ISSN 0005-1179, Automation and Remote Control, 2021, Vol. 82, No. 1, pp. 73–92. © Pleiades Publishing, Ltd., 2021.
Russian Text © The Author(s), 2021, published in Avtomatika i Telemekhanika, 2021, No. 1, pp. 95–118.

ROBUST, ADAPTIVE AND NETWORK CONTROL

Control of a Mobile Robot with a Trailer Based on Nilpotent Approximation

A. A. Ardentov[1*] **and A. P. Mashtakov**[1**]

[1] *Ailamazyan Program Systems Institute, Russian Academy of Sciences, Pereslavl-Zalessky, Yaroslavl oblast, 152020 Russia*
*e-mail: * aaa@pereslavl.ru, ** alexey.mashtakov@gmail.com*

Received October 13, 2019; revised May 14, 2020; accepted July 9, 2020

Abstract—We consider a kinematic model of a mobile robot with a trailer moving on a homogeneous plane. The robot can move back and forth and make a pivot turn. For this model, we pose the following optimal control problem: transfer the "robot–trailer" system from an arbitrarily given initial configuration into an arbitrarily given final configuration so that the amount of maneuvering is minimal. By a maneuver we mean a functional that defines a trade-off between the linear and angular robot motion. Depending on the trailer–robot coupling, this problem corresponds to a two-parameter family of optimal control problems in the 4-dimensional space with a 2-dimensional control.

We propose a nilpotent approximation method for the approximate solution of the problem. A number of iterative algorithms and programs have been developed that successfully solve the posed problem in the ideal case, namely, with no state constraints. Based on these algorithms, we propose a dedicated reparking algorithm that solves a particular case of the problem where the initial and final robot position coincide and takes into account a state constraint on the trailer's turning angle occurring in real systems.

Keywords: robot with trailer, kinematic model, optimal control, nilpotent approximation, sub-Riemannian problem

DOI: 10.1134/S0005117921010057

1. INTRODUCTION

Consider a wheeled mobile robot with a trailer moving on a horizontal plane. The friction of wheels and the mass and shape of the robot and trailer are disregarded. By a robot we mean a leading wheelpair centered at the point $(x, y) \in \mathbb{R}^2_{x,y}$ with the angle $\theta \in S^1_\theta$ defining the direction of wheel motion with respect to the abscissa axis. By a trailer we mean a passive wheelpair coupled with the leading one at some point (see Fig. 1). The trailer position is defined by the angle $\varphi \in S^1_\varphi$ of its orientation relative to the robot. Thus, the position of the robot with the trailer is defined by a point $q = (x, y, \theta, \varphi)$ in the space $M = \mathbb{R}^2_{x,y} \times S^1_\theta \times S^1_\varphi$. The parameters $l_r \geq 0$, $l_t > l_r$ prescribe the distances from the coupling point to the center of the robot and to the center of the trailer, therewith defining the configuration of the robot–trailer coupling.

The kinematic model is given by a differential system arising (see [1]) from a nonholonomic constraint for the no-slip motion of the robot and trailer wheels. The present paper continues the study of this model commenced in [2]. We propose a method for approximately solving the problem of control of the "robot–trailer" system that is based on constructing a nilpotent approximation, i.e., a simpler system preserving the important properties of the original system. Such an approach was used in [3] to control a "robot with two trailers" system.

We consider the problem of parking a robot with a trailer. This problem can be formalized as a two-point boundary value control problem (see [4]) in the state space M; namely, given boundary conditions, find a trajectory $q(t) \in M$, $t \in [0, t_1]$, satisfying these conditions, admissible in the sense of nonholonomic constraints (i.e., satisfying the differential system), and minimizing a given functional defining the weighted cost of angular and linear robot displacements.

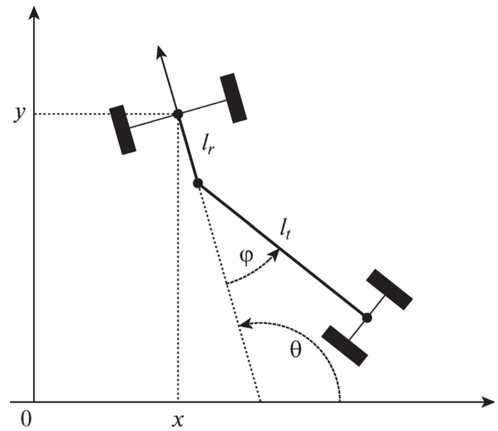

Fig. 1. Model of wheeled mobile robot with trailer and its parameters.

The problem of controlling nonholonomic systems is widely known in robotics [1]. The kinematic models of various mobile robots are described by control systems of the form

$$\dot{q} = \sum_{i=1}^{k} \mathbf{u}_i X_i(q), \quad q \in M, \ \dim M = n \geq k, \qquad \mathbf{u}_i \in \mathbb{R}, \tag{1}$$

where the X_i are smooth vector fields on the manifold M.

The solution of the problem for systems (1) of the general form remains unknown. A satisfactory solution is available only for systems of special form. The differential–geometric approach [5, 6] is one of the most efficient techniques.

The use of trigonometric controls is investigated in [7, 8] for nonholonomic systems of a certain kind, namely, for a class of systems transformable into a chain form. Because of its special form, there is a simple trigonometric control that changes a specific set of coordinates, while the other coordinates remain unchanged. Tilbury et al. [9] proposed to use trigonometric controls to move the system to the target state simultaneously along all coordinates for systems with two-dimensional control. In addition, they showed how polynomial controls can be used to achieve the target. Monaco and Norman-Cyrot [10] showed that piecewise constant controls provide an exact solution to the problem of controlling systems in chain form. The technique of chaining the system is described in [8, 11]. Note that systems in general position and, in particular, the system arising in our problem of controlling a robot with a trailer cannot be reduced to a chain form, but the nilpotentization method [12] described below generates a system in a chain form.

In exceptional cases, one can find an exact optimal control law (in the sense of the minimum of a given cost functional) for control systems. One possible approach to solving the fixed-time optimal control problem for systems linear in the control is given by the method developed in [13]. It is based on expanding the control function in a Fourier series and discarding terms above a certain order N. The solution of the new finite-dimensional problem converges to the solution of the original problem as N tends to infinity [13]. The solution thus established is said to be nearly optimal. The optimal control problem can also be solved using invariant geometric methods [6, 14] developed for solving optimal control problems on Lie groups. A numerical solution of systems simulating various types of mobile robots on the plane and in space is proposed in [15].

One class of control systems admitting exact solution is given by nilpotent systems. Recall that a control system is nilpotent if the Lie brackets of the control vector fields are zero starting from brackets of certain length. A method for controlling nilpotent systems was presented in [16]. It is based on the possibility of moving in the direction of an arbitrarily given holonomic curve (a priori not satisfying the nonholonomic constraint (1)) based on the Baker–Campbell–Hausdorff formula. This allows one to calculate admissible piecewise constant controls that transfer the nonholonomic system exactly to the final state.

Bellaïche et al. [17] developed a nilpotentization technique and later applied it to a control problem for nonholonomic systems. The paper [18] shows how to bring any control system to the canonical form corresponding to the nilpotent approximating system in a special triangular form that allows seeking trigonometric controls.

The solution of the optimal control problem presented in this paper also relies on the nilpotentization of the original system and is based on constructing an iterative process that solves the optimal control problem for the approximate system on each iteration. The ultimate control law is formed by a successive application of the controls found on each iteration. We say that such a control is *suboptimal*.

