma^\ell(g)\}$ with $1 \le \ell \le m$. Thus, the frame $\{\sigma^\ell(g)\}$ defines a canonical co-frame $\{\omega_\ell(g)\}$ with the following properties: $\omega_\ell(\sigma^{\ell_1})(g) = \delta_\ell^{\ell_1}$ and $\omega_j(\mathcal{D}_{\mathfrak{h}})(g) = 0$.

Connection Form

The connection one-form $\omega(g)$ on $\mathbb{G}$ is an $\mathfrak{a}$ valued one-form given by

$$\omega(g) = \sum_{j=1}^{n_1} \omega_k \otimes e^k(g) + \sum_{j=1+n}^{m} \omega_j \otimes e^j(g). \tag{2.1.4}$$

$\omega(g)$ is $\mathbb{A}$-invariant since $(L_a)_*\omega(g) = \omega(a * g)$. By definition $ker\ \omega(g) = \mathcal{D}_{\mathfrak{h}}(g)$ and $\omega \circ \sigma(g) = Id_{\mathfrak{a}}$.

The differential $d\pi$ of the canonical projection π has an inverse map.

Definition 13. *If $(v, u) = (v_1, \cdots, v_n, u_1, \cdots, u_{n_1})$ is in $T\mathbb{R}^{d_1}$, then we denote by $hor : T\mathbb{R}^{d_1} \to T\mathbb{G}$ the map given by $hor(v, u) := \sum_{i=1}^{n} v_i X^i + \sum_{k=1}^{n_1} u_k Y^k$;*

Then $d\pi \circ hor = Id_{\mathfrak{g}_1}$. we say that *hor* is a horizontal lift with respect to $d\pi$. The horizontal map *hor* defines a linear projection and an $\mathfrak{a}^*$-valued one-form $\mathcal{A}_{\mathbb{G}}$ on $\mathbb{R}^{d_1}$.

Definition 14. *We denote by $\Pi_{\mathbb{R}^{d_1}}$ the linear projection from T^*G to $T^*\mathcal{H}$ give by $\Pi_{\mathbb{R}^{d_1}}(\lambda) := \lambda \circ hor$. We define the $\mathfrak{a}^*$-valued one-form $\mathcal{A}_{\mathbb{G}} = \mathcal{A}_{\mathbb{G}}^M + \mathcal{A}_{\mathbb{G}}^E$ in $\Omega^1(\mathbb{R}^{d_1}, \mathfrak{a})$ by*

$$\mathcal{A}_{\mathbb{G}} := \Pi_{\mathbb{R}^{d_1}}(\omega)(g), \quad \mathcal{A}_{\mathbb{G}}^M := \Pi_{\mathbb{R}^{d_1}}(\omega)(g)|_{\mathcal{H}} \ \text{ and } \ \mathcal{A}_{\mathbb{G}}^E := \Pi_{\mathbb{R}^{d_1}}(\omega)(g)|_{\mathcal{V}}.$$

Where $\mathcal{A}_{\mathbb{G}}^{M}$ *is in* $\Omega^1(\mathcal{H}, \mathfrak{a}^*)$ *and* $\mathcal{A}_{\mathbb{G}}^{E}$ *is in* $\Omega^1(\mathcal{V}, \mathfrak{a}^*)$.

Let μ be in $\mathfrak{a}^*$, we define $\mathcal{A}_\mu$ as the pairing of $\mathcal{A}_{\mathbb{G}}$ with μ, that is,

$$\mathcal{A}_\mu(x) := \langle \mu, \mathcal{A}_{\mathbb{G}} \rangle = \mathcal{A}_\mu^M + \mathcal{A}_\mu^E, \quad \langle \mu, \mathcal{A}_{\mathbb{G}}^M \rangle := \mathcal{A}_\mu^M \text{ and } \langle \mu, \mathcal{A}_{\mathbb{G}}^E \rangle := \mathcal{A}_\mu^E. \tag{2.1.5}$$

Then $\mathcal{A}_\mu$ is a one-form on $\mathbb{R}^{d_1}$. If we write $\mathcal{A}_\mu$ in terms of these coordinates, then $\mathcal{A}_\mu$ depends only on x in a polynomial way.

2.1.5 Geodesic Flow and Symplectic Reduction

Let us consider cotangent bundle $T^*\mathbb{G}$ of a metabelian Carnot group endowed with the traditional symplectic structure and the canonical coordinates (p, g). Let P_{X^i} and P_{Y^k} be the momentum functions associated to the vector X^i in $\mathcal{D}_{\mathfrak{h}}$ and Y^k in $\mathcal{D}_{\mathfrak{v}}$, respectively, see [11, p.7] or [1, p. 67] for the formal definition of the momentum function. Then, the function governing the subRiemannian geodesic flow on $\mathbb{G}$ is given by

$$H_{sR} := \frac{1}{2}\left(\sum_{i=1}^{n} P_{X^i}^2 + \sum_{k=1}^{n_1} P_{Y^k}^2\right). \tag{2.1.6}$$

See [11, p.7] or [1, p. 67] for details subRiemannian geodesic flow. Where the condition $H_{sR} = 1/2$ implies that the geodesics are parameterized by arc length. The Hamiltonian action of $\mathbb{A}$ on $T^*\mathbb{G}$ is given by the Hamiltonian flow defined by the momentum function $P_\xi := P_{\sigma_\xi}$ associated to the Killing vector field σ_ξ. Let us consider P_ξ for some ξ in $\mathfrak{a}$, then the momentum map $J : T^*\mathbb{G} \to \mathfrak{a}^*$ is defined as μ in $\mathfrak{a}^*$ such that $P_\xi = \mu(\xi)$.

Definition 15. *We say* $\gamma(t)$ *is a geodesic parameterized by arc length and with momentum* μ, *if* $\gamma(t)$ *is the projection of subRiemannian geodesic flow and* $J(p(t), \gamma(t)) = \mu$.

Let us consider the canonical symplectic form $\Omega_{\mathbb{G}}$ on $T^*\mathbb{G}$. We remark that isotropy group $\mathbb{A}_\mu := \{a \in \mathbb{A} : Ad_g \mu = \mu\}$ is $\mathbb{A}$, then $\mathbb{A}$ acts freely and properly on $J^{-1}(\mu)$, so the reduce space $T^*\mathbb{G}//\mathbb{A}_\mu := J^{-1}(\mu)/\mathbb{A}_\mu$ is well defined. The trivialization $\mathbb{G} = \mathbb{A} \times \mathcal{H}$ implies that $T^*\mathbb{G} \simeq \mathbb{A} \times \mathfrak{a} \times T^*\mathcal{H}$. Then it follows $T^*\mathbb{G}//\mathbb{A}_\mu \simeq T^*\mathcal{H}$. Let $\pi_\mu^* : J^{-1}(\mu) \longrightarrow T^*\mathbb{G}//\mathbb{A}_\mu$ be the canonical projection , then, the unique symplecitc form Ω_μ on $T^*\mathbb{G}//\mathbb{A}_\mu$, characterized by $\pi_\mu^*\Omega_\mu = \Omega_\mathbb{G}|_{J^{-1}(\mu)}$, is the symplectic form $\Omega_\mathcal{H} - B_\mu$ where B_μ is a closed form. Then the reduce dynamics provided by the symplectic reduction theorem is given by the triple $(T^*\mathbb{G}//\mathbb{A}_\mu, \Omega_\mathcal{H} + B_\mu, \tilde{H}_\mu)$.

Since B_μ is a close 2-form on $\mathbb{R}^{d_1}$, then B_μ is exact and there exist 1-form $\mathcal{A}_\mu$ such that $d\mathcal{A} = -B_\mu$. The magnetic shift, $p_x + \mathcal{A}_\mu$, implies that the canonical projection $\Pi_\mathbb{A} : T^*\mathbb{G} \longrightarrow T^*\mathcal{H}$ restricted to $J^{-1}(\mu)$ is a symplectic diffeomorphism from $(J^{-1}(\mu), \Omega_\mathbb{G}|_{J^{-1}(\mu)})$ to $(T^*\mathcal{H}, \Omega_\mathcal{H})$.

2.1.6 The Reduced Space

Let $T^*\mathcal{H}$ be the cotangent bundle of $\mathcal{H}$, the Hamiltonian structure for the classical electromechanical system is given by a magnetic potential $\mathcal{A}$ and effective potential ϕ, see [17] or [21] for more details. Let (p_x, x) be the traditional coordinates for $T^*\mathcal{H} \subseteq T^*\mathbb{R}^{d_1}$, then the $\mathfrak{a}^*$-valued one-form $\mathcal{A}_\mu$, from Definition 14, defines a Hamiltonian H_μ function in $T^*\mathcal{H}$, given by

$$H_\mu(p_x, x) := \frac{1}{2}||p_x + \mathcal{A}_\mu(x)||^2_{(\mathbb{R}^{d_1})^*} = \frac{1}{2}||p_x + \mathcal{A}^M_\mu(x)||^2_{\mathcal{H}^*} + \frac{1}{2}\phi_\mu(x). \tag{2.1.7}$$

Where the effective potential $\frac{1}{2}\phi_\mu(x)$ is defined by the function $\phi_\mu(x) = ||\mathcal{A}^E_\mu(x)||^2_{\mathcal{V}^*}$, here $|| \quad ||_{(\mathbb{R}^{d_1})^*}$, $|| \quad ||_{\mathcal{H}^*}$, and $|| \quad ||_{\mathcal{V}^*}$ are the Euclidean norm in $(\mathbb{R}^{d_1})^*$, $\mathcal{H}^*$ and $\mathcal{V}^*$, respectively. Equation (2.1.7) shows that we can interpret $\mathcal{A}^M_\mu(x)$ and $\mathcal{A}^E_\mu(x)$ as the magnetic potential and effective potential of the reduced Hamiltonian H_μ.

Definition 16. *We say $\eta(t)$ is an $\mathcal{A}_{\mathbb{G}}$-curve for μ in $\mathcal{H}$, if $\eta(t)$ is the projection of the reduced Hamiltonian flow.*

The following result is a consequence of the symplectic reduction made with N. Paddeu and E. Le Donne, see [8].

Background Theorem. *Let $\mathbb{G}$ be a metabelian Carnot group and $\mathfrak{a}$ a choice of maximal abelian ideal ($[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{a}$). Then there exists an $\mathfrak{a}^*$ valued polynomial one-form $\mathcal{A}_{\mathbb{G}}(x)$ on $\mathbb{R}^{d_1} = \mathbb{G}/[\mathbb{G}, \mathbb{G}]$ given by $\mathcal{A}^M_{\mathbb{G}}(x) + \mathcal{A}^E_{\mathbb{G}}(x)$, $x \in \mathcal{H} := \mathbb{G}/\mathbb{A}$ with the following significance. If $\gamma(t)$ is a normal subRiemannian geodesic in $\mathbb{G}$ with momentum μ, then the curve $\eta(t) = \pi_{\mathbb{A}}(\gamma(t))$ is an $\mathcal{A}_{\mathbb{G}}$-curve for μ. Conversely, if $\eta(t)$ is an $\mathcal{A}_{\mathbb{G}}$-curve for μ, then its horizontal-lift is a normal subRiemannian geodesic in $\mathbb{G}$ with momentum μ.*

The first statement of **Background Theorem** was proved by showing that the symplectic reduction of the subRiemannian flow on $T^*\mathbb{G}$ yields the reduced Hamiltonian H_μ. In contrast, the converse statement was shown using the symplectic reconstruction. We reduce the study of subRiemannian geodesics in metabelian Carnot groups to the study of the $\mathcal{A}_{\mathbb{G}}$-curves. The **Background Theorem** justifies why it is enough to classify the reduced dynamics to classify the subRiemannian geodesic flow. See [4] or [5] for more details about the symplectic reduction and reconstruction.

In [2, 6, 7], A. Anzaldo-Meneses, and F. Monroy-Perez showed the bijection between normal geodesic and the pair (F_μ, I) in the context of the $J^k(\mathbb{R}, \mathbb{R})$. In [12], we used their approach to give our partial result of the conjecture 6. Later, we generalize the idea from A. Anzaldo-Meneses, and F. Monroy-Perez to make the syplectic reduction in the metabelian Carnot case.

2.1.7 Semidirect Product

We remark that the condition $[\mathfrak{h}, \mathfrak{h}] = 0$ implies $\mathcal{A}^M_{\mathbb{G}} = 0$ and $\mathcal{H}$ is a subgroup of $\mathbb{G}$, then $\mathbb{G} = \mathbb{A} \rtimes \mathcal{H}$. the reduced Hamiltonian is an n-degree of freedom system with polynomial potential given by

$$H_\mu(p_x, x) = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \frac{1}{2}\phi_\mu(x).$$

Geodesic lines are the geodesics associated with the constant polynomial $\phi_\mu(x)$. Let us assume $\phi_\mu(x)$ is not constant: there exists a closed set $hill(\mu) \subseteq \mathcal{H}$, called the hill region, where the dynamics take place. That is, if x is in $int\ (hill)$, then $0 \leq \phi(x) < 1$ and $p_x \neq 0$, while, if x is in $\partial\ (hill)$, then $\phi_\mu(x) = 1$ and $p_x = 0$. We say that $\eta(t)$ bounces at the boundary of $hill$, the dynamics of the harmonic oscillator is the simplest example of this phenomenon for the Heisenberg group.

The case $[\mathfrak{h}, \mathfrak{h}] = 0$ and $dim\ \mathfrak{v} = 1$ will be relevant since, in this general context, we will introduce the subRiemannian submersion π_F to prove that a geodesic in $\mathbb{G}$ is a metric line. We will see that $J^k(\mathbb{R}, \mathbb{R})$ and $\mathrm{Eng}(n)$ hold these conditions, see Sections 3.1 and 4.1.

Let $\mathcal{A}_{\mathbb{G}}$ be the $\mathfrak{a}^*$ valued one-form associated to the metabelian Carnot group $\mathbb{G}$. If θ_0 is the exponential coordinate associated to the left invariant vector field Y in $\mathcal{D}_{\mathfrak{v}}$ and μ is in $\mathfrak{a}$. Then, we define the polynomial $F_\mu(x)$ by equation $< \mu, \mathcal{A}_{\mathbb{G}} >:= F_\mu(x) d\theta_0$, so the reduced Hamiltonian is given by

$$H_\mu(p_x, x) = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \frac{1}{2}F_\mu^2(x).$$

In this case, $hill(F) := F_\mu^{-1}[-1, 1]$ and the left-invariant vector fields tangent to $\mathcal{D}$ are given by

$$X_i = \frac{\partial}{\partial x_i} \quad 1 \le i \le n, \text{ and } Y = \frac{\partial}{\partial \theta_0} + \sum_{\ell=1}^{m-1} \mathcal{A}_\ell^E(x) \frac{\partial}{\partial \theta_\ell}. \tag{2.1.8}$$

Here, $\mathcal{A}_\ell^E(x)$ are polynomial functions on $\mathcal{H}$, given by Proposition 12. Then the polynomial $F_\mu(x) = a_0 + \sum_{\ell=1}^{m-1} a_\ell \mathcal{A}_\ell^E(x)$, if $\mu = (a_0, \ldots, a_{m-1})$.

§ 2.2 Magnetic Space

Following the notation from sub-Section 2.1.7. Let $\mathbb{G}$ be a Carnot group such that $[\mathfrak{h}, \mathfrak{h}] = 0$, $dim\ \mathcal{H} = n$, and dim $\mathfrak{v} = 1$, then $\mathcal{D}$ is an $(n+1)$-rank distribution. Let us fix momentum μ in $\mathfrak{a}^*$ and consider the polynomial $F_\mu(x)$ defined in sub-Section 2.1.7. We will introduce an intermediate $(n+2)$-dimensional subRiemannian manifold denoted by $\mathbb{R}_F^{n+2}$ whose geometry depends on $F(x) := F_\mu(x)$, see [12] or [11] to understand the name "magnetic space".

2.2.1 Factoring a SubRiemannian Submersion

We denote by $\mathbb{R}_F^{n+2}$, the subRiemannian manifold with the following structure, let $(x_1, \cdots, x_n, y, z)$ be global coordinates, in short way (x, y, z). We define the $(n+1)$-rank non-integrable distribution $\mathcal{D}_F$ by the equation $dz - F(x)dy = 0$. To make $\mathbb{R}_F^{n+2}$ a subRiemannian manifold we define the subRiemannian metric on the distribution $\mathcal{D}_F$ given by $ds^2_{\mathbb{R}_F^{n+2}} = (\sum_{i=1}^n dx_i^2 + dy^2)|_{\mathcal{D}_F}$. We provide a subRiemannian submersion π_F factoring the subRiemannian submersion $\pi : \mathbb{G} \longrightarrow \mathbb{R}^{n+1}$, that is, $\pi = pr \circ \pi_F$, where the target of π_F is $\mathbb{R}_F^{n+2}$ and the target of pr is $\mathbb{R}^{n+1}$. If $\mu = (a_0, \ldots, a_{m-1})$, then the projections are given in coordinates by

$$\pi_F(x,\theta) = (x,\theta_0,\sum_{\ell=0}^{m-1} a_\ell\theta_\ell) = (x,y,z), \quad \text{and} \quad pr(x,y,z) := (x,y). \tag{2.2.1}$$

It follows that π_F maps the frame $\{X^1,\ldots,X^n,Y\}$ defined in 2.1.8 into the frame $\{\tilde{X}^1,\cdots,\tilde{X}^n,\tilde{Y}\}$, that is,

$$\tilde{X}^i := (\pi_F)_*X^i = \frac{\partial}{\partial x_i};\ 1 \le i \le n, \text{ and } \tilde{Y} := (\pi_F)_*Y = \frac{\partial}{\partial y} + F(x)\frac{\partial}{\partial z},$$

and $\mathcal{D}_F$ is globally framed by the orthonormal vector fields $\{\tilde{X}^1,\cdots,\tilde{X}^n,\tilde{Y}\}$.

We pause to explain why we've used the term "magnetic". For simplicity let us consider the case $n=1$. The motion of a particle of charge e moving non-relativistically in the Euclidean plane under the influence of a magnetic field of strength $B(x,y)$ orthogonal to the plane is given by $\ddot{c} = eB(c)\mathbb{J}\dot{c}$. Let $A = A_1(x,y)dx + A_2(x,y)dy$ be a vector potential for B, meaning that $dA = Bdx\wedge dy$. The Hamiltonian system on $T^*\mathbb{R}^2$ having Hamiltonian $H = \frac{1}{2}(p_x - eA_1(x,y))^2 + (p_y - eA_2(x,y))^2$ generates the motion of this particle. Introduce a third variable z with conjugate momentum p_z so that H becomes $H = \frac{1}{2}(p_x - p_zA_1(x,y))^2 + (p_y - p_zA_2(x,y))^2$ on $T^*\mathbb{R}^3$. This is the subRiemannian kinetic energy for the subRiemannian structure on $\mathbb{R}^3$ defined by the distribution $D = ker(dz - A)$ with inner product $dx^2 + dy^2|_D$. Since H is independent of z we have that p_z is constant along trajectories and we identify this constant with the charge e. See [11]. We call any subRiemannian structure of this form on $\mathbb{R}^3$ a magnetic subRiemannian structure. Our $\mathbb{R}^3_F$ is such a structure with $A = F(x)dy$.

2.2.2 Geodesics in the Magnetic Space

The Hamiltonian function governing the subRiemannian geodesic flow in $\mathbb{R}^{n+2}_F$ is

$$H_F(p_x, p_y, p_z, x, y, z) = \frac{1}{2}\sum_{i=1}^{n} p_{x_i}^2 + \frac{1}{2}(p_y + F(x)p_z)^2. \tag{2.2.2}$$

Since H_F does not depend on the coordinates y and z, they are cycle coordinates, so the momentum p_y and p_z are constant of motion, see [13] or [9] for the definition of cycle coordinate. This tells us that the translation $\varphi_{(y_0,z_0)}(x, y, z) = (x, y + y_0, z + z_0)$ is an isometry.

Definition 17. *We denote by* $dist_{\mathbb{R}^{n+2}_F}(\ ,\ )$ *and* $Iso(\mathbb{R}^{n+2}_F)$*, the subRiemannian distance and the isometry group in* $\mathbb{R}^{n+2}_F$*. In general, we denote by* $Iso(M)$ *the isometry group os the subRiemannian manifold* M*.*

For more details about these definitions see [11] or [1]. Then the translation $\varphi_{(y_0,z_0)}$ is in $Iso(\mathbb{R}^{n+2}_F)$.

Definition 18. *We say a curve* $c(t) = (x(t), y(t), z(t))$ *is a* $\mathbb{R}^{n+2}_F$*-geodesic parametrized by arc length in* $\mathbb{R}^{n+2}_F$*, if it is the projection of the subRiemannian geodesic flow with the condition* $H_F = \frac{1}{2}$*.*

Setting $p_y = a$ and $p_z = b$ inspired the following definition:

Definition 19. *We say that the two-dimensional linear space* Pen_F *is the pencil of* $F(x)$*, if* $Pen_F := \{G(x) = a + bF(x) : (a, b) \in \mathbb{R}^2\}$*.*

We define the lift of a curve in $\mathbb{R}^{n+2}_F$ to a curve in $\mathbb{G}$.

Definition 20. *Let $c(t)$ be a curve in $\mathbb{R}^{n+2}_F$. We say that a curve $\gamma(t)$ in $\mathbb{G}$ is the lift of $c(t) = (x(t), y(t), z(t))$ if $\gamma(t)$ solves*

$$\dot{\gamma}(t) = \sum_{i=1}^{n} \dot{x}_i(t) X^i(\gamma(t)) + G(x(t))Y(\gamma(t)).$$

Now we describe the $\mathbb{R}^{n+2}_F$-geodesics, their lifts, and their relation with the geodesics in $\mathbb{G}$.

Proposition 21. *Let $c(t)$ be a $\mathbb{R}^{n+2}_F$-geodesic for $G(x)$ in Pen_F, then its projection $x(t) := pr(c(t))$ satisfies the n-degree of freedom Hamiltonian equation*

$$H_{(a,b)}(p_x, x) := \frac{1}{2}\sum_{i=1}^{n} p_{x_i}^2 + \frac{1}{2}(a + bF(x))^2 = \frac{1}{2}\sum_{i=1}^{n} p_{x_i}^2 + \frac{1}{2}G^2(x).$$

Having found a solution $(p_{x_1}(t), \ldots, p_{x_n}(t), x_1(t), \ldots, x_n(t))$, the coordinates $y(t)$ and $z(t)$ satisfy

$$\dot{y} = G(x(t)) \ \ \text{and} \ \ \dot{z} = G(x(t))F(x(t)). \tag{2.2.3}$$

Moreover, every $\mathbb{R}^{n+2}_F$-geodesic is the π_F-projection of a geodesic in $\mathbb{G}$ corresponding to $G(x)$ in Pen_F. Conversely, the lifts of a $\mathbb{R}^{n+2}_F$-geodesic are precisely those geodesics corresponding to polynomials in Pen_F.

The proof is similar to the one exposed in [12, p. 161].

The subRiemannian geometry has two type of geodesics normal geodesics and abnormal geodesics. The following Lemma characterizes the abnormal geodesics in $\mathbb{R}^{n+2}_F$.

Lemma 22. *A curve $c(t)$ in $\mathbb{R}^{n+2}_F$ is an abnormal geodesic if and only if $c(t)$ is tangent to the vector field $\tilde{Y}$ and $pr(c(t)) = x^*$ is a constant point in $\mathcal{H}$ such that $dF|_{x^*} = 0$.*

For more details about abnormal geodesics, see [11], [1] or [14].

Corollary 23. *Let $\gamma(t)$ be a normal geodesic in $\mathbb{G}$ corresponding to the polynomial $F_\mu(x) = F(x)$ and let $c(t)$ be the curve given by $\pi_F(\gamma(t))$, then $c(t)$ is a $\mathbb{R}^{n+2}_F$-geodesic corresponding to the pencil $(a, b) = (0, 1)$.*

Proof. By construction, the pencil $(a, b) = (0, 1)$ correspond to the polynomial $F(x)$. □

2.2.3 Cost Map in Magnetic Space

Here we will define the *Cost* map, an auxiliary map to prove Theorems A, B and C.

Definition 24. *Let $c(t)$ be a $\mathbb{R}^{n+2}_F$-geodesic defined on the interval $[t_0, t_1]$. We define the function $\Delta : (c, [t_0, t_1]) \to [0, \infty] \times \mathbb{R}^2$ given by*

$$\begin{aligned}\Delta(c, [t_0, t_1]) &:= (\Delta t(c, [t_0, t_1]), \Delta y(c, [t_0, t_1]), \Delta z(c, [t_0, t_1])) \\ &:= (t_1 - t_0, y(t_1) - y(t_0), z(t_1) - z(t_0)).\end{aligned} \tag{2.2.4}$$

And the function $Cost : (c, [t_0, t_1]) \to [0, \infty] \times \mathbb{R}$ given by

$$\begin{aligned}Cost(c, [t_0, t_1]) &:= (Cost_t(c, [t_0, t_1]), Cost_y(c, [t_0, t_1]) \\ &:= (\Delta t(c, [t_0, t_1]) - \Delta y(c, [t_0, t_1]), \Delta y(c, [t_0, t_1]) - \Delta z(c, [t_0, t_1])).\end{aligned} \tag{2.2.5}$$

We call $Cost(c, [t_0, t_1])$ the cost function of $c(t)$.

Let us prove that $Cost(c, [t_0, t_1])$ is well-defined:

Proof. By construction, $|\Delta y(c, [t_0, t_1])| \leq \Delta t(c, [t_0, t_1])$, so $0 \leq Cost_t(c, [t_0, t_1])$. □

The function $Cost_t(c, [t_0, t_1])$ was defined in [12], We interpret $Cost_t(c, [t_0, t_1])$ as the time that takes to the geodesic $c(t)$ travel through the y-component. To give more meaning to this interpretation, we present the following Lemma:

Lemma 25. *Let $c(t)$ and $\tilde{c}(t)$ be two $\mathbb{R}_F^{n+2}$-geodesics. Let us assume that they travel from a point A to a point B in a time interval $[t_0, t_1]$ and $[\tilde{t}_0, \tilde{t}_1]$, respectively. If $Cost_t(c_1, [t_0, t_1]) < Cost_t(c_2, [\tilde{t}_0, \tilde{t}_1])$, then the arc length of $c(t)$ is shorter that the arc length of $\tilde{c}(t)$.*

Proof. We need to show that $\Delta t(c_1, [t_0, t_1]) < \Delta t(c_2, [\tilde{t}_0, \tilde{t}_1])$. Since $A = c(t_0) = \tilde{c}(\tilde{t}_0)$ and $B = c(t_1) = \tilde{c}(\tilde{t}_1)$, it follows that $\Delta y(c_1, [t_0, t_1]) = \Delta y(c_2, [\tilde{t}_0, \tilde{t}_1])$ which implies

$$\Delta t(c_1, [t_0, t_1]) - Cost_t(c_1, [t_0, t_1]) = \Delta t(c_2, [\tilde{t}_0, \tilde{t}_1]) - Cost_t(c_2, [\tilde{t}_0, \tilde{t}_1]),$$

so $0 < Cost_t(c_2, [\tilde{t}_0, \tilde{t}_1]) - Cost_t(c_1, [t_0, t_1]) = \Delta t(c_2, [\tilde{t}_0, \tilde{t}_1]) - \Delta t(c_1, [t_0, t_1])$. □

2.2.4 Sequence of Geodesics on the Magnetic Space

Let us present two classical results on metric spaces.

Lemma 26. *Let $c_n(t)$ be a sequence of minimizing geodesics on the compact interval $\mathcal{T}$ converging uniformly to a geodesic $c(t)$, then $c(t)$ is minimizing in the interval $\mathcal{T}$.*

Proof. Let $[t_0, t_1] \subseteq \mathcal{T}$, then $dist_{\mathbb{R}_F^{n+2}}(c_n(t_0), c_n(t_1)) = |t_1 - t_0|$ since $c_n(t)$ is sequence of minimizing geodesic. If $n \to \infty$ then $dist_{\mathbb{R}_F^{n+2}}(c(t_0), c(t_1)) = |t_1 - t_0|$, by the uniformly convergence. □

Proposition 27. *Let K be a compact subset of $\mathbb{R}_F^{n+2}$ and let $\mathcal{T}$ be a compact time interval. Let us define the following space of $\mathbb{R}_F^{n+2}$-geodesics*

$$Min(K, \mathcal{T}) := \left\{\mathbb{R}_F^{n+2}\text{-geodesics } c(t) : c(\mathcal{T}) \subseteq K \text{ and } c(t) \text{ is minimizing in } \mathcal{T}\right\}.$$

Then $Min(K, \mathcal{T})$ is a sequentially compact space with respect to the uniform topology.

Proof. We need to prove that every sequence of $\mathbb{R}_F^{n+2}$-geodesics $c_n(t)$ in $Min(K, \mathcal{T})$ has a uniformly convergent subsequence converging to a minimizing $\mathbb{R}_F^{n+2}$-geodesic $c(t)$ in $Min(K, \mathcal{T})$. The space of geodesics $Min(K, \mathcal{T})$ is uniformly bounded and smooth in compact interval $\mathcal{T}$, then $Min(K, \mathcal{T})$ is a equi-continuous family of geodesics. By Arzela-Ascoli theorem, every sequence $c_n(t)$ in $Min(K, \mathcal{T})$ has a convergent subsequence $c_{n_s}(t)$ converging uniformly to a smooth curve $c(t)$. By Lemma 26 $c(t)$ is minimizing in $\mathcal{T}$. □

A useful tool for the proof of Theorem A, B and C is the following.

Corollary 28. *Let $c_1(t)$ be a $\mathbb{R}_F^{n+2}$-geodesic in $Min(K, \mathcal{T})$ and let $c_2(t)$ be a $\mathbb{R}_F^{n+2}$-geodesic. If $\varphi(x, y, z)$ is an isometry such that $c_2(\mathcal{T}') \subseteq \varphi(c(\mathcal{T}))$, then $c_2(t)$ is iminimizing in $\mathcal{T}'$.*

Chapter 3

Metric lines in jet space

This Chapter is devoted to proving Theorems A and B.

§ 3.1 Jet Space as a SubRiemannian Manifold

Let $f(x)$ and $g(x)$ be real-valued functions: we say they are related up to order k at x_0 if $f(x) - g(x) = O(|x - x_0|^{k+1})$ holds on a neighborhood of x_0, this relation is an equivalence relation on the space of germs of smooth functions at x_0 and it is called a k-jet at x_0. We identify the k-jet of a function f at x_0 with its k-th order Taylor polynomial of f at x_0, that is, k-jet is determined by the list of its k first derivatives at x_0:

$$u_0 = f(x_0) \text{ and } u_j = \frac{d^\ell f}{dx^\ell}(x_0), \qquad \ell = 1, \cdots, k.$$

When we vary the point and the function, we sweep out the k-jet space $J^k(\mathbb{R}, \mathbb{R})$, a $(k+2)$-dimensional manifold with global coordinates x and u_ℓ with $0 \leq \ell \leq k$. When fix the function f and let the independent variable x vary, we get a curve $j^k f : \mathbb{R} \to J^k(\mathbb{R}, \mathbb{R})$

called the k-jet of f, sending $x \in \mathbb{R}$ to the k-jet of f at x. In coordinates this is given by

$$(j^k f)(x) = \{(x, u_k(x), u_{k-1}(x), \cdots, u_1(x), u_0(x)) : \qquad \frac{d^\ell f}{dx^\ell}(x) = u_\ell\}.$$

The k-jet curve itself is tangent to a rank two distribution $\mathcal{D} \subseteq TJ^k(\mathbb{R}, \mathbb{R})$ at every point, and the following two left-invariant vector fields globally frame the distribution $\mathcal{D}$:

$$X = \frac{\partial}{\partial x} + \sum_{\ell=1}^{k} u_\ell \frac{\partial}{\partial u_{\ell-1}} \qquad \text{and} \qquad Y = \frac{\partial}{\partial u_k}.$$

We remark that the subRiemannian structure on $J^k(\mathbb{R}, \mathbb{R})$ is given by the metric in coordinates $ds^2 = dx^2 + du_k^2|_{\mathcal{D}}$ and the frame $\{X, Y\}$ is orthonormal . The vector fields X and Y generate the following Lie algebra:

$$Y^1 := [X, Y], \; Y^2 := [X, Y^1], \; \dots, \; Y^k := [X, Y^{k-1}].$$

All the others brackts are zero. The Lie algebra $\mathfrak{a}$ is given by the trivialization of Y, $Y^1 \dots, Y^{k-1}$ and Y^k. In this case $\mathcal{H} = \mathbb{R}$, $\mathcal{V} = \mathbb{R}$ and $[\mathfrak{h}, \mathfrak{h}] = 0$, as we required is sub-Section 2.1.7. For more details about the jet space as the Carnot group, see [10] or [12].