The paper is structured as follows. Section 2 gives the statement of Problem 1, namely, the optimal control problem for a robot with a trailer without state constraints. In Sec. 3, using Algorithm 1, we establish relationship between Problem 1 and a simpler problem (defined by the polynomial system (10)), the so-called nilpotent sub-Riemannian problem on the Engel group [19],[1] which provides a nonlinear approximation to the original problem. In the Theorem, we describe the change of coordinates between these problems. A hybrid Algorithm 2 is used for solving the nilpotent problem. The known optimal solutions are compared with the suboptimal trajectories found for close boundary points. Section 4 refines the procedure for constructing a suboptimal (state-unconstrained) solution to Problem 1 in the situation of general position (for remote boundary points) with the help of Algorithm 3. Section 5 deals with the software implementation of the algorithms presented. We pose Problem 2 (with a constraint on the trailer turning angle) and also provide several examples of solving this problem with a modified Algorithm 3. Section 6 considers a special case of the constrained problem, the one of reparking the trailer. To solve this problem, we developed a dedicated algorithm, which was tested on a regular grid of values for the trailer angle and the ratio of the robot arm lengths l_r and l_t. The test results are listed in Table 1. The definitions of the main terms used in the paper are collected in the Appendix.

2. STATEMENT OF THE OPTIMAL CONTROL PROBLEM

Problem 1 (no constraints). Consider the control system

$$\dot{q} = \mathbf{u}_1 X_1(q) + \mathbf{u}_2 X_2(q), \tag{2}$$

$$q = (x, y, \theta, \varphi) \in M = \mathbb{R}^2_{x,y} \times S^1_\theta \times S^1_\varphi, \quad (\mathbf{u}_1, \mathbf{u}_2) \in \mathbb{R}^2. \tag{3}$$

Here q, x, y, θ, φ, $\mathbf{u}_1$, and $\mathbf{u}_2$ by default depend on the time parameter $t \in [0, t_1]$; the controls $\mathbf{u}_1$ and $\mathbf{u}_2$ are real-valued piecewise continuous functions prescribing the linear and angular velocity of motion of the robot with center at a point (x, y); and the vector fields by the controls have the form

$$\begin{aligned} X_1(q) &= \left(\cos\theta, \sin\theta, 0, -\frac{\sin\varphi}{l_t}\right), \\ X_2(q) &= \left(0, 0, 1, -\frac{l_r\cos\varphi}{l_t} - 1\right). \end{aligned} \tag{4}$$

[1] The multiplication law in the group can be found in [20, Ch. 15].

For some given $\mu > 0$, find a curve $q(t), t \in [0, t_1]$, that has prescribed boundary values

$$q(0) = q^0 = (x^0, y^0, \theta^0, \varphi^0), \qquad q(t_1) = q^1 = (x^1, y^1, \theta^1, \varphi^1) \tag{5}$$

at two points, satisfies system (2)–(5), and minimizes the functional

$$J = \int_0^{t_1} \sqrt{\mathbf{u}_1^2(t) + \mu^2 \mathbf{u}_2^2(t)}\ \mathrm{d}\,t, \tag{6}$$

where the coefficient μ defines a trade-off between the linear and angular displacements.

Remark 1. System (2)–(4) has the following symmetry (dilation):

$$\delta_\mu : \big(x, y, \theta, \varphi, l_t, l_r, \mathbf{u}_1, \mathbf{u}_2\big) \mapsto \big(\mu x, \mu y, \theta, \varphi, \mu l_t, \mu l_r, \mu \mathbf{u}_1, \mathbf{u}_2\big).$$

Therefore, minimizing (6) is equivalent to minimizing the sub-Riemannian length [21]

$$\int_0^{t_1} \sqrt{\mathbf{u}_1^2(t) + \mathbf{u}_2^2(t)}\ \mathrm{d}\,t \tag{7}$$

with recomputed boundary values and robot–trailer coupling parameters.

Remark 2. The invariance of Problem 1 under translations and rotations in the plane $\mathbb{R}^2_{xy}$ allows fixing $q^0 = (0, 0, 0, \varphi^0)$ without loss of generality.

Let us calculate the following commutators (Lie brackets):

$$X_3(q) = [X_1, X_2](q) = \left(\sin\theta, -\cos\theta, 0, -\frac{l_r + l_t\cos\varphi}{l_t^2}\right),$$
$$X_4(q) = \big[X_1, [X_1, X_2]\big](q) = \left(0, 0, 0, -\frac{l_t + l_r\cos\varphi}{l_t^3}\right).$$

It follows from the Chow–Rashevskii theorem [14] that system (2)–(5) is completely controllable, because, in view of the original assumption $l_t > l_r \geq 0$, one has

$$\det(X_1, X_2, X_3, X_4) = -\frac{l_t + l_r\cos\varphi}{l_t^3} \neq 0. \tag{8}$$

Problem 1 corresponds to a two-parameter family of *sub-Riemannian problems*, namely, optimal control problems with system (2) linear in the control, where the minimum of the sub-Riemannian length (7) serves as the optimality criterion. Solving any such problem with specific values of the parameters l_r and l_t is an open problem. The aim of the present paper is to develop a general scheme for constructing an approximate solution for the entire family of the problems.

3. SOLUTION OF THE LOCAL PROBLEM

Local approximation to a control system by another, simpler system is widely used in control theory. For the local approximation one usually takes the linearization of the control system. However, for systems (2) linear in the control, the linearization gives an approximation that is too rough. Since the control dimension is smaller than the state dimension, the linearization cannot be completely controllable. This follows from the Rashevskii–Chow theorem [14].

In the case of (2)–(4), the linearization has the form

$$\begin{aligned}\dot{q} &= \mathbf{u}_1 X_1^0(q) + \mathbf{u}_2 X_2^0(q),\\ X_1^0(q) &= \left(1, 0, 0, -\frac{\sin\varphi^0}{l_t}\right),\\ X_2^0(q) &= \left(0, 0, 1, -1 - \frac{l_r \cos\varphi^0}{l_t}\right).\end{aligned}$$

One can readily verify that $[X_1^0, X_2^0](q) = (0, 0, 0, 0)$, and therefore, the linearization is uncontrollable.

A natural replacement for the linear approximation is delivered in this case by the nilpotent approximation, the simplest system retaining the complete controllability property. For system (2)–(3), the nilpotent approximation is given by a control system of the form

$$\dot{\tilde{q}} = \mathbf{u}_1 \widehat{X}_1(\tilde{q}) + \mathbf{u}_2 \widehat{X}_2(\tilde{q}), \quad \tilde{q} \in \mathbb{R}^4,\ (\mathbf{u}_1, \mathbf{u}_2) \in \mathbb{R}^2, \tag{9}$$

where $\widehat{X}_1$, $\widehat{X}_2$ are the vector fields of the approximating system (see the Appendix for details).

The nilpotent approximation is constructed based on system (2) written in privileged coordinates (see Definition 6 in the Appendix) so that the vector fields $\widehat{X}_i$ of the approximating system define a nilpotent Lie algebra; i.e., for some $N \in \mathbb{N}$ one has

$$\left[\widehat{X}_{i_1}, \left[\widehat{X}_{i_2}, \ldots, [\widehat{X}_{i_N}, \widehat{X}_{i_{N+1}}], \ldots\right]\right] = 0,\ \forall i_1, \ldots, i_{N+1} \in \{1, 2\}.$$

In particular, commutators of order higher than 3 ($N = 3$) are zero for system (2)–(4),

$$\left[\widehat{X}_i, \left[\widehat{X}_j, [\widehat{X}_1, \widehat{X}_2]\right]\right] = 0, \quad \forall i, j \in \{1, 2\}.$$

Remark 3. Unlike the linear approximation, the nilpotent approximation preserves such an important invariant as the growth vector (see Definition 1 in the Appendix). For system (2)–(3) in general position the growth vector at a generic point is $(2, 3, 4)$; i.e., the vector fields $X_1, X_2, X_3 = [X_1, X_2]$ and $X_4 = [X_1, X_3]$ (or $X_4 = [X_2, X_3]$) form a basis of the tangent space $T_q M$ at each point $q \in M$. In particular, this is the case for the robot–trailer system (2)–(4).

The notion of nilpotent approximation to control systems was introduced for the first time by Stefani [22] and independently developed by Agrachev and Sarychev [23] and Hermes [24]. In the present paper, we use the algorithm for calculating the nilpotent approximation proposed by Bellaïche [25], refine it for systems (2)–(3), and supplement it with the transition to a coordinate system in which the nilpotent approximation has the canonical form (10).