3.1.1 The Cotangent Bundle of a Carnot Group

Consider the cotangent bundle $T^*J^k(\mathbb{R}, \mathbb{R})$ and its traditional coordinates p_x and p_{u_ℓ}. The momentum function associated to the vector fields X and Y are the following: $P_X := p_x + \sum_{\ell=1}^k u_\ell p_{u_{\ell-1}}$ and $P_Y := p_{u_k}$. The Hamiltonian function governing the geodesic flow is given by

$$H_{sR} = \frac{1}{2}P_X^2 + \frac{1}{2}P_Y^2 = \frac{1}{2}(p_x + \sum_{\ell=1}^{k} u_\ell p_{u_\ell})^2 + \frac{1}{2}p_{u_k}^2.$$

The jet space $J^k(\mathbb{R},\mathbb{R})$ has a natural definition using the coordinates x and u_ℓ with $0 \leq \ell \leq k$. However, these coordinates do not easily show the symmetries of the system, while the exponential coordinates of the second kind do. The left-invariant vector fields X and Y in the exponential coordinates of the second kind $(x, \theta_0, \ldots, \theta_n)$ have the following form:

$$X = \frac{\partial}{\partial x} \qquad \text{and} \qquad Y = \sum_{\ell=0}^{k} \frac{x^\ell}{\ell!}\frac{\partial}{\partial \theta_\ell}.$$

We rewrite the Hamiltonian function H_{sR} as:

$$H_{sR}(p,g) = \frac{1}{2}p_x^2 + \frac{1}{2}(\sum_{\ell=0}^{k} p_{\theta_\ell}\frac{x^\ell}{\ell!})^2.$$

Since H_{sR} does not depend on the variables θ_ℓ, they are cycle coordinates, and p_{θ_ℓ} are constant of motion. Then the $\mathfrak{a}^*$ valued one-form $\mathcal{A}_{J^k(\mathbb{R},\mathbb{R})}$ is given by

$$\mathcal{A}_{J^k(\mathbb{R},\mathbb{R})} = d\theta_0 \otimes (\sum_{\ell=0}^{k} e^\ell \frac{x^\ell}{\ell!}).$$

If $\mu = (a_0, \ldots, a_k)$ is in $\mathfrak{a}^*$, then the reduced Hamiltonian is given by

$$H_\mu(p_x, x) = \frac{1}{2}p_x^2 + \frac{1}{2}F_\mu^2(x) \quad \text{where} \quad F_\mu(x) = \sum_{\ell=1}^{m} a_\ell \frac{x^\ell}{\ell!}. \tag{3.1.1}$$

When $a_0 = a_2 = \cdots = a_m = 0$, the reduced system H_μ is the harmonic oscillator, and the corresponding geodesic $\gamma(t)$ in $J^k(\mathbb{R},\mathbb{R})$ is the lift of a geodesic in the Heisenberg group, see [12]. Let $\eta(t) = x(t)$ be a $\mathcal{A}_{J^k(\mathbb{R},\mathbb{R})}$-curve, then the lift equation is the following:

$$\dot{\gamma} = \dot{x}(t)X(\gamma(t)) + F_\mu(x(t))Y(\gamma(t)). \tag{3.1.2}$$

As we proved in [12], a geodesic in $J^k(\mathbb{R},\mathbb{R})$ is determined by a polynomial F_μ and a hill interval I. Let us formalize the hill interval definition:

Definition 29. *We say that a closed interval I is a hill interval associated to $F_\mu(x)$, if $|F_\mu(x)| < 1$ for every x in the interior of I and $|F_\mu(x)| = 1$ for every x in the boundary of I. If I is of the form $[x_0, x_1]$, then we call x_0 and x_1 the endpoints of the hill interval.*

We remark that the reduced dynamics occur in the hill interval and the hill region $hill(\mu)$ is union of all the hill intervals of $F_\mu(x)$. By definition, I is compact if and only if $F_\mu(x)$ is not a constant polynomial. In contrast, the constant polynomial $F_\mu(x)$ defines a geodesic line.

3.1.2 Classification of Geodesic in Jet Space

Let $\gamma(t)$ be a geodesic in $J^k(\mathbb{R}, \mathbb{R})$ corresponding to the pair (F_μ, I), where $F_\mu(x)$ is no constant polynomial and $I = [x_0, x_1]$, then $\gamma(t)$ is only one of the following options:

- We say $\gamma(t)$ is x-periodic if its reduced dynamics is periodic. The reduced dynamics is periodic if and only if x_0 and x_1 are regular points of $F_\mu(x)$.
- We say $\gamma(t)$ is homoclinic if its reduced dynamics is a homoclinic orbit. The reduced dynamics has a homoclinic orbit if and only if one of the points x_0 and x_1 is regular and the other is a critical point of $F_\mu(x)$.
- We say $\gamma(t)$ is heteroclinic if its reduced dynamics is a heteroclinic orbit. The reduced dynamics has a heteroclinic orbit if and only if both points x_0 and x_1 are critical of $F_\mu(x)$.
- We say a heteroclinic geodesic $\gamma(t)$ is turn-back if $F_\mu(x_0)F_\mu(x_1) = -1$.
- We say a heteroclinic geodesic $\gamma(t)$ is direct-type if $F_\mu(x_0)F_\mu(x_1) = 1$.

3.1.3 Unitary Geodesics

To prove Theorem A and B, we will introduce the concept of a unitary geodesic:

Definition 30. *We say a geodesic $\gamma(t)$ in $J^k(\mathbb{R},\mathbb{R})$ corresponding to the pair (F_μ, I) is unitary if $I = [0,1]$. We say a direct-type geodesic (or homoclinic) $\gamma(t)$ is unitary, if in addition $F_\mu(x(t)) \to 1$ when $t \to \pm\infty$.*

The reflection $R_{\theta_0}(x, \theta_0, \theta_1, \ldots, \theta_k) = (x, -\theta_0, \theta_1, \ldots, \theta_k)$ is in the isometry group $Iso(J^k(\mathbb{R},\mathbb{R}))$. If $\gamma(t)$ is a direct type or homoclinic geodesic such that $F_\mu(x(t)) \to -1$ when $t \to \pm\infty$, then $R_{\theta_0}(\gamma(t))$ is such that $F_\mu(x(t)) \to 1$ when $t \to \pm\infty$.

Corollary 31. *Let $\gamma(t)$ be a unitary direct-type geodesic for $F_\mu(x)$, then there exists $q(x)$ such that $F_\mu(x) = 1 - x^{k_1}(1-x)^{k_2}q(x)$, where $1 < k_1$, $1 < k_2$, and $q(x)$ is polynomial of degree $k - k_1 - k_2$ such that $0 < x^{k_1}(1-x)^{k_2}q(x) < 2$ if x is in $(0,1)$.*

Proof. By construction, $F_\mu(x)$ is such that $F_\mu(0) = F_\mu(1) = 1$, $F'_\mu(0) = F'_\mu(1) = 0$, and $|F_\mu(x)| < 1$ if x is in $(0,1)$, then using the Euclidean algorithm we find the desired result. □

Any geodesic in $J^k(\mathbb{R},\mathbb{R})$ is related to unitary geodesic by a Carnot dilatation and translation.

Proposition 32. *Let $\gamma(t)$ be a geodesic in $J^k(\mathbb{R},\mathbb{R})$ associated to the pair (F_μ, I) and let $h(\tilde{x}) = x_0 + u\tilde{x}$ be the affine map taking $[0,1]$ to $I = [x_0, x_1]$ with $u := x_1 - x_0$. If $\hat{F}_\mu(h(\tilde{x})) = F_\mu(x)$ and $\hat{\gamma}(t)$ is the geodesic in $J^k(\mathbb{R},\mathbb{R})$ corresponding to the pair $(\hat{F}_\mu, [0,1])$. Then $\gamma(t)$ is related to $\hat{\gamma}(t)$ by Carnot dilatation and translation, that is*

$$\gamma(t) = \delta_u \hat{\gamma}(\frac{t}{u}) * (x_0, 0 \ldots, 0),$$

where δ_u is the Carnot dilatation.

Proposition 32 and the reflection R_{θ_0} imply that it is enough to prove Theorem A and B for the unitary case.

3.1.4 The Three-Dimensional Magnetic Space

By classical mechanics, we get:

Proposition 33. *Let $c(t)$ be a x-periodic $\mathbb{R}^3_F$-geodesic for the pencil (a, b) with a hill interval I the period is given by*

$$L(G, I) := 2\int_I \frac{dx}{\sqrt{1 - G^2(x)}}. \tag{3.1.3}$$

Moreover, the changes $\Delta y(c, [t, t + L]) = \Delta y(G, I)$ and $\Delta z(c, [t, t + L]) = \Delta y(G, I)$ are given by

$$\Delta y(G, I) := 2\int_I \frac{G(x)dx}{\sqrt{1 - G^2(x)}} \quad \textit{and} \quad \Delta z(G, I) := 2\int_I \frac{G(x)F(x)dx}{\sqrt{1 - G^2(x)}}. \tag{3.1.4}$$

In [12, p. 162], we proved Proposition 33 using classical mechanics, see [13, Section 11]. In [30], we showed a similar statement using a generating function of the second type, see [9, Section 50]. $L(G, I)$, $\Delta y(G, I)$ and $\Delta z(G, I)$ are smooth functions with respect to the parameters (a, b) if and only if the corresponding geodesic $c(t)$ for (G, I) is x-periodic. We define an auxiliary map that will help us to prove Theorems A and B.

Definition 34. *The period map $\Theta : (G, I) \to [0, \infty] \times \mathbb{R}$ is given by*

$$\Theta(G, I) := (\Theta_1(G, I), \Theta_2(G, I)) := 2\left(\int_I \sqrt{\frac{1 - G(x)}{1 + G(x)}}dx, \int_I G(x)\frac{1 - F(x)}{\sqrt{1 - G^2(x)}}dx\right).$$

$\Theta_1(G,I)$ and $\Theta_2(G,I)$ are smooth function with respect the parameters (a,b) not only when the corresponding geodesic $c(t)$ for (G,I) is x-periodic, they are also smooth when $c(t)$ is a direct-type or homoclinic geodesic such that $G(x(t)) \to 1$ when $t \to \pm\infty$.

Corollary 35. *Let $G(x)$ be in Pen_F. Then:*

(1) $\Theta_1(G,I) = 0$ if and only if $G(x) = 1$.

(2) If $I = [x_0, x_1]$ is compact, then $\Theta_1(G,I)$ is finite if and only if x_0 and x_1 are not critical point of $G(x)$ with value -1.

We introduce an important concept called the travel interval:

Definition 36. *Let $c(t)$ be a $\mathbb{R}^3_F$-geodesic traveling during the time interval $[t_0, t_1]$. We say that $\mathcal{I}[t_0,t_1] := x([t_0,t_1])$ is the travel interval of $c(t)$, counting multiplicity.*

For instance, if $c(t)$ is a $\mathbb{R}^3_F$-geodesic with hill interval I such that its coordinate $x(t)$ is L-periodic, then $\mathcal{I}[t, t+L] = 2I$.

Corollary 37. *Let $c(t)$ be a $\mathbb{R}^3_F$-geodesic for $G(x)$ in Pen_F with travel interval $\mathcal{I}$. Then $\Delta(c,[t_0,t_1])$ from Definition 24 can be rewritten in terms of polynomial $G(x)$ and the travel interval $\mathcal{I}$ as follows;*

$$\Delta(c,[t_0,t_1]) = \Delta(G,\mathcal{I}) := \left(\int_{\mathcal{I}} \frac{dx}{\sqrt{1-G^2(x)}}, \int_{\mathcal{I}} \frac{G(x)dx}{\sqrt{1-G^2(x)}}, \int_{\mathcal{I}} \frac{G(x)F(x)dx}{\sqrt{1-G^2(x)}}\right).$$

In the same way, the map $Cost(c,[t_0,t_1])$ from Definition 24 can be rewritten as follows:

$$Cost(c,[t_0,t_1]) = Cost(G,\mathcal{I}) := \left(\int_{\mathcal{I}} \frac{1-G(x)}{\sqrt{1-G^2(x)}}dx, \int_{\mathcal{I}} \frac{(1-F(x))G(x)}{\sqrt{1-G^2(x)}}dx\right)$$

Same proof that Proposition 33.

Corollary 38. $\lim_{n\to\infty} Cost_t(c,[-n,n])$ *is finite if and only if* $\lim_{t\to\pm\infty} G(x(t)) = 1$.

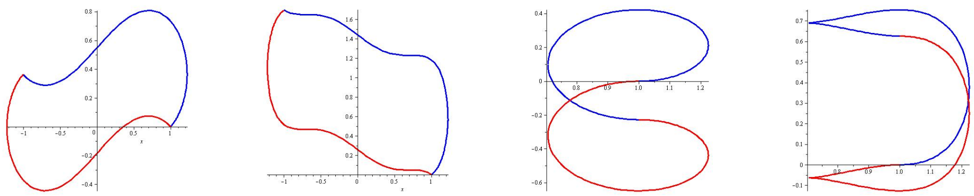

Figure 3.1: Typical x-periodic geodesics indicating Maxwell points. The first two images present two half-period curves share endpoints and the last two images present two complete-period curves share endpoints in the planes (x, y) and (x, z), respectively.

3.1.5 Upper bound of the cut point

Definition 39. *Let $\gamma : \mathbb{R} \to X$ be a geodesic in a length space (eg. a subRiemannian manifold) parameterized by arclength.*

- *The cut time of γ is*

$$t_{cut}(\gamma) := \sup\{t > 0 : \ \gamma|_{[0,t]} \ \textit{ is length-minimizing}\}.$$

- *A positive time $t = t_{MAX}$ is called a Maxwell time for γ if there is a geodesic distinct from γ which connects $\gamma(0)$ to $\gamma(t_{MAX})$ and whose length is t_{MAX}. We then call $\gamma(t_{MAX})$ a Maxwell point along γ.*

It is well-known that in subRiemannian and Riemannian metric spaces, geodesics fail to minimize when extended beyond their smallest Maxwell time t_{MAX}. Thus,

$$t_{cut}(\gamma) \leq \inf\{t : t \text{ is a Maxwell time for } \gamma\}$$

See for example, [31], Lemma 5.2, chapter 5 for the Riemannian case.

Proposition 40. *Let c be a x-periodic geodesic on R^3_F with x-period L. Then*

1.- $t_{cut}(c) \leqslant \frac{L(G,I)}{2}$ if F is even and c's Hill interval contains 0.

2.- $t_{cut}(c) \leqslant L(G, I)$ in all cases.

The prove of this Proposition is in [12]

§ 3.2 Direct-Type Geodesic

This section is devoted to proving Theorem A. Let $\gamma_d(t)$ be an arbitrary unitary direct-type geodesic in $J^k(\mathbb{R}, \mathbb{R})$ for a unitary polynomial $F_d(x)$ given by Corollary 31. We will consider the space $\mathbb{R}^3_{F_d}$ and the $\mathbb{R}^3_{F_d}$-geodesic $c_d(t) := \pi_{F_d}(\gamma_d(t))$. Then we will prove the following Theorem:

Theorem 41. *The $\mathbb{R}^3_{F_d}$-geodesic $c_d(t)$ is a metric line $\mathbb{R}^3_{F_d}$.*

The strategy to prove Theorem 41 is the following: We take an arbitrary T and build a $\mathbb{R}^3_{F_d}$-geodesic $c_\infty(t)$ in $Min(K, \mathcal{T})$ and isometry φ in $Iso(\mathbb{R}^3_{F_d})$ such that $c([-T, T]) = \varphi(c_\infty(\mathcal{T}))$, where K is a compact subset of $\mathbb{R}^3_{F_d}$ and $\mathcal{T}$ is a compact interval. By corollary 28, $c_d(t)$ is minimizing in $[-T, T]$. Since T is arbitrary, $c_d(t)$ is a metric line.

Let $c_d(t) = (x(t), y(t), z(t))$. Without loss of generality, we can assume that $0 \leq \dot{x}(t)$ and $c_d(0) = (x(0), 0, 0)$ for some $x(0)$ in $(0, 1)$ since the proof for the case $0 \geq \dot{x}(t)$ is similar and we can use the t, y, and z translations.