Remark 4. The tangent space of a sub-Riemannian manifold is a sub-Riemannian manifold itself. It can be defined as a metric space using Gromov's definition in [26]. Moreover, it has the algebraic structure of a nilpotent Lie group. The key observation is that the structure of $T_q M$ is similar to the structure of a real vector space with the group being nilpotent rather than abelian.

3.1. Calculation of the Nilpotent Approximation

The Bellaïche method [25] for constructing nilpotent approximation in privileged coordinates is refined for systems with the growth vector $(2, 3, 4)$ in the following Algorithm 1. This algorithm derives an explicit expression for the vector fields of the approximating system in the canonical privileged coordinates that allow seeking an optimal control for the nilpotent approximation.

Algorithm 1. The change of coordinates $\tau = \Phi \circ \mathcal{A}$ bringing system (2)–(3) to canonical form is constructed as follows.

1. Calculate the privileged coordinates $\tilde{q}$ in terms of the original coordinates q,
$$\mathcal{A}:\ \tilde{q} = g(q) - \frac{1}{2}\Big(0,0,0,\sigma_1\big(g_1(q)\big)^2 + 2\sigma_2 g_1(q) g_2(q) + \sigma_3\big(g_2(q)\big)^2\Big),$$
where $g(q) = (g_1(q),\dots,g_4(q)) = \Gamma^{-1}(q-q^0)$ and Γ is the 4×4 matrix with entries Γ_{ij} determined by the relation $X_j(q^0) = \sum_{i=1}^4 \Gamma_{ij}\ \frac{\partial}{\partial q_i}|_{q^0}$, all the coefficients σ_i being calculated by the formulas
$$\sigma_1 = X_1\big(X_1(g_4)\big)(q^0),\quad \sigma_2 = X_1\big(X_2(g_4)\big)(q^0),\quad \sigma_3 = X_2\big(X_2(g_4)\big)(q^0).$$
Under such a change, q^0 moves to the origin $\mathbf{0} = (0,0,0,0)$, and the vector fields X_i are taken to the fields $\tilde{X}_i = \mathcal{A}_* X_i$ that form a privileged basis.
2. Using the Maclaurin series expansion of the vector fields $\tilde{X}_i(\tilde{q})$, construct the nilpotent approximation in the coordinates $\tilde{q}$,
$$\dot{\tilde{q}} = \mathbf{u}_1\widehat{X}_1(\tilde{q}) + \mathbf{u}_2\widehat{X}_2(\tilde{q}),\quad \tilde{q}\in\mathbb{R}^4,\ (\mathbf{u}_1,\mathbf{u}_2)\in\mathbb{R}^2,$$
$$\hat{X}_i = \tilde{X}_i^1(\mathbf{0})\partial_{\tilde{q}_1} + \tilde{X}_i^2(\mathbf{0})\partial_{\tilde{q}_2} + \sum_{k=1}^2 \frac{\partial\tilde{X}_i^3}{\partial\tilde{q}_k}(\mathbf{0})\,\tilde{q}_k\partial_{\tilde{q}_3}$$
$$+\left(\frac{\partial\tilde{X}_i^4}{\partial\tilde{q}_3}(\mathbf{0})\tilde{q}_3 + \frac{1}{2}\sum_{k=1}^2 \frac{\partial^2\tilde{X}_i^4}{\partial\tilde{q}_k^2}(\mathbf{0})\tilde{q}_k^2 + \frac{\partial^2\tilde{X}_i^4}{\partial\tilde{q}_1\partial\tilde{q}_2}(\mathbf{0})\tilde{q}_1\tilde{q}_2\right)\partial_{\tilde{q}_4}.$$
3. Calculate the change of variables $\Phi:\tilde{q}\to\bar{q}:\bar{q} = e^{T_4\bar{X}_4}\circ\dots\circ e^{T_1\bar{X}_1}(\mathbf{0})$, where the parameters $T_i\geq 0$ are found from the condition $\tilde{q} = e^{T_4\hat{X}_4}\circ\dots\circ e^{T_1\hat{X}_1}(\mathbf{0})$, to pass from the privileged coordinates $\tilde{q}$ to the coordinates $\bar{q}$ in terms of which the nilpotent approximation has the canonical form
$$\begin{cases}\dot{\bar{q}}_1 = \mathbf{u}_1,\\ \dot{\bar{q}}_2 = \mathbf{u}_2,\\ \dot{\bar{q}}_3 = \frac{1}{2}(\bar{q}_1\mathbf{u}_2 - \bar{q}_2\mathbf{u}_1),\\ \dot{\bar{q}}_4 = \frac{1}{2}(\bar{q}_1^2+\bar{q}_2^2)\mathbf{u}_2,\end{cases}\qquad \bar{q}\in\mathbb{R}^4; \tag{10}$$
i.e., the vector fields $\bar{X}_1$ and $\bar{X}_2$ have the form
$$\bar{X}_1 = \left(1,0,-\frac{\bar{q}_2}{2},\frac{\bar{q}_1}{2}\right),\quad \bar{X}_2 = \left(0,1,0,\frac{\bar{q}_1^2+\bar{q}_2^2}{2}\right).$$

Theorem. *For system* (2)–(4), *the mapping* τ *has the form*
$$\begin{aligned}
\bar{q}_1 &= x,\\
\bar{q}_2 &= \theta,\\
\bar{q}_3 &= \frac{1}{2}\,x\,\theta - y,\\
\bar{q}_4 &= \Big(l_t\Big(\big(l_r^2+2\big)\theta^3 + 6\,\theta\,x\big(x-l_r\big) + 6\,y\big(2l_r - x\big) - 12l_t^2\big(\varphi-\varphi^0+\theta\big)\Big)\\
&\quad + \sin(\varphi^0)\Big(-3x\big(l_r^2\theta^2+4l_t^2\big) - 6l_rl_t^2\theta^2 + x^3\Big)\\
&\quad + \cos(\varphi^0)\Big(l_r\theta\Big(\big(l_r^2+2\big)\,\theta^2 + 2l_t^2\,\big(\theta^2-6\big) + 9x^2\Big) + 12y\big(l_t^2 - l_rx\big)\Big)\\
&\quad + 3l_t\cos(2\varphi^0)\Big(l_r^2\theta^3 + 2l_r\theta x - 2xy\Big) + 3l_t\sin(2\varphi^0)\Big(-l_r^2\theta^2 + l_r\theta\big(\theta x + 2y\big) + x^2\Big)\\
&\quad - x\sin(3\varphi^0)\Big(x^2 - 3l_r^2\theta^2\Big) + l_r\theta\cos(3\varphi^0)\Big(l_r^2\theta^2 - 3x^2\Big)\Big)\Big/\Big(12\,\big(l_r\cos(\varphi^0) + l_t\big)\Big).
\end{aligned}\tag{11}$$

Proof. Calculate the commutators

$$X_3 = \left(\sin\theta, -\cos\theta, 0, -\frac{l_r + l_t\cos\varphi}{l_t^2}\right), \quad X_4 = \left(0, 0, 0, -\frac{l_t + l_r\cos\varphi}{l_t^3}\right).$$

Note that by virtue of (8), the system of vector fields $\tilde{X}_i$ forms a basis at each point. Calculate the coefficients σ_i,

$$\sigma_1 = -\frac{l_t\cos\varphi^0\sin\varphi^0}{l_t + l_r\cos\varphi^0}, \quad \sigma_2 = \frac{l_r l_t\sin^2\varphi^0}{l_t + l_r\cos\varphi^0}, \quad \sigma_3 = l_r l_t\sin\varphi^0.$$

It is well known [28] that two arbitrary nilpotent systems with the growth vector $(2,3,4)$ are diffeomorphic. The change of variables that takes one such system to the other is constructed as follows. Let $\hat{X}_1, \hat{X}_2$ be the vector fields of the first system and $\bar{X}_1, \bar{X}_2$, the vector fields of the second system with the growth vector $(2,3,4)$. By computing the commutators $\hat{X}_3, \hat{X}_4$ and $\bar{X}_3, \bar{X}_4$, one can construct a diffeomorphism that sends the fields $\hat{X}_i$ in a neighborhood of a point $\hat{q}^0$ into the fields $\bar{X}_i$ in a neighborhood of $\bar{q}^0$,

$$\Phi : O(\hat{q}^0) \to O(\bar{q}^0), \quad \Phi_*(\hat{X}_i) = \bar{X}_i.$$

Define mappings F and G as the composition of the flows of the vector fields $\hat{X}_i$ and $\bar{X}_i$, respectively, for time T_i,

$$F(T_1,\ldots,T_4) = e^{T_4\hat{X}_4}\circ\cdots\circ e^{T_1\hat{X}_1}(\hat{q}^0),$$
$$G(T_1,\ldots,T_4) = e^{T_4\bar{X}_4}\circ\cdots\circ e^{T_1\bar{X}_1}(\bar{q}^0).$$

Then the desired diffeomorphism has the form $\Phi = G\circ F^{-1}$.

Let us apply items 2 and 3 in Algorithm 1 to the point $q^0 = (0,0,0,\varphi^0)$. Straightforward calculations and elementary simplifications lead to the change of variables (11) for the passage to the canonical privileged coordinates. ■

3.2. Search for the Roots of the Nilpotent Problem

Algorithm 1 allows finding an approximate solution to Problem 1 as a solution to the nilpotent sub-Riemannian problem on the Engel group defined by the optimal control problem for the differential system (10) in which the cost criterion is the minimum of the sub-Riemannian length functional (7). This problem has been extensively studied recently [19, 29–32]; its optimal synthesis is described in [30]. In the general case, the problem reduces to solving a four-dimensional system of algebraic equations in terms of the Jacobi elliptic functions $\mathrm{sn}, \mathrm{cn}, \mathrm{dn}$ and the elliptic integrals of the first and second kind (F and E). The left-hand side of this system is determined from a parametrization of extremal paths defined with the so-called *exponential* mapping

$$\mathrm{Exp}(u_1,u_2,k,\alpha) = \big(\bar{q}_1(u_1,u_2,k,\alpha), \bar{q}_2(u_1,u_2,k,\alpha), \bar{q}_3(u_1,u_2,k,\alpha), \bar{q}_4(u_1,u_2,k,\alpha)\big).$$

Closed-form expression can be calculated directly from the formulas given in [19]. The right-hand side of the system is determined by the terminal point $\bar{q}(t_1) = \bar{q}^1 = (\bar{q}_1^1, \bar{q}_2^1, \bar{q}_3^1, \bar{q}_4^1)$. Moreover, the dilation symmetry with respect to the parameter α present in the system allows eliminating this parameter and reducing the system to a three-dimensional one. As a result, to solve the problem in the general case of $\bar{q}_1^1\bar{q}_3^1 \neq 0$, one needs to solve a three-dimensional system of the following form:

$$\begin{cases} \dfrac{\bar{q}_2(u_1,u_2,k,1)}{\bar{q}_1(u_1,u_2,k,1)} = \dfrac{\bar{q}_2^1}{\bar{q}_1^1}, \\ \dfrac{\bar{q}_3(u_1,u_2,k,1)}{\big(\bar{q}_1(u_1,u_2,k,1)\big)^2} = \dfrac{\bar{q}_3^1}{(\bar{q}_1^1)^2}, \\ \dfrac{\bar{q}_4(u_1,u_2,k,1)}{\big(\bar{q}_1(u_1,u_2,k,1)\big)^3} = \dfrac{\bar{q}_4^1}{(\bar{q}_1^1)^3}. \end{cases} \tag{12}$$

Here the desired vector (u_1, u_2, k) is unique and lies in some subset of the bounded set $(0, \pi) \times (0, 2\pi) \times (0, 1)$. The appropriate subsets are described in detail in [30]. Various available numerical methods such as the Newton method and the chord method were used to find the root (u_1, u_2, k) of system (12) with a fixed right-hand side. Since system (12) is given by nonelementary functions, one needs initial approximations close to the desired root for the convergence of the standard methods. Using stochastic methods combined with multistart does not produce the desired effect. Therefore, we have developed the following algorithm for the approximate solution of system (12).

Algorithm 2 (hybrid). Consider the system of algebraic equations $Q(\nu) = Q^1$, which has a unique solution $\nu \in \Omega \subset \mathbb{R}^n$ for each $Q^1 \in \Xi \subset \mathbb{R}^n$. The hybrid algorithm of numerical search for the solution with a certain accuracy $\epsilon_e > 0$ and some constants $m_1, m_2 \in \mathbb{N}$ includes the following steps.

1. Define sets $\Omega_j \subset \Omega$, $j = 0, 1$, each of which consists of a discrete collection of random points ν_i^j, $i = 1, \ldots, m_1$.
2. Choose a point having the least error $d_e(\nu) = (Q(\nu) - Q^1)w(Q(\nu) - Q^1)^T$ from each set Ω_j, $j = 0, 1$, where $w \in \mathbb{R}^{n \times n}$ is the matrix of weight coefficients. Denote the respective points by $\nu_j = \arg \min\limits_{\nu_i^j \in \Omega_j} d_e(\nu_i^j)$.
3. On the selected points ν_0 and ν_1, run the chord method [33] to solve system $Q(\nu) = Q^1$. The calculations result in a point ν_2.
4. For $i = 1, \ldots, m_2 - 1$, iteratively run the Newton method [33] for the modified system $Q(\nu) = \frac{Q^1 + (m_2 - i)Q(\nu_{i+1})}{m_2 - i + 1}$ with the initial approximation $\nu = \nu_{i+1}$. The result is a point ν_{i+2}, which is supplied as the initial approximation for the next iteration. After performing $m_2 - 1$ iterations, we obtain a point ν_{m_2+1}. In this case, if the corresponding error is sufficiently small, $d_e(\nu_{m_2+1}) < \epsilon_e$, then the desired root has been found; i.e., $\nu = \nu_{m_2+1}$ is the approximate solution to the original system $Q(\nu) = Q^1$. Otherwise go to step 1 of the algorithm and repeat all steps over again until the desired accuracy is achieved.

Remark 5. Algorithm 2 was implemented in the Wolfram Mathematica programming system to solve system (12) after selecting the diagonal matrix of weight coefficients $w = \mathrm{diag}\big(\bar{q}_1^1, (\bar{q}_1^1)^2, (\bar{q}_1^1)^3\big)$. The software finds a vector $\nu = (u_1, u_2, k)$. Here two general cases $\nu \in N_1 \cup N_2$ that define formulas for the optimal control and for the corresponding curve $\bar{q}(t)$ are possible [19].

3.3. Construction of the Control for the Original Problem

We have thus proposed Algorithm 1, which permits locally reducing Problem 1 to a nilpotent sub-Riemannian problem on the Engel group. To construct an optimal control in the nilpotent problem in the general case, Algorithm 2 is used to solve the system of algebraic equations (12). The resulting root $\nu = (u_1, u_2, k) \in N_c$, $c \in \{1, 2\}$, determines the desired control as follows:

– Calculate the parameters $p_1 = F(u_1, k)$, $p_2 = F(u_2, k)$, and $\alpha = \frac{\bar{q}_1(u_1,u_2,k,1)|\bar{q}_1(u_1,u_2,k,1)|}{\bar{q}_1^1 |\bar{q}_1^1|}$, where F is the normal Legendre elliptic integral of the first kind; the parameter k determines the modulus of the elliptic integral.