3.2.1 The Magnetic Space

Corollary 42. *Let q_{max} be equal to $\max_{x\in[0,1]}\{x^{k_1}(1-x)^{k_2}q(x)\}$, where $q(x)$, k_1 and k_2 are given by Corollary 31. The set of all the direct-type $\mathbb{R}^3_{F_d}$-geodesic with hill interval $[0, 1]$ is given by*

$$Pen_d := \{(a, b) = (s, 1-s) : \quad s \in (\frac{2}{q_{max}}, 1)\} \cup \{(a, b) = (-s, s-1) : \quad s \in (\frac{2}{q_{max}}, 1)\}.$$

Moreover, the map $\Theta_2(G,[0,1]) : Pen_d \to \mathbb{R}$ is one to one, and $Cost(c_d,[t_0,t_1])$ is bounded by $\Theta_d := \Theta_1(F_d,[0,1])$ for all $[t_0,t_1]$.

Proof. Since $F_d(x) \neq -1$ if x is in $[0,1]$, the constant Θ_d is finite. Let us prove that $Cost(c_d,[t_0,t_1])$ is bounded by Θ_d for all $[t_0,t_1]$. Using Corollary (37) and the condition $|F_d(x)| \leq 1$ for x in $[0,1]$, we find that:

$$|Cost_y(c_d,[t_0,t_1])| < Cost_t(c_d,[t_0,t_1]) < 2\int_{[0,1]} \sqrt{\frac{1-F_d(x)}{1+F_d(x)}}dx =: \Theta_1(F_d,[0,1]).$$

To prove that $\Theta_2(G,[0,1]) : Pen_{F_d} \to \mathbb{R}$ is one to one, we notice that the multiplication by minus sends $(s,1-s)$ into $(-s,s-1)$ and $\Theta_2(G,I) = -\Theta_2(-G,I)$. Then, it is enough to consider the case $(a,b) = (s,1-s)$. We consider the one-parameter family of polynomials $G_s(x) = s + (1-s)F_d(x)$. Thus, $\Theta_2(G_s,[0,1]) : (0,q_{max}) \to \mathbb{R}$ is one variable function, let us calculate its derivative:

$$\frac{d}{ds}\Theta_2(G_s,[0,1]) = \frac{d}{ds}\int_{[0,1]} \frac{(1-F_d(x))G_s(x)}{\sqrt{1-G_s^2(x)}}dx = \int_{[0,1]} \frac{1-F_d(x)}{(1-G_s^2(x))^{\frac{3}{2}}}dx.$$

Since $0 < 1 - F_d(x)$, then $0 < \frac{d}{ds}\Theta_2(G_s,[0,1])$. □

Lemma 43. *Let $\Omega(F_d) = hill(F_d)\times\mathbb{R}^2$ be the region and let $S_+(x,y,z) : \Omega \to \mathbb{R}$ be the calibration for $c_d(t)$ function given by Proposition 95, then $c_d(t)$ is minimizing between the curves that lay in the region Ω.*

Proof. The proof follows by Proposition 94, since $c_d(t)$ never touches the hill interval boundary in finite time. □

Corollary 44. *There exist $T_d^* > 0$ such that $y_d(t) > 0$ if $T_d^* < t$, and $y_d(t) < 0$ if $-T_d^* > t$.*

Proof. By construction, $\lim_{t\to\infty} y_d(t) = \infty$ and $\lim_{t\to-\infty} \Delta y_d(0) = -\infty$. □

Definition 45. *If $\mathcal{T} := [t_0, t_1]$, we define the following set*

$$Com([0,1]) := \left\{(c(t), \mathcal{T}) : c(t) \text{ is a } \mathbb{R}^3_{F_d}\text{-geodesic},\ x(t_0) \in [0,1] \text{ and } x(t_1) \in [0,1]\right\}.$$

Lemma 46. *Let us consider a sequence of pair $(c_n(t), [-n, n])$ in $Com([0,1])$. If $Cost(c_n, [-n, n])$ is uniformly bounded, then there exists a compact subset $K_{\mathcal{H}}$ of $\mathcal{H}$ such that $\mathcal{I}_n[-n, n] \subseteq K_{\mathcal{H}}$ for all n.*

The proof is Appendix 1.1.

3.2.2 Proof of Theorem 41

Set up the Proof of Theorem 41

Let T be arbitrarily large and consider the sequence of points $c_d(-n)$ and $c_d(n)$ where $T < n$ and n is in $\mathbb{N}$. Let $c_n(t) = (x_n(t), y_n(t), z_n(t))$ be a sequence of minimizing $\mathbb{R}^3_{F_d}$-geodesics, in the interval $[0, T_n]$ such that:

$$c_n(0) = c_d(-n), \quad c_n(T_n) = c_d(n) \quad \text{and} \quad T_n \leq n. \tag{3.2.1}$$

We call the equations and inequality from 3.2.1 the endpoint conditions and the shorter condition, respectively. If $c_n(t)$ is geodesic for the polynomial $G_n(x)$ and a hill interval I_n, then Proposition 40 implies $T_n \leq L(G_n, I_n)$. Since the endpoint condition holds for all n, then the sequence $c_n(t)$ holds asymptotic conditions:

$$\lim_{n\to\infty} c_n(0) = (0, -\infty, -\infty), \quad \lim_{n\to\infty} c_n(T_n) = (1, \infty, \infty), \tag{3.2.2}$$

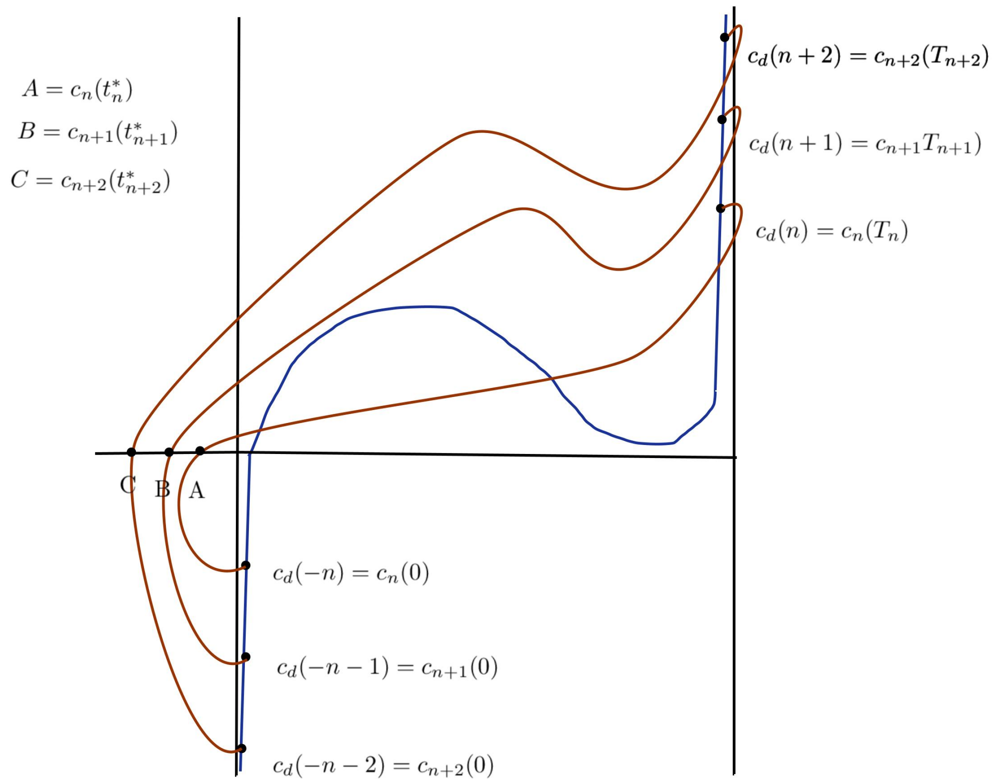

Figure 3.2: The images show the projection to $\mathbb{R}^2$, with coordinates (x, y), of direct type geodesic $c_d(t)$ and the sequence of geodesics $c_n(t)$.

and the asymptotic period condition:

$$\lim_{n\to\infty} Cost_y(c_n, [0, T_n]) = \frac{1}{2}\Theta_2(F_d, [0, 1]). \tag{3.2.3}$$

Corollary 47. *The sequence of $\mathbb{R}^3_{F_d}$-geodesics $c_n(t)$ is not a sequence of geodesic lines and does not converge to a geodesic line. In particular, $c_n(t)$ does not converge to an abnormal geodesic.*

Proof. The Calibration function from Lemma 43 implies that if $c_n(t)$ is shorter than $c_d(t)$, then $c_n(t)$ must leave the region $[0, 1] \times \mathbb{R}^2$ and come back, then $c_n(t)$ is a geodesic for non-constant polynomial $G_n(x)$, and $c_n(t)$ is not a geodesic line.

Let $\mathcal{I}_n$ travel interval of $c_n(t)$, then $c_n(t)$ cannot converge to a geodesic line, since $\lim_{n\to\infty} \mathcal{I}_n = [0,1]$ and the only line in the plane (x, θ_0) that travel from $\theta_0 = -\infty$ into $\theta_0 = \infty$ in a fine travel interval is the vertical line, but the vertical line has travel interval $[0,0]$. In particular, Lemma 22 implies $c_n(t)$ cannot converge to an abnormal geodesic. □

The construction of the $\mathbb{R}^3_{F_d}$-geodesic c_n is such that the initial condition $c_n(0)$ is not bounded. The following Proposition provides a bounded initial condition.

Proposition 48. *Let n be a natural number larger than T_d^*, where T_d^* is given by Corollary 44, and let $K_0 := K_{\mathcal{H}} \times [-1,1] \times K_z$ be the compact set, where $K_{\mathcal{H}}$ is the compact set from Lemma 46 and $K_z := [-\Theta_d, \Theta_d]$. Then there exist a time $t_n^* \in (0, T_n)$ such that $c_n(t_n^*)$ is in K_0 for all $n > T_d^*$.*

Proof. Let n be a natural number larger than T_d^*. By construction, $y_n(0) < 0$ and $y_n(T_n) > 0$, the intermediate value theorem implies that exist a t_n^* in $(0, T_n)$ such that $y_n(t_n^*) = 0$. Since $Cost(c_n, [0, T_n])$ is bounded, by Lemma 46, there exists a compact set $K_{\mathcal{H}}$ such that $x_n(t)$ is in $K_{\mathcal{H}}$ for all t in $[0, T_n]$.

Let us prove that $|z_n(t_n^*)| \leq \Theta_d$: the endpoint conditions imply

$$\Delta y(c_d, [-n,n]) = \Delta y(c_n, [0,T_n]) \quad \text{and} \quad \Delta z(c_d, [-n,n]) = \Delta z(c_n, [0,T_n]).$$

So $Cost_y(c_d, [-n,n]) = Cost_y(c_n, [0,T_n])$ and Corollary 42 tells us $Cost_y(c_n, [0,T_n])$ is bounded. By definition of $Cost_y$, it follows that:

$$z_n(t_n^*) - z_n(0) = \Delta z(c_n, [0,t_n^*]) = \Delta y(c_n, [0,t_n^*]) - Cost_y(c_n, [0,t_n^*]),$$

$$z_d(0) - z_d(-n) = \Delta z(c_d, [-n,0]) = \Delta y(c_d, [-n,0]) - Cost_y(c_d, [-n,0]).$$

By construction, $\Delta y(c_n, [0, t_n^*]) = \Delta y(c_d, [-n, 0])$, $z_d(0) = 0$ and $z_n(0) = z_d(-n)$, then

$$|z_n(t_n^*)| = |Cost_y(c_n, [0, t_n^*]) - Cost_y(c_d, [-n, 0])| \leq \Theta_d.$$

We just proved $c_n(t_n^*)$ is in K. □

Let us consider the sequence of minimizing $\mathbb{R}^3_{F_d}$-geodesics $\tilde{c}_n(t) := c_n(t + t_n^*)$ in the interval $\mathcal{T}_n := [-t_n^*, T_n - t_n^*]$. $\tilde{c}_n(0)$ is bounded and minimizing $\mathbb{R}^3_{F_d}$-geodesics in the interval $\mathcal{T}_n$.

Corollary 49. *There exists a subsequence $\mathcal{T}_{n_j}$ such that $\mathcal{T}_{n_j} \subseteq \mathcal{T}_{n_{j+1}}$.*

Proof. Since $\tilde{c}_n(0)$ is bounded and $c(-t_n^*)$ and $c(T_n - t_n^*)$ are unbounded, it follows that $[-t_n^*, T_n - t_n^*] \to [-\infty, \infty]$ when $n \to \infty$. We can take a subsequence of intervals $\mathcal{T}_{n_j}$ such that $\mathcal{T}_{n_j} \subseteq \mathcal{T}_{n_{j+1}}$. □

For simplicity, we will use the notation $\mathcal{T}_n$ for the subsequence $\mathcal{T}_{n_j}$.

Lemma 50. *Let N be a natural number larger than T_d^*. Then there exist compact set $K_N \subseteq \mathbb{R}^3_F$ such that $c_n(t)$ is in $Min(K_N, \mathcal{T}_N)$ if $n > N$.*

Proof. Since $\tilde{c}_n(t)$ is minimizing on the interval $\mathcal{T}_n$, it follows that $\tilde{c}_n(t)$ is minimizing on the interval $\mathcal{T}_N \subseteq \mathcal{T}_n$ if $n > N$. Moreover, there exists a compact set K_N such that $\tilde{c}_n(\mathcal{T}_N) \subseteq K_N$, since $c_n(0)$ is in K_0 and $c_n(t)$ is a family of smooth functions defined on the compact set $\mathcal{T}_N$. □

Therefore, $\tilde{c}_n(t)$ has a convergent subsequence $\tilde{c}_{n_j}(t)$ converging to a $\mathbb{R}^3_{F_d}$-geodesic $c_\infty(t)$. Corollary 47 implies that $c_\infty(t)$ is a normal $\mathbb{R}^3_{F_d}$-geodesic for a polynomial $G(x)$ in Pen_{F_d}. The following Lemma provides the uniqueness of $G(x) = F_d(x)$:

Lemma 51. *$G(x) = F_d(x)$ is the unique polynomial in the pencil of $F_d(x)$ satisfying the asymptotic conditions given by (3.2.2) and (3.2.3).*

Proof. By Proposition 27, $\tilde{c}_n(t)$ has a convergent subsequence $\tilde{c}_{n_s}(t)$ converging to a minimizing geodesic $\tilde{c}(t)$ on the interval $\mathcal{T}_N$. Being a $\mathbb{R}^3_{F_d}$-geodesic, $c(t)$ is associated to a polynomial $G(x) = a + bF_d(x)$. $G(0) = a + b$ must be equal 1, to satisfy the asymptotic conditions given by (3.2.2). Then (a, b) is in Pen_d, the set defined in Corollary 42. Since the map $\Theta_1(a, b) : Pen_d \to \mathbb{R}$ is one to one, the unique polynomial in Pen_d satisfying the condition (3.2.2) and (3.2.3) is $G(x) = F_d(x)$. □

Proof of Theorem 41

Proof. Let $\tilde{c}_n(t)$ be the sequence of geodesics defined by the endpoint conditions (3.2.1). By Lemma 50, for all $N > T^*_d$ there exist a compact set K_N such that $\tilde{c}(t)$ is in $Min(K_N, \mathcal{T}_N)$ if $n > N$. By Proposition 27, there exist a subsequence $\tilde{c}_{n_j}(t)$ converging to a $\mathbb{R}^3_{F_d}$-geodesic $c_\infty(t)$ in $Min(K_N, \mathcal{T}_N)$. Corollary 47 implies that $c_\infty(t)$ is a normal geodesic for a polynomial $G(x)$ in Pen_{F_d}. Lemma 50 tells that $G(x) = F_d(x)$.

Since $c_\infty(t)$ and $c_d(t)$ are $\mathbb{R}^3_{F_d}$-geodesics for $F_d(x)$ with the same hill interval, there exists a translation $\varphi_{(y_0,z_0)}$, in $Iso(\mathbb{R}^3_{F_d})$ sending $c_\infty(t)$ to $c_d(t)$. Using N is arbitrary and $c_d([-T, T])$ is bounded, we can find compact sets $K := K_N$ and $\mathcal{T} := \mathcal{T}_N$ such that $c_d([-T, T]) \subseteq \varphi_{(y_0,z_0)}(c_\infty(\mathcal{T}))$ and c_∞ is in $Min(K, \mathcal{T})$. Corollary 28 implies that $c_d(t)$ is minimizing in $[-T, T]$ and T is arbitrarily. Therefore, $c_d(T)$ is a metric line in $\mathbb{R}^3_{F_d}$. □

Proof of Theorem A

Proof. By Theorem 41, $c_d(t)$ is a metric line. Since π_{F_d} is a subRiemannian submersion and $\gamma_d(t)$ is the lift of $c_d(t)$, then Proposition 3 implies that the direct type geodesic

$\gamma_d(t)$ is a metric line in $J^k(\mathbb{R}, \mathbb{R})$. □

§ 3.3 Homoclinic Geodesics in Jet Space

This chapter is devoted to proving Theorem B. Let $\gamma_h(t)$ be the homoclinic geodesic in $J^k(\mathbb{R}, \mathbb{R})$ for $F_h(x) := 1 - 2x^{2n}$. We will consider the space $\mathbb{R}^3_{F_h}$ and the geodesic $c_h(t) := \pi_{F_h}(\gamma_h(t))$, then we will prove the following Theorem:

Theorem 52. *The geodesic $c_h(t)$ is a metric line $\mathbb{R}^3_{F_h}$.*

The following Theorem shows that the method used to prove Theorem 52 cannot be used to prove the odd case $F(x) := 1 - 2x^{2n+1}$.