– The parameters t_1 and ϕ_0 are determined using the expressions [19]

$$
\begin{aligned}
c = 1 \qquad &\Rightarrow \qquad t_1 = \frac{2p_1}{\sqrt{|\alpha|}}, \ \phi_0 = \frac{p_2 - p_1}{\sqrt{|\alpha|}}, \\
c = 2 \qquad &\Rightarrow \qquad t_1 = \frac{2kp_1}{\sqrt{|\alpha|}}, \ \phi_0 = \frac{k(p_2 - p_1)}{\sqrt{|\alpha|}}.
\end{aligned}
$$

– The optimal control is determined by the formulas

$$
\begin{aligned}
c=1 \quad &\Rightarrow \quad
\begin{aligned}
\mathbf{u}_1 &= -\frac{2k\,\mathrm{sign}\,\alpha}{\mu}\mathrm{sn}\left(\sqrt{|\alpha|}(\phi_0+t),k\right)\mathrm{dn}\left(\sqrt{|\alpha|}(\phi_0+t),k\right),\\
\mathbf{u}_2 &= -\mathrm{sign}\,\alpha\left(1-2\mathrm{dn}^2\left(\sqrt{|\alpha|}(\phi_0+t),k\right)\right);
\end{aligned}\\
c=2 \quad &\Rightarrow \quad
\begin{aligned}
\mathbf{u}_1 &= \mp\frac{2\,\mathrm{sign}\,\alpha}{\mu}\mathrm{sn}\left(\frac{\sqrt{|\alpha|}(\phi_0+t)}{k},k\right)\mathrm{cn}\left(\frac{\sqrt{|\alpha|}(\phi_0+t)}{k},k\right),\\
\mathbf{u}_2 &= -\mathrm{sign}\,\alpha\left(1-2\mathrm{cn}^2\left(\frac{\sqrt{|\alpha|}(\phi_0+t)}{k},\,k\right)\right),
\end{aligned}
\end{aligned}
\tag{13}
$$

where the functions sn, cn, and dn are essentially the elliptic sine, the elliptic cosine, and the delta amplitude.

3.4. Comparison with the Optimal Control

Problem 1 has not been solved in the general case, but there are known optimal (in the sense of the minimum of functional (7)) controls for parking a robot (without a trailer); see [34–36]. The corresponding sub-Riemannian problem is embedded in Problem 1: having taken the optimal controls $(\mathbf{u}_1(t),\mathbf{u}_2(t)), t\in[0,t_1]$ for the robot (without a trailer) and substituted them into system (2), one can numerically integrate this system for fixed values of φ^0, l_r, and l_t to obtain a path $q(t), t\in[0,t_1]$ leading to the point $q^1=q(t_1)$. Note that this path will be optimal in Problem 1 for $\mu=1$.

This class of optimal trajectories was used for comparison with suboptimal trajectories constructed in the present paper by the nilpotent approximation method. To this end, the resulting values of $q^1=(x^1,y^1,\theta^1,\varphi^1)$ were substituted into formula (11) to calculate the terminal point $\bar{q}^1$ in the nilpotent problem. Algorithm 2 was used to find the corresponding value of the root ν determining the desired control by formulas (13). Finally, the suboptimal solution to Problem 1 was calculated by integrating system (2) with the resulting control.

To compare the resulting solutions, let us introduce the following measure of closeness between the terminal point and a point on the determined suboptimal trajectory:

$$
\begin{aligned}
d\big(q(t),q^1\big) := \Big(&\big(x(t)-x^1\big)^2+\big(y(t)-y^1\big)^2+\big(\theta(t)-\theta^1\big)^2\\
&+\big(\varphi(t)-\varphi^1\big)^2+4l_r^2\sin^2\frac{\theta(t)-\theta^1}{2}+4l_t^2\sin^2\frac{\varphi(t)-\varphi^1}{2}\Big).
\end{aligned}
$$

In addition, we choose the final time T on the suboptimal trajectory,

$$
T=\arg\min_t d\big(q(t),q^1\big). \tag{14}
$$

Remark 6. When choosing the positive integer parametrization $\mathbf{u}_1^2+\mathbf{u}_2^2\equiv 1$ for $\mu=1$, we have $J=t_1$ for the optimal solution and $J=T$ for the suboptimal one.

Figures 2–4 give a comparison of the trajectories. The dashed line denotes the optimal solution; the solid line, the suboptimal one; the grey circumference of small size defines the terminal point q^1; the grey dot corresponds to the original final point on the suboptimal solution; and the black dot is the $q(T)$ calculated by formula (14). Figures 2 and 3, with a small time $t_1=2$, demonstrate a rather tight correspondence between the optimal and suboptimal solutions, but, as can be seen from Fig. 4, such correspondence is violated for a sufficiently large t_1, hence the necessity for creating a global solution algorithm.

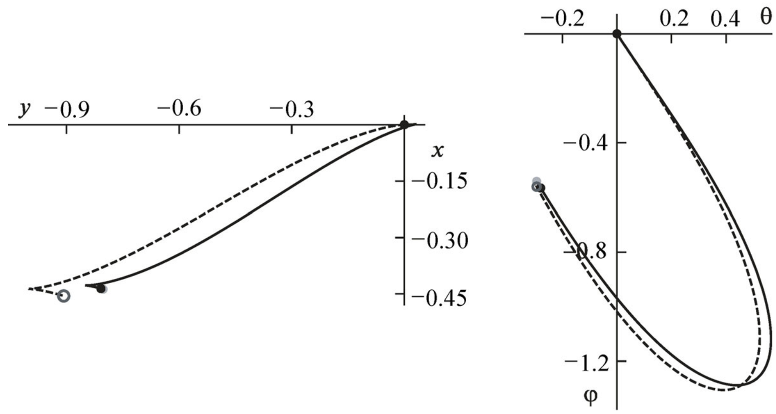

Fig. 2. $q^1 = (-0.90724; -0.45537; -0.29433; -0.5608)$, accuracy $d\big(q(T), q^1\big) = 0.1025$, for the optimal solution $J = 2$ (for the suboptimal solution $J = 1.975927$).

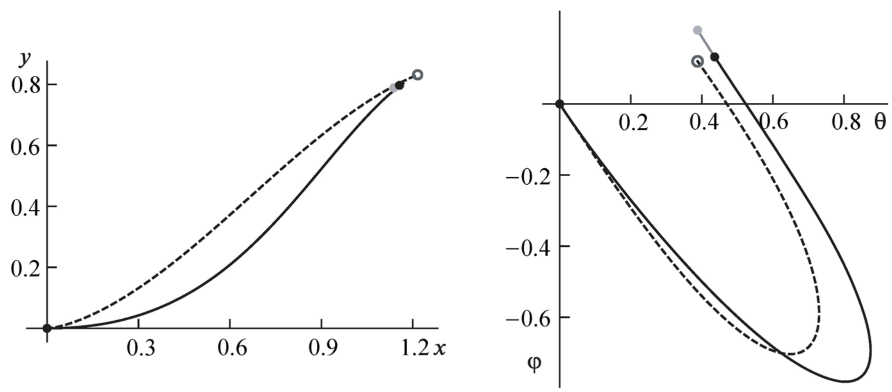

Fig. 3. $q^1 = (1.21571; 0.831468; 0.387799; 0.119918)$, accuracy $d\big(q(T), q^1\big) = 0.08905$, for the optimal solution $J = 2$ (for the suboptimal solution $J = 2.10794$).

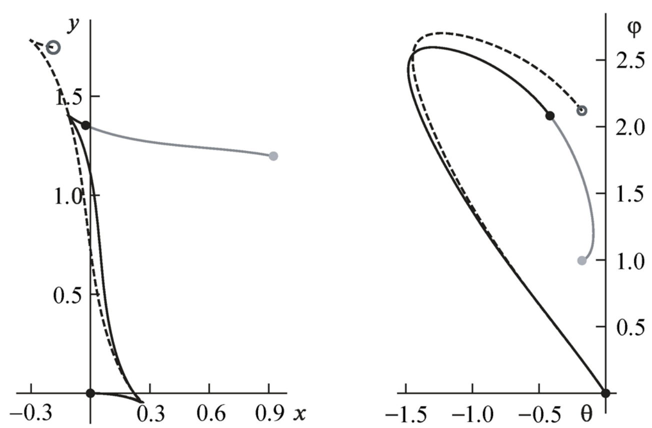

Fig. 4. $q^1 = (-0.18761; 1.74623; -0.178081; 2.12067)$, accuracy $d\big(q(T), q^1\big) = 0.50614$, for the optimal solution $J = 4$ (for the suboptimal solution $J = 3.51347$).