Theorem 53. *Let $\gamma(t)$ be the homoclinic geodesic in $J^k(\mathbb{R}, \mathbb{R})$ for $F(x) := 1 - 2x^{2n+1}$ and $c(t) := \pi_{F_h}(\gamma(t))$ be the homoclinic $\mathbb{R}^3_{F_h}$-geodesic. Then $c(t)$ is not a metric line $\mathbb{R}^3_{F_h}$.*

The proof of Theorem 53 is in Section 2.

3.3.1 The Magnetic space For the Homoclinic Geodesics

Without loss of generality, $c_h(0) = (1, 0, 0)$, by use of the t, y and z translations. By the time reversibility of the reduced Hamiltonian h_μ given by (3.1.1), it follows that $x(-n) = x(n)$ and $\Delta x(c_h, [-n, n]) := x(n) - x(-n) = 0$ for all n.

Lemma 54. *Let $c_h(t)$ be the homoclinic $\mathbb{R}^3_{F_h}$-geodesic for $F_h(x) := 1 - 2x^{2n}$, then*

$$\Theta_2(F_h, [0, 1]) < 0.$$

Proof. By construction, $-xF_h'(x) = (2n-1)(1-F_h(x))$. Using integration by parts it follows that

$$\begin{aligned}\Theta_2(F_h,[0,1]) &= \frac{-2}{2n-1}\int_{[0,1]}\frac{xF_h'(x)F(x)dx}{\sqrt{1-F_h^2(x)}}\\ &= \frac{2}{2n-1}x\sqrt{1-F_h^2(x)}|_0^1 - \frac{2}{2n-1}\int_{[0,1]}\sqrt{1-F_h^2(x)}dx.\end{aligned}$$

$x\sqrt{1-F_h^2(x)}|_0^1 = 0$ implies the desired result. $\square$

Corollary 55. *The set of all the homoclinic $\mathbb{R}^3_{F_h}$-geodesics is given by*

$$Pen_h := \{(a,b) = (s, 1-s) : \ \ s \in (1,\infty)\} \cup \{(a,b) = (-s, s-1) : \ \ s \in (1,\infty)\}.$$

Moreover, the map $\Theta_2(G,[0,1]) : Pen_h \to \mathbb{R}$ is one to one and $Cost(c_h,[t_0,t_1])$ is bounded by $\Theta_1(F_h,[0,1]) := \Theta_h$ for all $[t_0,t_1]$.

Proof. The proof's first part is the same as the one from 42. To prove that $\Theta_1(a,b)$: $Pen_h \to \mathbb{R}$ is one to one, we notice the multiplication by minus sends $(s, 1-s)$ to $(-s, s-1)$ and $\Theta_2(G,I) = -\Theta_2(-G,I)$. It is enough we consider the one-parameter family of homoclinic polynomial $G_s(x) := s-(1-s)F_h(x)$ with hill interval $[0, \sqrt[2n]{\frac{1}{s}}]$. Thus, $\Theta_1(G_s,[0,\sqrt[2n]{\frac{1}{s}}]) : (0,\infty) \to \mathbb{R}$ is a one variable function and it is enough to show it is a monotone increasing function. Let us set up the change of variable $x = \sqrt[2n]{\frac{1}{s}}\tilde{x}$ so that $F(\tilde{x}) = 1-2\tilde{x}^{2n} = F_h(\tilde{x})$ and

$$\Theta_2(G_s,[0,\sqrt[2n]{\frac{1}{s}}]) = \int_{[0,\sqrt[2n]{\frac{1}{s}}]}\frac{2x^{2n}G_s(x)}{\sqrt{1-G_s^2(x)}}dx = (\sqrt[2n]{\frac{1}{s}})^{n+1}\Theta_2(F_h,[0,1]).$$

Since $\frac{1}{s}$ is monotone decreasing and $\Theta_2(F_h,[0,1])$ is negative. Then $\Theta_2(G_s,[0,\sqrt[2n]{\frac{1}{s}}])$ is a monotone increasing function with respect to s. $\square$

Corollary 56. *There exist $T_h^* > 0$ such that $y_h(t) > 0$ if $T_h^* < t$ and $y_h(t) < 0$ if $-T_h^* > t$. Moreover, $Cost_y(c_h, [-t, t]) < 0$ if $T_h^* < t$.*

Proof. Since $Cost_y(c_h, [-t, t]) \to \Theta_2(F_h, [0, 1])$ as $t \to \infty$ and $\Theta_2(F_h, [0, 1]) < 0$, we can find the desired T_h^*. The rest of the proof is equal to Corollary 44. □

3.3.2 Set up the proof of Theorem 52

Let T be arbitrarily large and consider the sequence of points $c_h(-n)$ and $c_h(n)$ where $T < n$ and n is in $\mathbb{N}$. Let $c_n(t) = (x_n(t), y_n(t), z_n(t))$ be a sequence of minimizing $\mathbb{R}^3_{F_h}$-geodesics in the interval $[0, T_n]$ such that:

$$c_n(0) = c_h(-n), \quad c_n(T_n) = c_h(n) \quad \text{and} \quad T_n \le n. \tag{3.3.1}$$

We call the equations and inequality from (3.3.1) the endpoint conditions and the shorter condition, respectively. If $c_n(t)$ is geodesic for the polynomial $G_n(x)$ and a hill interval I_n, then Proposition 40 implies $T_n \le L(G_n, I_n)$. Since the endpoint condition holds for all n, the sequence $c_n(t)$ has the asymptotic conditions:

$$\lim_{n\to\infty} c_n(0) = (0, -\infty, -\infty), \quad \lim_{n\to\infty} c_n(T_n) = (0, \infty, \infty), \tag{3.3.2}$$

and the asymptotic period condition

$$\lim_{n\to\infty} Cost_y(c_n, [0, T_n]) = \Theta_2(F_h, [0, 1]). \tag{3.3.3}$$

The following Corollary tells us $c_n(t)$ is not a sequence of line geodesics. We remark that applying the calibration function from proposition 95 is impossible.

Corollary 57. *Let n be larger than T_h^*, where T_h^* is given by Corollary 56, then the* *sequence of geodesics $c_n(t)$ neither is a sequence of geodesic lines, nor converge to a geodesics line. In particular, $c_n(t)$ does not converge to an abnormal geodesic.*

Proof. Let us assume that $c_n(t)$ is a sequence of geodesic lines. Since $\Delta x(c_h, [-t,t])) = 0$ for all n and $\Delta y(c_h, [-t,t])) > 0$ for all $n > T_h^*$, the unique geodesic line satisfying these conditions is the one generated by the polynomial $G_n(x) = 1$. Since $1 - F_h(x) > 0$ for all x, then $(1 - F_h(x))G_n(x) > 0$ for all x and it follows that:

$$Cost_y(c_n, [0, T_n]) = \int_0^T (1 - F_h(x(t)))G_n(x(t))dt > 0.$$

This contradicts the endpoint conditions given by (3.3.1) since $Cost_y(c_h, [-t,t]) < 0$ if $T_h^* < t$. The same proof follows if $c_n(t)$ converges to a geodesics line $c(t)$ generated by $G(x) = 1$, since there exists N big enough that $G_n(x) > \frac{1}{2}$ for $n > N$. □

Notice that this proof cannot be done in the case $F_h(x) = 1 - 2x^{2n+1}$. In Section 2 under the hypothesis $F_h(x) = 1 - 2x^{2n+1}$, we will find a sequence of curves $c_n(t)$ shorter than $c_h(t)$ than $c_h(t)$ that converges to the abnormal geodesic.

The following Proposition provides the bounded initial condition.

Proposition 58. *Let n be a natural number larger than T_h^*, where T_h^* is given by Corollary 56, and let $K_0 = K_{\mathcal{H}} \times [-1,1] \times [-C_h, C_h]$ the compact set, where $K_{\mathcal{H}}$ and C_h is a compact set and the constant defined by Lemma 46 and Corollary 55, respectively. Then there exist a time $t_n^* \in (0, T_n)$ such that $c_n(t_n^*)$ is in K_0 for all $n > T_1^*$.*

Same proof as Proposition 48. Consider the sequence of minimizing $\mathbb{R}^3_{F_h}$-geodesic $\tilde{c}_n(t) := c_n(t + t_n^*)$ in the interval $\mathcal{T}_n := [-t_n^*, T_n - t_n^*]$, so $\tilde{c}_n(0)$ is bounded and minimizing on the interval $\mathcal{T}_n$.

Corollary 59. *There exists a subsequence $\mathcal{T}_{n_j}$ such that $\mathcal{T}_{n_j} \subseteq \mathcal{T}_{n_{j+1}}$.*

The proof of Corollary 59 is equal as the of Corollary 49. For simplicity, we will use the notation $\mathcal{T}_n$ for the subsequence v_{n_j}.

Lemma 60. *There exist compact set $K_N \subseteq \mathbb{R}^3_F$ such that $c_n(t)$ is in $Min(K_N, \mathcal{T}_N)$ if $n > N$.*

The proof of Lemma 60 is equal to the of Lemma 50. Therefore, $\tilde{c}_j(t)$ has a convergent subsequence $\tilde{c}_{j_i}(t)$ converging to a $\mathbb{R}^3_{F_h}$-geodesic $c_\infty(t)$. Corollary 57 implies that $c_\infty(t)$ is a normal $\mathbb{R}^3_{F_h}$-geodesic for a polynomial $G(x)$ in Pen_{F_h}. The following Lemma provides the uniqueness of $G(x) = F_h(x)$.

Lemma 61. *$G(x) = F_h(x)$ is the unique polynomial in the pencil of $F_h(x)$ satisfying the asymptotic conditions given by* (3.3.2) *and* (3.3.3)*.*

Proof. By Proposition 27 $\tilde{c}_n(t)$ has a convergent subsequence $\tilde{c}_{n_s}(t)$ converging to a minimizing geodesic $\tilde{c}(t)$ on the interval $\mathcal{T}_N$. Being a geodesic in $\mathbb{R}^3_{F_h}$, $c(t)$ is associated to a polynomial $G(x) = a+bF_h(x)$. $G(0) = a+b$ must be equal 1, to satisfy the asymptotic conditions given by (3.3.2). Then (a, b) is in Pen_h, the set defined in Corollary 55. Since the map $\Theta_1(G, I) : Pen_h \to \mathbb{R}$ is one to one, the unique polynomial in Pen_h satisfying the condition (3.3.2) is $G(x) = F_h(x)$. □

The proof of Theorems 52 and B are the same as the proof of Theorems 41 and A, respectively.

Chapter 4

Metric lines in Engel type

This Chapter is devoted to proving Theorem C.

§ 4.1 The Engel Type Group as subRiemannian Manifold

Let $\mathrm{Eng}(n)$ be the Carnot group with growth vector $(n+1, 2n+1, 2n+2)$ and whose first layer $\mathfrak{g}_1$, framed by $\{E^1, \cdots, E^n, E^0_{\mathfrak{a}}\}$, generates the following Lie algebra:

$$E^i_{\mathfrak{a}} := [E^i, E^0] \quad i = 1, \cdots, n, \quad \text{and} \quad E^{n+1}_{\mathfrak{a}} := [E^i, E^i_{\mathfrak{a}}]. \tag{4.1.1}$$

Otherwise, zero. The Lie algebra $\mathfrak{a}$ is given by E^0, $E^1_{\mathfrak{a}}, \cdots$, $E^n_{\mathfrak{a}}$ and $E^{n+1}_{\mathfrak{a}}$. In this case $\mathcal{H} \simeq \mathbb{R}^n$, $\mathcal{V} \simeq \mathbb{R}$ and $[\mathfrak{h}, \mathfrak{h}] = 0$. The $\mathfrak{a}^*$ valued one-form $\mathcal{A}_{\mathrm{Eng}(n)}$ is given by

$$\alpha_{\mathrm{Eng}(n)} = d\theta \otimes (e_0 + \sum_{i=1}^{n} x_i e_i + \frac{1}{2}||x||^2_{\mathcal{H}} e_{i+1}).$$

If $\mu = \sum_{\ell=0}^{n+1} a_\ell e^\ell$ in $\mathfrak{a}^*$ then the reduced Hamiltonian H_μ is given by

$$H_\mu(p_x, x) = \frac{1}{2}||p_x||^2_{\mathcal{H}} + \frac{1}{2}F^2_\mu(x) \text{ where } F_\mu(x) = a_0 + \sum_{i=1}^{n} a_i x_i + a_{n+1}\frac{1}{2}||x||^2_{\mathcal{H}}. \quad (4.1.2)$$

Let us consider the case $a_{n+1} \neq 0$, if we set up the change of coordinates

$$(\hat{x}_1, \cdots, \hat{x}_n) = (\frac{a_1}{a_{n+1}} + x_1, \ldots, \frac{a_1}{a_{n+1}} + x_n) \text{ and define } (b_1, b_2) = (a_0 - \frac{1}{2}\sum_{i=1}^{n} a_i^2, \frac{a_{n+1}}{2}).$$

Then

$$H_\mu(p_x, x) = \frac{1}{2}||p_x||^2_{\mathcal{H}} + \frac{1}{2}F^2_\mu(r) \text{ where } F_\mu(r) = b_1 + b_2 r^2. \quad (4.1.3)$$

Where $r := ||\hat{x}||$. We conclude that after a translation, the reduced Hamiltonian H_μ is the radial an-harmonic oscillator.

4.1.1 History of the notation

In [16], E. Le Donne and F. Tripaldi used the notation $N_{6,3,1a^*}$ to denote the Carnot group $\mathrm{Eng}(2)$. After making the symplectic reduction in the general context, we used it to find the reduced Hamiltonian H_μ for particular examples from [16], one of these was $N_{6,3,1a^*}$. We consider the subRiemannian geodesic flow on $N_{6,3,1a^*}$ and found that the reduced Hamiltonian H_μ is the plane radial an-harmonic oscillator. We were inspired to define the Carnot group $\mathrm{Eng}(n)$ by the work of R. Montgomery, in [28]. Where he considered the subRiemannian geodesic flow in Eng , and he showed that the reduced Hamiltonian H_μ is the an-harmonic oscillator. Latter, we discovered the relation between the homoclinic geodesics in $\mathrm{Eng}(n)$ and the Euler-Soliton.

§ 4.2 Geodesics in Engel Type

We split the dynamics of reduced Hamiltonian H_μ, given by (4.1.2), into two cases, when $p_{\theta_{n+1}} = a_{n+1} = 0$ and $p_{\theta_{n+1}} = a_{n+1} \neq 0$. In the first case, the Hamiltonian H_μ has a quadratic potential on the x coordinates, so the problem is a small oscillation system, see [9, Chapter 5]. In the second case, the reduced dynamics correspond to the radial an-harmonic oscillator, see (4.1.3). In sub-Section 4.2.2, we will reduce, again, the radial an-harmonic oscillator into a Hamiltonian $H_{(\mu,\ell)}(p_r, r)$ with one degree of freedom and effective potential $V_{ef}(r)$, see (4.2.2). We use the classification of one degree of freedom systems to classify the case $a_{n+1} \neq 0$, as we did in $J^k(\mathbb{R}, \mathbb{R})$.

Definition 62. *Let $\gamma(t)$ be a geodesic in* $\mathrm{Eng}(n)$ *different than a geodesic line, then $\gamma(t)$ is only one of the following options:*

1. *We say a geodesic $\gamma(t)$ is oscillatory if $a_{n+1} = 0$.*
2. *we say a geodesic $\gamma(t)$ is radial if $a_{n+1} \neq 0$.*
3. *We say a geodesic $\gamma(t)$ is r-periodic if the dynamics of reduced system* (4.2.2) *is periodic.*
4. *We say a geodesic $\gamma(t)$ is r-homoclinic if the dynamics of reduced system* (4.2.2) *is a homoclinic orbit.*

The following Theorem tells oscillatory and r-periodic geodesic are not metric lines:

Theorem 63. *The oscillatory and r-periodic geodesics are not metric lines in* $\mathrm{Eng}(n)$.

The proof is in Appendix 5.

4.2.1 Radial Case

Proposition 64. *Let $SO(n)$ be the group of rotation of $\mathcal{H}$, then the Lie algebra $\mathfrak{eng}(n)$ is invariant under the action of $SO(n)$ given by*

$$\tilde{E}^j = \sum_{i=1}^{n} Q_{j,i} E^i, \quad \tilde{E}^0_{\mathfrak{a}} = E^0_{\mathfrak{a}}, \quad \tilde{E}^j_{\mathfrak{a}} = \sum_{i=1}^{n} Q_{j,i} E^i_{\mathfrak{a}}, \quad \tilde{E}^{n+1}_{\mathfrak{a}} = E^{n+1}_{\mathfrak{a}},$$

where $Q := (Q_{j,i})$ is in $SO(n)$. Moreover, the action on $\mathfrak{eng}(n)$ induces an isometric action φ_Q on $\mathrm{Eng}(n)$. If (x, θ) are exponential coordinates of the second type defined in 4.1, then $\varphi_Q(x, \theta) = (\tilde{x}, \tilde{\theta})$ is given by

$$\tilde{x}_j = \sum_{i=1}^{n} Q_{j,i} x_i, \quad \tilde{\theta}_0 = \theta_0, \quad \tilde{\theta}^2_j = \sum_{i=1}^{n} Q_{j,i} \theta^2_i, \quad \tilde{\theta}^3_1 = \theta^3_1 \;\; \textit{where} \;\; Q := (Q_{j,i}) \in SO(n).$$

Proof. Let us prove that vectors $\{\tilde{E}^1, \dots, \tilde{E}^n, \tilde{E}^0_{\mathfrak{a}}, \dots, \tilde{E}^{n+1}_{\mathfrak{a}}\}$ satisfy the bracket relations given by (4.1.1): Let us start with the first layer $\mathfrak{g}_1$,

$$[\tilde{E}^j, \tilde{E}^0_{\mathfrak{a}}] = \sum_{i=1}^{n} Q_{j,i} [E^j, E^0_{\mathfrak{a}}] = \sum_{i=1}^{n} Q_{j,i} E^j_{\mathfrak{a}} = \tilde{E}^j_{\mathfrak{a}}.$$

Let us verify that the bracket relations hold for the second layer $\mathfrak{g}_2$,

$$[\tilde{E}^j, \tilde{E}^k_{\mathfrak{a}}] = \sum_{i=1, i'=1}^{n} Q_{j,i} Q_{j,i'} [E^j, E^k_{\mathfrak{a}}] = \sum_{i=1, i'=1}^{n} Q_{j,i} Q_{j,i'} \delta^k_j E^{n+1}_{\mathfrak{a}} = E^{n+1}_{\mathfrak{a}} = \tilde{E}^{n+1}_{\mathfrak{a}}.$$

□

Definition 65. *Let $M_{(x_1,x_2)}$ be the 6 dimensional sub-manifold of $\mathrm{Eng}(n)$ given by*

$$M_{(x_1,x_2)} := \{(x, \theta) \in \mathrm{Eng}(n) : (x, \theta) = (x_1, x_2, 0, \dots, 0, \theta_0, \theta_1, \theta_2, 0, \dots, 0, \theta_n)\}$$

Lemma 66. *Let $\gamma(t)$ be a geodesic in $\mathrm{Eng}(n)$ such that $\gamma(0)$ is in $M_{(x_1,x_2)}$ and $\dot{\gamma}(0)$ is in $T_{\gamma(0)} M_{x_1,x_2}$, then $\gamma(t)$ lies in M_{x_1,x_2} for all t.*

Proof. By Hamilton equation we have $\dot{x}_i = p_{x_i}(t)$, $\dot{p}_{x_i} = 2b_2 x_i(t) F_\mu(r(t))$ and $\dot{\theta}_i(t) = x_i(t) F_\mu(r(t))$. The initial condition implies $\dot{x}_i(0) = p_{x_i}(0) = 0$, $\dot{p}_{x_i}(0) = 0$ and $\dot{\theta}_i(0) = 0$ for all $2 < i \leq n$. Therefore, $\dot{x}_i(t) = 0$, $\dot{p}_{x_i}(t) = 0$ and $\theta_i(t) = 0$ for all t and $2 < i \leq n$. □

Corollary 67. *Any geodesic in* $\mathrm{Eng}(n)$ *with* $p_{\theta_{n+1}} \neq 0$ *has the form* $\gamma(t) = \varphi_Q(\gamma_0(t))$, *where* φ_Q *is given by 64 and* $\gamma_0(t)$ *is a geodesic in* $M_{(x_1,x_2)}$.

Then, it is enough to understand the dynamics of the plane an-harmonic oscillator to describe the dynamics of the radial an-harmonic oscillator.

4.2.2 The plane radial an-harmonic oscillator

The reduced Hamiltonian H_μ defined by equation (4.1.3) in polar coordinates is given by

$$H_\mu(p_x, p_\theta, r, \theta) := \frac{1}{2}(p_r^2 + \frac{p_\theta^2}{r^2}) + \frac{1}{2}F_\mu^2(r). \tag{4.2.1}$$

index$V_{ef}(r)$ Since the potential is radial, θ is a cyclic coordinate, and p_θ is constant. If $p_\theta = \ell$, then the effective potential is $\frac{1}{2}V_{ef}(r)$, where $V_{ef}(r) := \frac{\ell^2}{r^2} + F_\mu^2(r)$ and the reduced Hamiltonian H_μ can be reduced, again, to one-degree of freedom Hamiltonian system given by

$$H_{(\mu,\ell)}(p_x, r) := \frac{1}{2}p_r^2 + \frac{1}{2}V_{ef}(r). \tag{4.2.2}$$

Fixing the energy level $H_{(\mu,\ell)} = \frac{1}{2}$ and using Hamilton equation $\dot{r} = p_r$, we reduced the calculations to a quadrature in the radial coordinate.

Definition 68. *We say an interval $R = [r_{min}, r_{max}]$ is the radial hill interval of the effective potential V_{ef}, if $V_{ef}(r_{min}) = V_{ef}(r_{max}) = 1$ and $V_{ef}(r) < 1$ for all r in (r_{min}, r_{max}).*

Definition 69. *We denote by $hill(\mu, \ell)$ the closed annulus given by*

$$hill(\mu, \ell) := \{(x_1, x_2) \in \mathbb{R}^2 : r_{min}^2 \leq x_1^2 + x_1^2 \leq r_{max}^2\}.$$

We call $hill(\mu, \ell)$ the hill region of the reduced Hamiltonian $H_{(\mu,\ell)}$ $F_\mu(r)$, where r_{min} and r_{max} are given by Corollary 68.

Corollary 70. *The plane an-harmonic oscillator has an equilibrium at $r = 0$ if and only if $\ell = 0$ and $F_\mu(0) = \pm 1$.*

Proof. Let us assume $(p_r, p_\theta, r, \theta) = (0, 0, 0, \theta_0)$ is an equilibrium point. By, Hamilton's equations for H_μ and $p_\theta = 0$ imply $\ell = 0$. Then, we can read the conservation of the energy $\frac{1}{2} = H_\mu$ as $\frac{1}{2} = \frac{1}{2}(p_r^2 + F_\mu^2(r))$. If we plug $(p_r, p_\theta, r, \theta) = (0, 0, 0, \theta_0)$ into H_μ we have that $F_\mu(0) = 1$.

Conversely, let us assume $\ell = 0$ and $F_\mu(0) = \pm 1$: then the reduced Hamilton equation for H_μ with the conditions $\ell = 0$ imply $\dot{p}_\theta = 0$. The conservation of energy tells $\dot{p}_r = 0$ at $r = 0$. □

§ 4.3 The radial Magnetic Space

$SO(n) \times \mathbb{R}^2$ acts on $\mathbb{R}_F^{n+2}$ by rotation and translation. If Q is in $SO(\mathcal{H})$ and (y_0, z_0) is in $\mathbb{R}^2$, then $\varphi_{(Q,y_0,z_0)}(x, y, z) = (Qx, y + y_0, z + z_0)$ and $\varphi_{(Q,y_0,z_0)}$ is in $Iso(\mathbb{R}_F^{n+2})$.

Lemma 71. *If $\mathbb{R}^2_{(x_1,x_2)} := \{(x_1, \ldots, x_n, y, z) \in \mathbb{R}^{n+2}_{F_h} : 0 = x_3 = \cdots = x_n\}$, then every geodesic $\mathbb{R}_F^{n+2}$-geodesic $c(t)$ with $b \neq 0$ has the form $\varphi_{(Q,0,0)}(c_0(t))$ where $c_0(t)$ is a*

$\mathbb{R}^{n+2}_F$-geodesic in $\mathbb{R}^2_{(x_1,x_2)}$ for all t.

Therefore, it is enough to work on $\mathbb{R}^4_{F_h}$. If $F(r)$ is given by (4.1.3) and H_F is the Hamiltonian defined by equation (2.2.2), then $V_{ef}(r) := \frac{\ell^2}{r^2} + G^2(r)$ is the effective potential of the reduced system. This inspires the following definition.

Definition 72. *We say that the three-dimensional space Pen_V is the pencil of $F(r)$, if $Pen_V := \{V_{ef}(r) = \frac{\ell^2}{r^2} + G^2(r) : G(r) \in Pen_F\}$.*

The following Proposition is the analogous to Proposition 33

Proposition 73. *Let $c(t)$ be a r-periodic $\mathbb{R}^4_F$-geodesic for the pair (G, ℓ) with a radial hill interval R then the r and θ periods are given by*

$$L(G, \ell, R) := 2\int_R \frac{dr}{\sqrt{1 - V_{ef}(r)}} \quad \text{and} \quad \Delta\theta(G, \ell, R) := 2\int_R \frac{\ell dr}{r^2\sqrt{1 - V_{ef}(r)}}. \tag{4.3.1}$$

Moreover, the changes $\Delta y(c, [t, t+L]) = \Delta y(G, \ell, R)$ and $\Delta z(c, [t, t+L]) = \Delta y(G, \ell, R)$ are given by

$$\Delta y(G, \ell, R) := 2\int_R \frac{G(r)dr}{\sqrt{1 - V_{ef}(r)}} \quad \text{and} \quad \Delta z(G, \ell, R) := 2\int_R \frac{G(r)F(r)dr}{\sqrt{1 - V_{ef}(r)}}. \tag{4.3.2}$$

We define an axillary map that will help us prove Theorems C.

Definition 74. *The period map $\Theta : (G, \ell, R) \longrightarrow [0, \infty] \times \mathbb{R}$ is given by*

$$\Theta(G, \ell, R) := (\Theta_1(G, \ell, R), \Theta_2(G, \ell, R)) := 2\Big(\int_R \frac{1 - G(r)}{\sqrt{1 - V_{ef}(r)}}dr, \int_R \frac{G(r)(1 - F(r))}{\sqrt{1 - V_{ef}(r)}}dr\Big).$$

,

Corollary 75. *Let $G(r)$ be in Pen_F and let ℓ be the angular momentum. Then:*

(1) $\Theta_1(G, \ell, R) = 0$ if and only if $G(r) = 1$ and $\ell = 0$.

(2) If R is compact, then $\Theta_1(G, \ell, R)$ is finite if and only if 0 is in $\mathcal{R}$ and $G(0) = -1$.

We introduce an important concept called the radial travel interval:

Definition 76. *Let $c(t)$ be a $\mathbb{R}^4_F$-geodesic traveling in the time interval $[t_0, t_1]$. We say $\mathcal{R}[t_0, t_1] := r([t_0, t_1])$ is the travel interval of the $c(t)$, counting multiplicity.*

For instance, if $c(t)$ is an $\mathbb{R}^4_F$-geodesic with hill interval R such that its coordinate r is L-periodic then $\mathcal{R}[t, t+L] = 2R$.

Corollary 77. *Let $c(t)$ be an $\mathbb{R}^4_F$-geodesic for $V_{ef}(r)$ in Pen_V and let $\mathcal{R}$ be its radial travel interval. Then $\Delta(c, [t_0, t_1])$ from Definition 24 can be rewritten in terms of the effective potential $V_{ef}(r)$ and the travel radial interval $\mathcal{R}$ as follows;*

$$\Delta(c, [t_0, t_1]) = \Delta(G, \ell, \mathcal{R}) := (\int_{\mathcal{R}} \frac{dr}{\sqrt{1 - V_{ef}(r)}}, \int_{\mathcal{R}} \frac{G(r)dr}{\sqrt{1 - V_{ef}(r)}}, \int_{\mathcal{R}} \frac{G(r)F(r)dr}{\sqrt{1 - V_{ef}(r)}}).$$

In the same way, the map $Cost(c, [t_0, t_1])$ from Definition 24 can be rewritten as follows:

$$Cost(c, [t_0, t_1]) = Cost(G, \ell, \mathcal{R}) := 2(\int_{\mathcal{R}} \frac{1 - G(r)}{\sqrt{1 - V_{ef}(r)}} dr, \int_{\mathcal{R}} \frac{(1 - F(r))G(r)}{\sqrt{1 - V_{ef}(r)}} dr).$$

The proof of Corollary 77 is the same proof of Proposition 33.

4.3.1 Upper bound of the cut point

Proposition 78. *Let $c(t)$ be a r-periodic geodesic on R^4_F with r-period $L(G, \ell, R)$. Then $t_{cut}(c) \leqslant L(G, \ell, R)$.*

Let $c(t) = (r(t), \theta(t), y(t), z(t))$ be the geodesic, let (G, ℓ) be its pair and R be its radial Hill-interval. Write $c(0) = (r_i, \theta_i, y_i, z_i)$. If r_i is interior to the Hill interval, then there are exactly two magnetic geodesics passing through $c(0)$ and associated to (G, ℓ), namely, the given one $c(t)$ and $\tilde{c} = (\tilde{r}(t), \tilde{\theta}(t), \tilde{y}(t), \tilde{z}(t))$ characterized by $\dot{r}(0) = -\dot{\tilde{r}}(0)$.

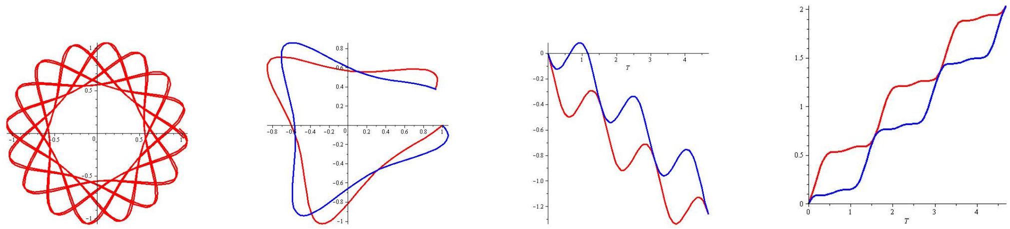

Figure 4.1: The first panel displays a typical r-periodic solution of an-harmonic oscillator. The last three panels show the Maxwell point in the plane (x_1, x_2), (t, y) and (t, z), respectively.

Then $\tilde{r}(t) = r(-t)$ for all t. By r-periodicity we have $r(L(G,\ell,R)) = \tilde{r}(L(G,\ell,R)) = r_i$. Proposition 73 tells us that $c(t)$ and $\tilde{c}(t)$ have the same θ, y and z, periods, $\Delta\theta(G,\ell,R)$, $\Delta y(G,\ell,R)$, $\Delta z(G,\ell,R)$. Thus

$$c(L) = (r_i, \theta_i + \Delta\theta(G,\ell,R), y_i + \Delta y(G,\ell,R), z_i + \Delta z(G,\ell,R)) = \tilde{c}(L).$$

The geodesics curves are distinct, showing that L is a Maxwell time for $c(t)$ and so $t_{cut}(c) \leqslant L(G,\ell,R)$. In case r_i is in the boundary of the radial Hill interval R, we can find a Jacobi vector field in the same way we did in [12] or [8].

§ 4.4 Homoclinic Geodesics in Engel Type

This section is devoted to proving C. Without loss of generality, let $\gamma_h(t)$ be the homoclinic geodesic in $\mathrm{Eng}(2)$ for $F_h(x) := 1 - 2r^2$, whose reduced dynamics has initial condition $x = (1,0)$. Using Carnot dilatation and rotation, it is enough to prove this case.