4. GENERAL SOLUTION SCHEME: REDUCTION TO A SERIES OF LOCAL PROBLEMS

Based on Algorithm 1, which solves Problem 1 locally, we have developed a global algorithm for the general case of remote boundary points q^0 and q^1.

Algorithm 3. Consider the control problem (2)–(5) in which the condition that the trajectory $q(t)$ hits the terminal point $q(t_1) = q^1$ has been replaced with the condition of entering some ϵ-neighborhood of the point q^1, $d\big(q(t_1), q^1\big) < \epsilon$.

Let $m_3 \in \mathbb{N}$ be some constant. Then the suboptimal control of the problem under consideration is constructed as follows.

1. Use Algorithm 1 and calculate the point $\bar{q}^1$ using formula (11) for $q = q^1$. Use Algorithm 2 to find the root ν^1 of system (12) and construct the control by formulas (13). Further, substituting it into system (2)–(5) and, numerically integrating the resulting expressions, obtain the trajectory $q_0(t), t \in [0, t_1^0]$, with the initial condition $q_0(0) = q^0$.
2. If $d_0 = d\big(q_0(t_1^0), q^1\big) < \epsilon$, then the trajectory $q_0(t)$, $t \in [0, t_1^0]$, delivers an approximate solution with the desired accuracy. Otherwise, the interval $[0, t_1^0]$ is divided into m_3 identical parts and we successively start the calculation of nilpotent approximations at the points $q_0\left(\frac{it_1^0}{m_3}\right)$, $i = 1, \ldots, m_3$, just as it has been done in the first item for the point q^0. Denote the corresponding trajectory for which the initial condition $q_i(0) = q_0\left(\frac{it_1^0}{m_3}\right)$ is satisfied by $q_i(t), t \in [0, t_1^i]$, and the corresponding distances, by $d_i = d\big(q_i(t_1^i), q^1\big)$. Then we select a number $j = \arg\min\limits_{i=0\ldots m_3} d_i$, and the procedure described in this item is now applied to the trajectory $q_j(t), t \in [0, t_1^j]$.

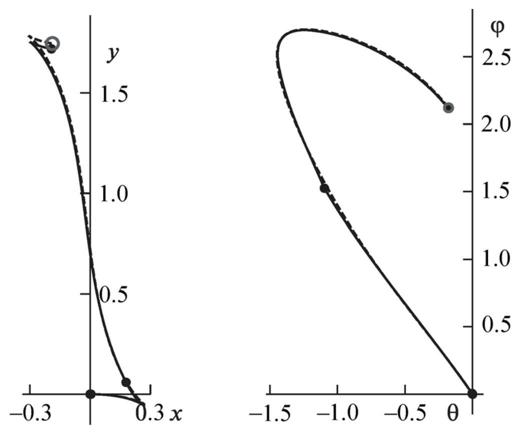

Fig. 5. $q^1 = (-0.18761; 1.74623; -0.178081; 2.12067)$, accuracy $d\big(q(t_1), q^1\big) = 0.02739$, for the optimal solution $J = 4$ (for the suboptimal solution $J = 3.9688$).

Figure 5 gives an example of refining the solution with Algorithm 3; the black dot on the trajectory corresponds to some intermediate point $q_i(0)$ at which the additional approximation has been calculated. As is seen from the figure, in this case one extra iteration suffices to produce an acceptable proximity between the optimal and suboptimal solutions.

5. SOFTWARE IMPLEMENTATION AND EXAMPLES OF PARKING ALGORITHM OPERATION

We developed software for constructing a suboptimal control for the approximate solution of Problem 1 based on nilpotent approximation in the Wolfram Mathematica environment. Proceeding from the input data l_r, l_t, φ^0, x^1, y^1, θ^1, φ^1, ϵ, the software constructs control functions $\mathbf{u}_1$, $\mathbf{u}_2$ and the corresponding robot–trailer motion path $q(t)$ issuing from the point $q^0 = (0, 0, 0, \varphi^0)$ and arriving in an ϵ-neighborhood of the point $q^1 = (x^1, y^1, \theta^1, \varphi^1)$.

The parking software was tested for various robot configurations $0 \le l_r < l_t$. The results indicate that in most tests the software constructs a control under which the jackknifing effect [37] occurs, meaning a collision of the robot and the trailer. Such solutions are inadmissible in practice, and therefore, we additionally considered the following problem, taking this natural restriction on the trailer turning angle into account.

Problem 2 (with constraints). Consider the robot–trailer system (2)–(4). The following constraint is given for trailer's turning angle:

$$|\varphi(t)| \le \varphi_{\max} < \pi. \tag{15}$$

Find a curve $q(t) = \big(x(t), y(t), \theta(t), \varphi(t)\big)$ satisfying the boundary conditions (5) and minimizing the cost functional (6).

To solve Problem 2, we used the natural modification of Algorithm 3 in which the corresponding paths $q_i(t)$, $i = 0, 1, \ldots$, are constructed on the trimmed interval $t \in [0, \tilde{t}_1^i], \tilde{t}_1^i \le t_1^i$, on which condition (15) is satisfied. In the software implementation, the parameter μ is considered to be free and its value is chosen at the start of the program so as to ensure the least error $d\big(q_0(t_1^0), q^1\big)$. Figures 6–8 present several examples of parking with the software developed. The black dashed line indicates the initial robot position; the required final position is designated with solid black color; the black line corresponds to the robot motion path constructed with the software; the grey dashed line denotes some intermediate states of the robot with a trailer along the constructed trajectory (including the terminal state). Note that Problem 1 still remains open, and the existing methods for solving the control problem (2)–(5) do not take into account the optimality criterion (6). A comparison of the determined parking paths with the trajectories obtained by other methods requires separate consideration. The known methods [37, 38] for solving the control problem (2)–(5) are mainly aimed at allowance for the phase constraints in the plane (x, y), which often determine the desired trajectory. Namely, in the plane (x, y) one seeks the so-called holonomic trajectory connecting the boundary value (5) with no allowance for nonholonomic constraints represented by the differential system (2). Then a collection of intermediate values is isolated on the holonomic trajectory and a nonholonomic trajectory connecting pairwise close intermediate values is sought with the help of various local methods. In general position, such an approach leads to a solution with a large scale of maneuver.

Next, consider the special case of Problem 2 in which the initial and final robot positions coincide.

6. TESTING TRAILER REPARKING ALGORITHM

Based on the general parking algorithm, we developed a dedicated algorithm for trailer reparking, i.e., for parking under the condition $x^1 = y^1 = \theta^1 = 0$. The final point for the nilpotent approximation of reparking is calculated in the canonical form using the formula

$$\bar{q}^1 = \left(0, 0, 0, \frac{l_t^3(\varphi^1 - \varphi^0)}{l_t + l_r \cos\varphi^1}\right).$$

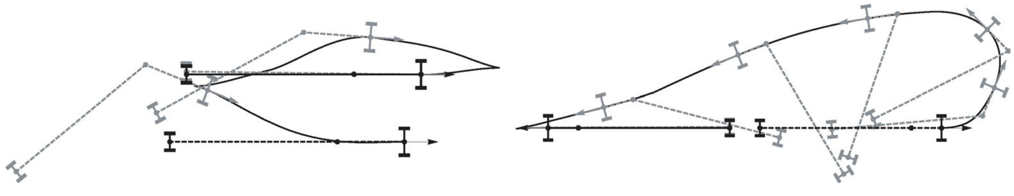

Fig. 6. Example of parking.
On the left: $(l_t, l_r) = (10, 4)$, $x^1 = 1$, $y^1 = 4$, $\theta^1 = \varphi^1 = \varphi^0 = 0$, $\varphi_{\max} = \dfrac{\pi}{2}$ with $\mu = 0.15$ and $d\big(q(t_1), q^1\big) = 0.061644$.
On the right: $(l_t, l_r) = (10, 2)$, $x^1 = -26$, $\theta^1 = \pi$, $y^1 = \varphi^1 = \varphi^0 = 0$, $\varphi_{\max} = \dfrac{3\pi}{4}$ with $\mu = 0.05$ and $d\big(q(t_1), q^1\big) = 0.0264575$.