Let $\gamma_h(t)$ be the homoclinic geodesic in $\mathrm{Eng}(n)$ for $F_h(x) := 1 - 2r^2$. We consider the geodesic $c_h(t) := \pi_{F_h}(\gamma_h(t))$ in the space $\mathbb{R}^4_{F_h}$, and will prove the following Theorem.

Theorem 79. *The direct type geodesic $c_h(t)$ is a metric line $\mathbb{R}^4_{F_h}$.*

Without loss of generality, we take the initial condition $c_h(0) = (1, 0, 0, 0)$. By construction, $c_h(t) = (x_1(t), 0, y(t), z(t))$. Moreover, $\Delta x(c_h, [-n, n])) = 0$ for all n, since $x_1(-t) = x_1(t)$.

4.4.1 The Magentic Space

Lemma 80. *Let $c_h(t)$ be the homoclinic $\mathbb{R}^4_{F_h}$-geodesic for $F_h(r) := 1 - 2r^2$, then*

$$\Theta_2(F, [0, 1]) < 0. \tag{4.4.1}$$

There exist $T^*_h > 0$ such that $y_h(t) > 0$ if $T^*_h < t$ and $y_h(t) < 0$ if $-T^*_h > t$. Moreover, $Cost_y(c_h, [-t, t]) > 0$ if $T^*_h < t$.

Same proof as Lemma 54 and Corollary 56.

Corollary 81. *The set of all the homoclinic $\mathbb{R}^4_F$-geodesic $Pen_h \subseteq Pen_V$ is given by*

$$Pen_h := \{(a, b, \ell) = (s, 1-s, 0) : \ \ s \in (1, \infty)\} \cup \{(a, b, \ell) = (-s, s-1, 0) : \ \ s \in (1, \infty)\}.$$

Moreover, the map $\Theta_2(G, \ell, \mathcal{R}) : Pen_h \to \mathbb{R}$ is one to one, and $Cost(c_h, [t_0, t_1])$ is bounded by $\Theta_1(F, 0, [0, 1]) := \Theta_h$ for all $[t_0, t_1]$.

Same proof as Corollary 55.

Definition 82. *Let $B_{\mathcal{H}}$ be the ball of radius one on $\mathcal{H}$. We define the following set*

$$Com(B_{\mathcal{H}}) := \left\{(c(t), [t_0, t_1]) : c(t) \text{ is a } \mathbb{R}^4_F\text{-geodesic}, \ x(t_0) \in B_{\mathcal{H}} \text{ and } x(t_1) \in B_{\mathcal{H}}\right\}.$$

Lemma 83. *Let us consider a sequence of pairs* $(c_n(t), [-n, n])$ *in* $Com(B_{\mathcal{H}})$. *If* $Cost(c_n, [-n, n])$ *is uniformly bounded then there exists a compact subset* $K_{\mathcal{H}}$ *of* $\mathcal{H}$ *such that* $x([-n, n]) \subseteq K_{\mathcal{H}}$ *for all pair* n.

Same proof as Lemma 46.

4.4.2 Set up for the proof of Theorem 79

Let T be arbitrarily large and consider the sequence of points $c_h(-n)$ and $c_h(n)$ where $T < n$ and n is $\mathbb{N}$. Let $c_n(t) = (x_n(t), y_n(t), z_n(t))$ be a sequence of minimizing geodesics in the interval $[0, T_n]$ such that

$$c_n(0) = c_h(-n), \quad c_n(T_n) = c_h(n) \;\text{ and }\; T_n \le n. \tag{4.4.2}$$

We call the equations and inequality from (4.4.2) the endpoint conditions and the shorter condition, respectively. If $c_n(t)$ is geodesic for the pair (G_n, ℓ_n) and a hill interval R_n, then Proposition 40 implies $T_n \le L(G_n, \ell_n, R_n)$. Since the endpoint condition holds for all n, the sequence $c_n(t)$ has the asymptotic conditions:

$$\lim_{n\to\infty} c_n(0) = (0, 0, -\infty, -\infty), \quad \lim_{n\to\infty} c_n(T_n) = (0, 0, \infty, \infty), \tag{4.4.3}$$

and the asymptotic period condition

$$\lim_{n\to\infty} Cost_y(c_n, [0, T_n]) = \Theta_2(F_h, 0, [0, 1]). \tag{4.4.4}$$

The following Proposition provides the bounded initial condition.

Proposition 84. *Let n be a natural number larger than* T_h^*, *where* T_h^* *is given by Corollary 80, and let* $K = K_{\mathcal{H}} \times [-1, 1] \times [C_h, C_h]$ *be the compact set, where* $K_{\mathcal{H}}$ *and* C_h *is a* *compact set and the constant defined by Lemma 83 and Corollary 81, respectively. Then there exists a time $t_n^* \in (0, T_n)$ such that $c_n(t_n^*)$ is in K^* for all $n > T_1^*$.*

Same proof that Proposition 48. Consider the sequence of minimizing geodesics $\tilde{c}_n(t) := c_n(t + t_n^*)$ on the interval $\mathcal{T}_n := [-t_n^*, T_n - t_n^*]$ so that $\tilde{c}_n(0)$ is bounded and minimizing on the interval $\mathcal{T}_n$.

Corollary 85. *There exists a subsequence $\mathcal{T}_{n_j}$ such that $\mathcal{T}_{n_j} \subseteq \mathcal{T}_{n_{j+1}}$.*

The proof of Corollary 85 is equal as the of Corollary 49. For simplicity we will use the notation $\mathcal{T}_n$ for the subsequence $\mathcal{T}_{n_j}$.

Lemma 86. *There exists compact set $K_N \subseteq \mathbb{R}^4_F$ such that $c_n(t)$ is in $Min(K_N, \mathcal{T}_N)$ if $n > N$.*

The proof of Lemma 86 is equal as the of Corollary 50. Therefore $\tilde{c}_n(t)$ has a convergent subsequence $\tilde{c}_{n_j}(t)$ converging to a $\mathbb{R}^4_{F_h}$-geodesic $c_\infty(t)$, then $c_\infty(t)$ is a $\mathbb{R}^4_{F_h}$-geodesic for a polynomial $G(x)$ in Pen_{F_h}. The following Lemma provides the uniqueness of $G(x) = F(x)$.

Lemma 87. *$(G, \ell) = (F_h, 0)$ is the unique pair satisfying the asymptotic conditions given by (4.4.3) and (4.4.4).*

Proof. Corollary 70 tells that the reduced system has an equilibrium point if and only if $G(0) = 1$ and $\ell = 0$. The rest of the proof is the same from Lemma 61. □

4.4.3 Proof of Theorem 79

Proof. Let $\tilde{c}_n(t)$ be the sequence of geodesics defined by the endpoint conditions (4.4.2). By Lemma 86, for all $N > T_d^*$ there exist a compact set K_N such that $\tilde{c}_n(t)$ is in

$Min(K_N, \mathcal{T}_N)$ if $n > N$. By Proposition 27, there exist a subsequence $\tilde{c}_{n_j}(t)$ converging to a $\mathbb{R}^4_{F_h}$-geodesic $c_\infty(t)$ in $Min(K_N, \mathcal{T}_N)$. Corollary 47 implies that $c_\infty(t)$ is a normal geodesic for a polynomial $G(x)$ in Pen_{F_h}. Lemma 50 tells that $G(x) = F_h(x)$.

Since $c_\infty(t)$ and $c_h(t)$ are $\mathbb{R}^4_{F_h}$-geodesics for $F_h(r)$ with the same radial hill interval, there exists an isometry $\varphi_{(Q,y_0,z_0)}(x, y, z) = (Qx, y + y_0, z + z_0)$ sending $c_\infty(t)$ to $c_h(t)$. Using N is arbitrary and $c_d([-T, T])$ is bounded, we can find compact sets $K := K_N$ and $\mathcal{T} := \mathcal{T}_N$ such that $c_d([-T, T]) \subseteq \varphi_{(Q,y_0,z_0)}(c_\infty(\mathcal{T}))$ and c_∞ is in $Min(K, \mathcal{T})$. Corollary 28 implies that $c_h(t)$ is minimizing in $[-T, T]$ and T is arbitrarily. Therefore, $c_h(T)$ is a metric line in $\mathbb{R}^4_{F_h}$. $\square$

The proof of Theorem C is the same as the proof of Theorem A.

Chapter 5

Conclusion

(1) We developed a new method to prove that a geodesic is a metric line. Theorem A proves the Conjecture 6 for the direct-type case, and the problem remains open for the homoclinic case. Theorem 53 says we cannot use the space R^3_F to prove the Conjecture. However, Theorem 53 does not imply that the Conjecture is false. The homoclinic case can be solved by showing the corresponding period map in $J^k(\mathbb{R}, \mathbb{R})$ restricted to the homoclinic geodesics is one-to-one.

(2) In [29], R. Montgomery, M. Shapiro and A. Stolin proved the subRiemannian geodesic flow on the group of all 4 by 4 lower triangular matrices with 1's on the diagonal is not integrable. In [16], E. Le Donne and F. Tripaldi used the notation $N_{6,3,1}$ to denote this Carnot group. $N_{6,3,1}$ has one family of homoclinic geodesics up to a dilatation. This family is related to the Euler-Soliton: there exists a two-plane inside $\mathbb{R}^3$ such that the projection to the homoclinic geodesic is the Euler-Soliton, as we say in $\mathrm{Eng}(n)$. In future work, we will prove that this family's homoclinic geodesics are metric lines in $N_{6,3,1}$.

Appendix

§ 1 Prelude to the Proof of Lemma 46

Definition 88. *Let $\mathcal{P}(k)$ be the vector space of polynomial on $\mathcal{H} = \mathbb{R}$ of degree bounded by k, and let $||F||_\infty := \sup_{x\in[0,1]} |F(x)|$ be the uniform norm. We denote by $B(k)$ the closed ball of radius 1.*

Proposition 89. *$B(k)$ is a compact set.*

Proof. Since $B(k)$ is a bounded subset of the finite-dimensional space $\mathcal{P}(k)$, it is enough to prove that $B(k)$ is closed, indeed, by Arzela-Ascoli theorem we just need to prove that $B(k)$ is an equi-continuous set: let $F(x)$ be a polynomial in $C(k)$, then the Markov brothers' inequality implies $|F'(x)| \leq k^2$, so $|F(x_1) - F(x_2)| < k^2|x_1 - x_2|$. □

Definition 90. *We say a polynomial F is unitary if F has a hill interval $[0, 1]$, and let $\mathcal{P}_N(k)$ be the set of unitary polynomials. Let $F_\mu(x)$ be a polynomial with hill interval $[x_0, x_1]$ and let $u := x_1 - x_0$ be the length of the hill interval.*

Corollary 91. *If $G_n(x)$ is a sequence of non-constant polynomials in Pen_F with hill interval $I_n = [x_n, x_n']$ such that $G_n(x_n) = G_n(x_n') = 1$, $\lim_{n\to\infty} x_n = -\infty$ and $\lim_{n\to\infty} x_n' = \infty$, then $F(x)$ must be even degree.*

Proof. Let $G_n(x)$ be equal to $a_n + b_n F(x)$. There exists K_x a compact set containing all the roots of $F(x)$, and let n be large enough that $K_x \subseteq I_n$. Let us assume $F(x)$ is an odd degree. Without loss of generality, let us assume $F(x'_n) > 0$ and $F(x_n) < 0$, then $0 = G(x'_n) - G(x_n) = b_n(F(x'_n) - F(x_n))$ and $b_n = 0$ since $F(x'_n) - F(x_n) > 0$, which is a contradiction to the assumption that $G_n(x)$ is a sequence of non-constant polynomials. $\square$

1.1 Proof of Lemma 46

Proof. Let $c_n(t) = (x_n(t), y_n(t), z_n(t))$ be a sequence of $\mathbb{R}^3_F$-geodesics traveling during a time interval $[(t_0)_n, (t_1)_n]$ and with travel interval $\mathcal{I}_n[(t_0)_n, (t_1)_n]$ such that $x_n((t_0)_n)$ and $x_n((t_1)_n)$ are in $[0, 1]$ for all n. Then we will prove that if $\mathcal{I}_n$ is unbounded, then $\Theta(c, [t_0, t_1])$ is unbounded.

The sequence of $c_n(t)$ of $\mathbb{R}^3_F$-geodesics, induces a sequence of $G_n(x)$ polynomials, which induces a sequence of unitary polynomials $\hat{G}_n(\tilde{x}) := G_n(h_n(\tilde{x}))$ where $h_n(\tilde{x})$ is the affine map given by Definition 90, that is, $h_n(\tilde{x}) = (x_0)_n + u_n\tilde{x}$ where $u_n := (x_0)_n - (x_1)_n$. Since $\hat{G}_n(\tilde{x})$ is in $C(k)$. There exists a subsequence $\hat{G}_{n_s}(\tilde{x})$ converging to $\hat{G}(\tilde{x})$. Let us proceed by the following cases: case $\hat{G}(\tilde{x}) \neq 1$ or case $\hat{G}(\tilde{x}) = 1$.

Case $\hat{G}(\tilde{x}) \neq 1$: by Fatou's lemma $0 < Cost(\hat{G}) \leq \liminf_{n\to\infty} Cost(\hat{G}_n)$. Then $u_n \longrightarrow \infty$ implies $Cost(c, \mathcal{I}_n)$ is unbounded.

Case $\hat{G}(\tilde{x}) = 1$: let $K'_{\mathcal{H}}$ be a compact set such that all the roots of $1 - F(x)$ are in $K'_{\mathcal{H}}$. There exists $n^* > 0$ such that $\hat{G}(\tilde{x}) > \frac{1}{2}$ for all $\tilde{x}$ in $[0, 1]$ if $n_s > n^*$. We split the integral for $\Delta z(c, \mathcal{I}_n)$ given by Corollary 37 in the following way

$$\int_{\mathcal{I}_n} \frac{(1 - F(x))G_n(x)}{\sqrt{1 - G_n^2(x)}} dx = \int_{K'_{\mathcal{H}} \cap \mathcal{I}} \frac{(1 - F(x))G_n(x)}{\sqrt{1 - G_n^2(x)}} dx + \int_{(K'_{\mathcal{H}})^c \cap \mathcal{I}} \frac{(1 - F(x))G_n(x)}{\sqrt{1 - G_n^2(x)}} dx.$$

Since the first integral of the right side is finite, it is enough to focus on the second integral.

We proceed by cases: Case $(x_0)_n$ and $(x_1)_n$ are both unbounded and cases (x_0) is bounded and (x_1) is unbounded or (x_0) is unbounded and (x_1) is bounded.

Case $(x_0)_n$ and $(x_1)_n$ unbounded: by Corollary 91 we can assume that $F(x)$ is even, then the condition $\hat{G}(\tilde{x}) > \frac{1}{2}$ implies $|G_n(x)| > \frac{1}{2}$ in the travel interval $\mathcal{I}_n$ and $(1 - F(x))G_n(x)$ does not change sign in the set $\mathcal{I}_n \backslash K'_{\mathcal{H}}$, therefore

$$|\int_{\mathcal{I}_n \backslash K'_{\mathcal{H}}} \frac{(1 - F(x))G_n(x)}{\sqrt{1 - G_n^2(x)}} dx| > \frac{1}{2} \int_{\mathcal{I}_n \backslash K'_{\mathcal{H}}} |F(x)| dx \longrightarrow \infty \text{ when } n \longrightarrow \infty.$$

A similar proof follows if $(x_0)_n$ is bounded and $(x_1)_n$ is unbounded or $(x_0)_n$ is unbounded, and $(x_1)_n$ is bounded. □

§ 2 Proof of Theorem 53

For simplicity, we will prove Theorem 53 for the case $F(x) = 1 - 2x^3$. Let $c(t)$ be a $\mathbb{R}^3_F$-geodesic for $F(x) = 1 - 2x^3$ with initial point $c(0) = (1, 0, 0)$ and hill interval $[0, 1]$. Let us consider the travel interval $\mathcal{I}(x) = 2[x, 1]$, then by 34, the relation between the travel interval and the time is given by

$$2T = 2 \int_{[x,1]} \frac{dx}{\sqrt{1 - F^2(x)}}.$$

By equation 34, the change in $\Delta y(c, t)$ and $\Delta z(c, t)$ are given by

$$\Delta y(F, x) := 2 \int_{[x,1]} \frac{F(x)dx}{\sqrt{1 - F^2(x)}} \text{ and } \Delta y(F, x) = 2 \int_{[x,1]} \frac{F^2(x)dx}{\sqrt{1 - F^2(x)}}.$$

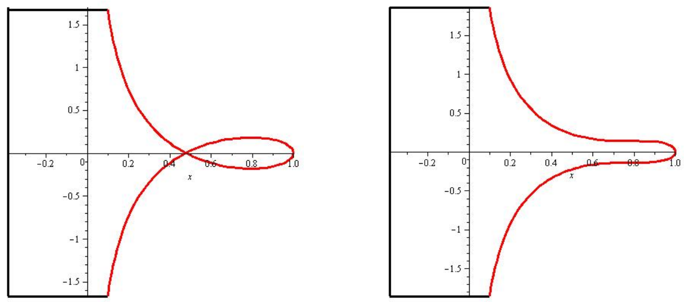

Figure 1: Both images show the projection of the geodesic $c(t)$ for $F(x) = 1 - 2x^3$ and the curve $\tilde{c}(t)$ to the (x, y) and (x, z) planes, respectively.

Therefore

$$c(-T) = (x, -\frac{\Delta y(F,t)}{2}, -\frac{\Delta z(F,t)}{2}) \text{ and } c(T) = (x, \frac{\Delta y(F,t)}{2}, \frac{\Delta z(F,t)}{2}).$$

Corollary 92. *If* $F(x) = 1 - 2x^3$, *then* $\Delta y(F,x) < \Delta z(F,x)$ *and* $\lim_{x\to 0} \frac{\Delta z(F,x)}{\Delta y(F,x)} = 1$.

Proof. If $F(x) = 1 - 2x^3$, then the same integration by parts, used in the proof of Corollary 54, shows the integral $\Delta y(F,x) - \Delta z(F,x)$ is positive. L'Hopital rules implies $\lim_{x\to 0} \frac{\Delta z(F,x)}{\Delta y(F,x)} = 1$. □

2.1 Proof of Theorem 53

Proof. Let us consider $0 < \epsilon < \frac{1}{2}$ and find a x^* such that $\Delta z(F,x) = (1+\epsilon)\Delta y(F,x)$. There exists $\delta < 0$ such that $F(\delta) = 1 + \epsilon$. If $\delta_1 = x^* + \delta$ and $\delta_2 = \delta_1 + \Delta y(F,t)$, then we define the following curve $\tilde{c}(t)$ in $\mathbb{R}^3_F$.

$$\tilde{c}(t) = \begin{cases} c(-n) + (-t, 0, 0) \text{ where } t \in [0, \delta_1] \\ c(-n) + (-\delta_1, t - \delta_1, 0) \text{ where } t \in [\delta_1, \delta_2] \\ c(-n) + (-\delta_1 + t - \delta_2, \Delta y(F,t), \Delta z(F,t)) \text{ where } t \in [\delta_2, \delta_1 + \delta_2]. \end{cases}$$

See figure 1. The by construction, $c(-T) = \tilde{c}(0)$ and $c(T) = \tilde{c}(\delta_1 + \delta_2)$, the relation between the T and $\Delta y(F, x^*)$ is given by $2T = \Delta y(F, x^*) + Cost_t(F, x^*)$, while, the relation between $\delta_1 + \delta_2$ and $\Delta y(F, x^*)$ is given by $\delta_1 + \delta_2 = 2(\delta + x^*) + \Delta y(F, x)$. If $x^* \to 0$, then $Cost_t(F, x^*) \to \Theta_1(F, [0,1]) > 0$, while, $2(\delta + x^*) \to 0$. Thus there exists an x_1 such that $Cost_t(F, x_1) > 2(\delta + x^*)$ and $\tilde{c}(t)$ is shorter that $c(t)$.

□

§ 3 The Calibration Method

Definition 93. *Let $c(t)$ be an $\mathbb{R}^3_F$-geodesic and let be $\Omega \subseteq \mathbb{R}^3_F$ a simple connected domain, we say that a functions $S : \Omega \to \mathbb{R}$ is a calibration function for $c(t)$, if $dS(\dot{c}) = 1$ and $dS(v) = (\dot{c}, v)_{\mathbb{R}^3_F}$ for all v tangent to $\mathcal{D}_F$, where $(\cdot,\cdot)_{\mathbb{R}^3_F}$ is the subRiemannian inner-product in $\mathbb{R}^3_F$.*

A classical application of a calibration one-form is the following.

Proposition 94. *Let $c(t)$ be a $\mathbb{R}^3_F$-geodesic in $\mathbb{R}^3_F$, if $S : \Omega \to \mathbb{R}$ is a calibration function for $c(t)$, then the $\mathbb{R}^3_F$-geodesic $c(t)$ is a globally minimize within Ω.*

Proof. Let S be calibration function for $c(t)$, let A and B be two points in Ω such that $c(t)$ travel from A to B with arc length $\ell(c)$. Let us assume $\tilde{c}(t)$ is a curve tangent to $\mathcal{D}_F$ and join the points A and B with arc length $\ell(\tilde{c})$, then by Stoke's theorem, the fact that S is a calibration for $c(t)$ and Cauchy-Schwarz inequality we have

$$\ell(c) = \int_c dS = \int_{\tilde{c}} dS = \int_{\tilde{c}} (\dot{c}, \dot{\tilde{c}})_{\mathbb{R}^3_F} dt \leq \int_{\tilde{c}} ||\dot{\tilde{c}}||_{\mathbb{R}^3_F} dt = \ell(\tilde{c}).$$

By Cauchy-Schwarz inequality we know $\ell(c) = \ell(\tilde{c})$ if and only if $\dot{\tilde{c}}$ is parallel to $\dot{c}$ a.e.. □

A canonical method to find a calibration function is to solve the Hamilton-Jacobi equation defined for the subRiemannian Hamiltonian function, see [11] or [12, Section 5]. In the context of the space $\mathbb{R}^3_F$ we have the following Proposition.

Proposition 95. *A calibration function $S_\pm$ for a $\mathbb{R}^3_F$-geodesic $c(t)$, generated by $G(x) = a + bF(x)$ in the pencil of $F(x)$, is given by*

$$S_\pm(x, u, z) = \pm \int^x \sqrt{1 - G^2(\tilde{x})} d\tilde{x} + ay + bz.$$

$S_\pm$ is smooth inside the region $\Omega(G) := hill(G) \cup \mathbb{R}^2$, where $hill(G) := G^{-1}([-1, 1])$ is the hill region of $G(x)$ and C^1 on the boundary of the hill region, and the abnormal curve does not cross from one connected set to another.

Proof. By Proposition , the $\mathbb{R}^3_F$-geodesic $c(t)$ has derivative

$$\dot{c}(t) = \pm\sqrt{1 - G^2(x(t))}\frac{\partial}{\partial x} + G(x(t))(\frac{\partial}{\partial y} + F(x(t))\frac{\partial}{\partial z}),$$

we notice $dS_\pm = \pm\sqrt{1 - G^2(x)}dx + ady + bdz$, then

$$dS_\pm(\dot{c}) = 1 - G^2(x) + G(x(t))(a + bF(x(t))) = 1 - G^2(x) + G^2(x(t)) = 1.$$

□

We notice calibration function provided by Proposition 95 is globally defined if and only if $G(x)$ is a constant polynomial, otherwise, it is defined in region $\Omega(G)$ of $\mathbb{R}^3_F$ and the geodesic is minimizing in the region $\Omega(G)$ until it touches its boundary. It is worth seeing how this argument looks in each of our three cases.

Recall geodesics in J^k come in three "flavors": heteroclinic, homoclinic and x-periodic. It is worth going into details around the time interval $\mathcal{T}$, the domain of the geodesic, for each of the three cases:

(x-Periodic). Choose a time origin so that $x(0) = x_0$ and $x(L/2) = x_1$. Then $\mathcal{T} = (0, L/2)$ or $(L/2, L)$ up to a period shift. The minimizing arcs correspond to half periods of the x-periodic geodesic. The domain $\Omega(G)$ projects onto the interior (a, b) of the hill interval.

(Heteroclinic.) If γ is heteroclinic then $\mathcal{T} = \mathbb{R}$ and $c : \mathbb{R} \to \Omega(G)$ is globally minimizing within $\Omega(G)$.

(Homoclinic). In this case, the x curve bounces once off the non-critical endpoint of the hill interval.

§ 4 Geodesics in Engel type

Here we will introduce the necessary tools to prove Theorems 63. Since we are interested in studying geodesics different that geodesic lines, we will be restricted to the case $\mu \neq 0$. As we said before, the dynamics are split into two cases; when $p_{\theta_{n+1}} = a_{n+1}$ equals zero or not. Let us start with the case $p_{\theta_{n+1}} = a_{n+1} = 0$.

4.1 Small oscillations

The condition $p_{\theta_{n+1}} = a_{n+1} = 0$ implies the reduced Hamiltonian H_μ from (4.1.3) has potential $\frac{1}{2}(a_0 + \sum_{i=1}^n a_i x_i)^2$. Using the translation $x_1 \to x_1 - \frac{a_0}{a_1}$, the reduced Hamiltonian is given by

$$H_\mu = \frac{1}{2}(p_x, p_x) + \frac{1}{2}(Bx, x)_{\mathcal{H}}, \tag{4.1}$$

where $(\ ,\ )_{\mathcal{H}}$ is the Euclidean product on $\mathcal{H}$, B is a n by n matrix.

Lemma 96. *For any* $(a_1, \ldots, a_n) \neq 0$*, the matrix* B *has rank one.*

Proof. The quadratic form $(Bx, x)_{\mathcal{H}}$ is the square of a lineal form $(a_1, \ldots, a_n) \cdot (x_1, \ldots, x_n)$. □

We know by linear algebra that the pair of quadratic forms (p_x, p_x) and $(Bx, x)_{\mathcal{H}}$, where the first one is positive-definite, can be reduced to principal axes by a linear change of coordinates $\tilde{x} = Qx$ and the reduced Hamiltonian H_μ, given by (4.2.2), in the new coordinates $\tilde{x}$ is the following

$$\begin{aligned} H_\mu(p_{\tilde{x}}, \tilde{x}) &= H_{line}(p_{x_1}, \cdots, p_{x_{n_1+1}}) + H_{osc}(p_{x_{n_1+1}}, \cdots, p_{x_n}, x_{n_1+1}, \ldots, x_n) \\ H_{line}\frac{1}{2} &= \sum_{i=1}^{n-1} p_{\tilde{x}_i}^2 \quad \text{and} \quad H_{osc} = \frac{1}{2} p_{\tilde{x}_n}^2 + \frac{1}{2}\lambda \tilde{x}_{n+2}^2. \end{aligned} \tag{4.2}$$

Then at $H_\mu(p_{\tilde{x}}, \tilde{x})$ has $n-1$ cycle coordinate and $n-1$ constant of motion, namely, x_i and $p_{\tilde{x}_1}$ for $1 \leq i \leq n-1$. Let us built a solution with initial point the origin. The solution of $\tilde{x}_i$ is $\tilde{x}_i = p_{\tilde{x}_i} t$ for $1 \leq i \leq n-1$, and the energy level $H_\mu(p_{\tilde{x}}, \tilde{x}) = \frac{1}{2}$ implies $H_{line}\frac{1}{2} \leq \frac{1}{2}$. Moreover, $H_{line}\frac{1}{2} = \frac{1}{2}$ implies that the corresponding geodesic is a geodesic line. In contrast, using Hamilton equations for x_n, we find that $\ddot{\tilde{x}}_n = \lambda x_n$, so $\tilde{x}_n(t) = \sqrt{2H_{osc}} \sin(\omega t)$, where $\omega = \frac{1}{\sqrt{\lambda}}$.

$$\tilde{x}(t) = (p_{\tilde{x}_i} t, \ldots, p_{\tilde{x}_i} t, \sqrt{2H_{osc}} \sin(\omega t)). \tag{4.3}$$

Corollary 97. *The solution to the Hamiltonian 4.2 is bounded in the coordinates* $\tilde{x}_n$ *and unbounded in the coordinates* $\tilde{x}_i$ *such that* $1 \leq \lambda_i \leq n-1$ *and* $p_{\tilde{x}_i}$ *is not zero.*

§ 5 Proof of Theorem 63

Prelude to Proof of Theorem 63

The following proofs rely on the method of blowing-down geodesics as explained by E. Hakavuouri and E. Le Donne, in [27]. Suppose that $\gamma : \mathbb{R} \to G$ is a rectifiable curve in a Carnot group G. For $h \in \mathbb{R}^+$ form

$$\gamma_h(t) = \delta_{\frac{1}{h}} \gamma(ht).$$

where $\delta_h : G \to G$ is the Carnot dilation. One easily checks that if γ is a geodesic then so is γ_h for any $h > 0$.

Definition 98 (blow-down)**.** *A blow-down of γ is any limit curve $\tilde{\gamma} = \lim_{k\to\infty} \gamma_{h_k}$ where $h_k \in \mathbb{R}$ is any sequence of scales tending to infinity with k, and the limit being uniform on compact sub-intervals.*

In [27], E. Hakavouri and E. Le Donne proved the following powerful lemma

Lemma 99. *If γ is globally minimizing geodesic parameterized by arc length then every blow-downs $\tilde{\gamma}$ of $\gamma(t)$ is also a globally minimizing geodesic parameterized by arc length.*

Proof of Theorem 63

Proof. The strategy of the proof is the same in both cases, we will consider a small oscillation or an r-periodic geodesic $\gamma(t)$, and we will compute one of its blow-down $\tilde{\gamma}(t) = \lim_{k\to\infty} \gamma_{h_k}$ and check $\tilde{\gamma}(t)$ is not parameterized by arc length.

Case r-periodic geodesics: Let L be the period, let us consider the sequence $h_n = nL$ and the compact interval $[0, 1]$. We compute the change undergone by the coordinate

θ_0 of the geodesic $\gamma_{nL}(t)$ after time changes by 1:

$$\Delta y(\gamma_{nL}, [0,1]) := \frac{1}{nL}\Delta y(\gamma, [0, nL]) = \frac{1}{L}\Delta y(\gamma, [0, L]) < 1. \tag{5.1}$$

Since $\Delta y(\gamma_{nL}, [0,1])$ is constant for all n. The change undergone by the coordinate after time change by 1 for the geodesic is equal to $\frac{1}{L}\Delta y(\gamma, [0,1])$. Being $\gamma_{nL}(t)$ a geodesic in $\mathrm{Eng}(n)$, there exists an μ_n in $\mathfrak{a}$ such that $\gamma_{nL}(t)$ has momentum μ_n. The relation between hill regions $h(\mu_n, \ell)$ and $hill(\mu, \ell)$, given by Definition 69, of the geodesics $\gamma_{nL}(t)$ and $\gamma(t)$ is $hill(\mu_n, \ell) = \frac{1}{nL} hill(\mu, \ell)$. Since $hill(\mu, \ell)$ is bounded, $\tilde{\gamma}(t)$ has hill region equal to $r = 0$.

Therefore, $\tilde{\gamma}(t)$ is a curve tangent to the vector field Y, $\tilde{\gamma}(t)$ is a line. Instead of being parametrized by arc-length, moving one unit along the line requires a time of $L/\Delta y(\gamma, [0, L]) > 1$ of the blow-down time. We conclude that $\tilde{\gamma}(t)$ is not parameterized by arc length.

Case small oscillations geodesics: The fact that $\gamma(t)$ is not a line implies H_{osc} is a constant different from zero. Let us consider $\gamma(t)$ the geodesic corresponding the solution $\tilde{x}(t)$ given by 4.3. We define $\gamma_n(t) = \delta_n \gamma(n\omega t)$, by construction the reduced dynamics of $\tilde{x}_n(t)$ is the following

$$\tilde{x}_n(t) = (p_{\tilde{x}_1} t, \dots, p_{\tilde{x}_{n-1}} t, \frac{\sqrt{2H_{osc}}}{n} \sin(n\omega t)). \tag{5.2}$$

If we take the limit $n \to \infty$, then $\lim_{n\to\infty} \tilde{x}_n(t) = t(p_{\tilde{x}_1}, \dots, p_{\tilde{x}_{n-1}}, 0)$. However, $||\dot{\tilde{x}}_n(0)|| = 2H_{lin} + 2H_{osc} = 1$ for all n. Therefore, $\tilde{\gamma}(t)$ is a curve tangent to $\sum_{i=1}^{n-1} p_{\tilde{x}_i} X_i$, so $\tilde{\gamma}(t)$ is a line. Rather that being parameterized by arc length, moving one unit along the line requires a time $\frac{1}{2H_{lin}}$ of the blow-down time. We conclude that $\tilde{\gamma}(t)$ is not parameterized by arc length. □

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