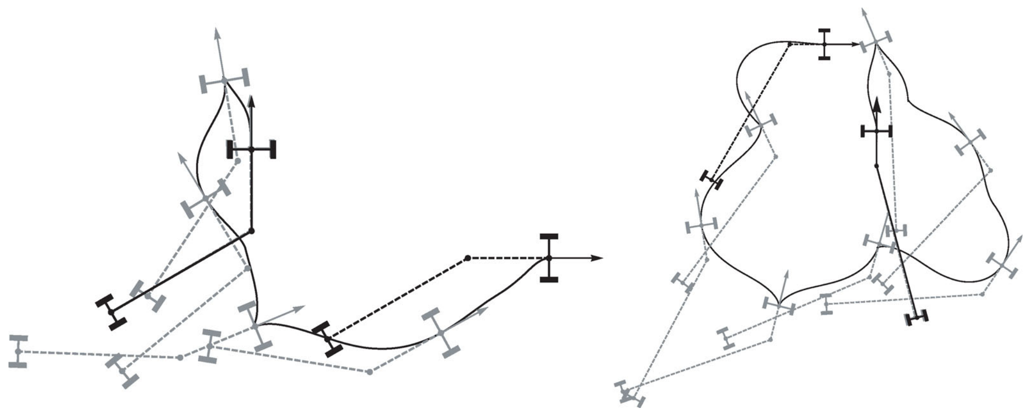

Fig. 7. Example of parking.
On the left: $(l_t, l_r) = (6, 3)$, $x^1 = -11$, $y^1 = 4$, $\theta^1 = \frac{\pi}{2}$, $\varphi^1 = -\frac{\pi}{3}$, $\varphi^0 = \frac{\pi}{6}$, $\varphi_{\max} = \frac{3\pi}{4}$ with $\mu = 0.5$ and $d\big(q(t_1), q^1\big) = 0.057446$.
On the right: $(l_t, l_r) = (9, 2)$, $x^1 = 3$, $y^1 = -5$, $\theta^1 = \pi/2$, $\varphi^1 = \frac{\pi}{12}$, $\varphi^0 = \frac{\pi}{3}$, $\varphi_{\max} = \frac{3\pi}{4}$ with $\mu = 0.512$ and $d\big(q(t_1), q^1\big) = 0.046904$.

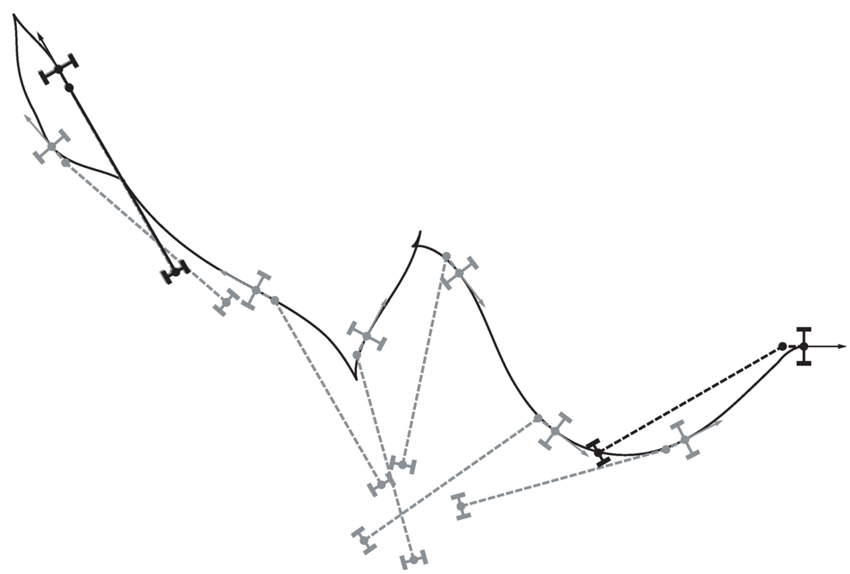

Fig. 8. Example of parking: $(l_t, l_r) = (10, 1)$, $x^1 = -35$, $y^1 = 13$, $\theta^1 = \frac{2\pi}{3}$, $\varphi^1 = 0$, $\varphi^0 = \frac{\pi}{6}$, $\varphi_{\max} = \frac{3\pi}{4}$ with $\mu = 0.5$ and $d\big(q(t_1), q^1\big) = 0.042426$.

This case is of interest in view of the fact that, for a nilpotent problem, each point of such a form is hit by a one-parameter family of optimal trajectories (as opposed to the situation of general position, in which there exists only a unique optimal solution)—an eight-curve shaped elastica (Bernoulli's lemniscate). Along with the variation in the parameter μ, in this case we have a two-parameter family of nilpotent approximations among which one can find a solution that sufficiently precisely transfers the original system from the initial position to the final one and satisfies the state constraint (15). A solution to this problem is found with Algorithm 3 in the form of the trajectory $q_0(t)$ without considering the trajectories $q_i(t)$, $i = 1, \dots, m_3$.

Note that by selecting the scale we can arbitrarily fix the value of $l_t > 0$ without loss of generality.

In the Wolfram Mathematica software system, we developed a program that constructs a reparking problem solution $q(t)$, $t \in [0, t_1]$ with accuracy $d\big(q(t_1), q^1\big) < 1/10$ and with the additional condition

$$\max_{t\in[0,t_1]} |\varphi(t)| - \varphi_{\max} < 1/10$$

on the maximum trailer angle. The program for seeking a suitable solution makes use of the standard functions ParametricNDSolveValue and FindMinimum of the software system.

We considered the following collection of tests. Let $l_t = 10$ and $l_r = 0, 1/2, 1, 3/2, \dots, 5$. Set the maximum angle $\varphi_{\max} = 3\pi/4$ as well as the trailer's angle mesh

$$\varphi^0, \varphi^1 \in \varPhi = \{\pi i/12 \mid i \in \{-6, \dots, 6\}\}, \quad \varphi^0 \neq \varphi^1.$$

Denote the trajectory that the software constructs based on the parameters l_r, φ^0, and φ^1 by

$$q_{(l_r,\varphi^0,\varphi^1)}(t), \quad t \in [0, t_1].$$

(Formally, the value of t_1 also depends on l_r, φ^0, and φ^1; this will be meant in the sequel).

Let us determine the maximum error

$$d_\varPhi(l_r) = \max_{\varphi^0,\varphi^1\in\varPhi} d\big(q_{(l_r,\varphi^0,\varphi^1)}(t_1), q^1\big),$$

as well as the maximum deviation from the maximum angle

$$\max\nolimits_\varPhi(l_r) = \max_{\varphi^0,\varphi^1\in\varPhi}\big(\max_{t\in[0,t_1]} |\varphi_{(l_r,\varphi^0,\varphi^1)}(t)| - \varphi_{\max}\big),$$

where $\varphi_{(l_r,\varphi^0,\varphi^1)}(t)$ is the corresponding component of the trajectory $q_{(l_r,\varphi^0,\varphi^1)}(t)$.

Table 1 lists the results of testing in the form of maximum error and maximum deviation from $\varphi_{\max}$.

Table 1. Testing trailer reparking

l_r	0	1/2	1	3/2	2	5/2	3	7/2	4	9/2	5
$d_\varPhi$	0.027	0.028	0.027	0.03	0.078	0.03	0.026	0.03	0.038	0.071	0.058
$\max_\varPhi$	0	0	0	0	0	0	0	0	0.059	0.085	0.038

The tests have shown that for any $l_t > 2l_r$ and $\varphi^0, \varphi^1 \in [-\pi/2, \pi/2]$ there exists a nilpotent approximation that allows transferring the robot with coupling (l_t, l_r) from the state $(0, 0, 0, \varphi^0)$ into the state $(0, 0, 0, \varphi^1)$ with an error $d_\varPhi(l_r) < 1/10$ and with the maximum absolute value of trailer's turning angle $3\pi/4 + 1/10$. For specific values of (l_t, l_r) calculated from the actual robot model, one can develop a more precise specialized reparking algorithm.

There is a known solution [34–36] of the sub-Riemannian problem describing the optimal control of a robot without a trailer. The corresponding sub-Riemannian length provides a lower bound for the sub-Riemannian length (7) in the problem considered in the present paper. At the same time, the case of reparking permits one to describe the upper bound for the sub-Riemannian length (7), a fact that will help in the future analysis of the problem in hand.

7. CONCLUSIONS

In the present paper, we have considered the problem of controlling a mobile robot with a trailer moving on a plane. The following main new results have been produced:

– An algorithm is proposed for constructing a nilpotent approximation for the "robot–trailer" system in the canonical coordinates (Algorithm 1).

– The change of variables that leads the nilpotent approximation to the canonical form (the Theorem) has been found in closed form; in this case, the change of optimal control by the nilpotent approximation to the robot–trailer system is reduced to the nilpotent problem on the Engel group.
– A hybrid algorithm and software for constructing a solution to the system of algebraic equations that determine the optimal control in the nilpotent sub-Riemannian problem on the Engel group have been developed (Algorithm 2).
– An algorithm and software for globally solving the problem of parking a robot with a trailer with no constraints on the state variables have been developed (Algorithm 3).
– Based on the algorithms developed, a dedicated algorithm and reparking software have been created that solve a particular case of the problem and take into account the state constraint on the trailer turning angle.

The optimal control in the nilpotent sub-Riemannian problem is sought using Algorithm 2 for solving the system of three equations depending on elliptic integrals of the first and second kind. It should be noted that when introducing state constraints on the trailer angle, there can be cases where the global Algorithm 3 modified for solving the constrained Problem 2 requires too many iterations. To overcome this difficulty, it looks promising to develop an algorithm that uses nilpotent approximation in the sub-Riemannian case only for trailer reparking: first, the robot with the trailer is transferred to the final position while disregarding the trailer and when moving only forward (for example, along the path of Dubins' car, consisting of a combination of circular arcs and straight-line segments [39], or along the Euler elastica [40]), and then the trailer is transferred into the required position using the reparking algorithm. In the future, we plan to compare the proposed algorithms according to the criterion (6) and test the developed software for controlling a real model of a robot in the framework of solving Problem 2 with a constraint on trailer's turning angle based on the specific model of the robot.

In this paper, it is meant that the robot has a capability of setting an arbitrary velocity of tug wheels and, accordingly, its linear and angular velocity. The mathematical model allowing for a constraint on the velocities of robot's wheels naturally leads to the notion of a sub-Finsler structure on M. The corresponding sub-Finsler problem is by definition given by system (2)–(5) with the minimization of time $t_1 \to \min$ (or the length of the path on the plane (x, y)) and restriction of control to some convex set $(\mathbf{u}_1, \mathbf{u}_2) \in \Omega \subset \mathbb{R}^2$ [41]. Note that the considered Problem 1 is equivalent to a sub-Finsler problem for $\Omega = \Omega_\mu := \{(\mathbf{u}_1, \mathbf{u}_2) \in \mathbb{R}^2 \mid \mathbf{u}_1^2 + \mu^2 \mathbf{u}_2^2 \leq 1\}$. In the sub-Finsler case, the set Ω is determined based on the particular model, for example, the case where Ω is a convex polygon (quadrangle) represents an important subclass of sub-Finsler problems in which the optimal control is, as a rule, piecewise constant.

To conclude with, note that the proposed Algorithm 1 and the Theorem can be used not only for approximating sub-Riemannian problems, i.e., problems with no constraints on control and with the minimization of the sub-Riemannian length (7), but also for a more general class of sub-Finsler problems.

APPENDIX

Let M be a smooth manifold of dimension $\dim M = n$.

Denote by T_qM the tangent space to M at a point $q \in M$.

Suppose that on M we are given a family $\mathcal{F} = \{X_1, X_2\}$ of two smooth vector fields X_1, $X_2 \in \operatorname{Vec}(M)$ satisfying the full rank conditions

$$\operatorname{Lie}_q \mathcal{F} = T_qM, \quad \forall q \in M,$$

where $\mathrm{Lie}_q\,\mathcal{F}$ denotes the Lie algebra generated by the system $\mathcal{F}$ at the point q,

$$\mathrm{Lie}_q\,\mathcal{F} = \mathrm{span}\,(X_1(q), X_2(q), [X_1, X_2](q), \ldots, [X_i, [\ldots, [X_1, X_2]\ldots]](q) \mid X_i \in \mathcal{F})\,.$$

Here the brackets designate the commutator (Lie bracket) of vector fields,

$$[X_1, X_2](q) = \frac{d}{dt}\Big|_{t=0} \left(e^{-\sqrt{t}X_2} \circ e^{-\sqrt{t}X_1} \circ e^{\sqrt{t}X_2} \circ e^{\sqrt{t}X_1}(q)\right) \in \mathrm{Vec}(M),$$

where $e^{tX_i}(q)$ stands for the flow of the vector field $X_i \in \mathrm{Vec}\,(M)$ from the point $q \in M$ in time t, i.e., the solution of the Cauchy problem $\dot\gamma(t) = X_i(\gamma(t))$, $\gamma(0) = q$.

By $\mathbf{L}^s(q)$, $s \in \mathbb{N}$, we denote the vector spaces generated by the values of the Lie brackets of the fields X_1, X_2 of length $\leq s$ at the point q (the fields X_i themselves are brackets of length 1):

$$\begin{aligned}
\mathbf{L}^1(q) &= \mathrm{span}\big(X_1(q), X_2(q)\big),\\
\mathbf{L}^2(q) &= \mathrm{span}\big(\mathbf{L}^1(q) + [\mathbf{L}_1, \mathbf{L}_1](q)\big),\\
&\ldots\ldots\ldots\ldots\ldots\\
\mathbf{L}^s(q) &= \mathrm{span}\left(\mathbf{L}^{s-1}(q) + [\mathbf{L}^1, \mathbf{L}^{s-1}](q)\right).
\end{aligned}$$

The full rank condition guarantees that for any point $q \in M$ there exists a least integer $r = r(q)$ such that $\dim \mathbf{L}^r(q) = n$. In other words, the system $\mathcal{F}$ defines a distribution in the tangent space with the flag

$$\mathbf{L}^1(q) \subseteq \mathbf{L}^2(q) \subseteq \cdots \subseteq \mathbf{L}^{r-1}(q) \subset \mathbf{L}^r(q) = T_qM. \tag{A.1}$$

Definition 1. A *growth vector* of the system $\mathcal{F}$ at a point q is the vector

$$\big(\dim \mathbf{L}^1(q), \ldots, \dim \mathbf{L}^r(q)\big).$$

Fix the dimension $\dim M = 4$ and consider the control system

$$\dot q = \mathbf{u}_1 X_1\big(q\big) + \mathbf{u}_2 X_2\big(q\big), \tag{A.2}$$

where the trajectory $q = q(t) \in M$, $t \geq 0$, is a piecewise smooth curve, the controls $\mathbf{u}_1$, $\mathbf{u}_2$ are real-valued piecewise continuous functions, and the smooth vector fields X_1, $X_2 \in \mathrm{Vec}\,(M)$ form a system with growth vector $(2, 3, 4)$,

$$\mathrm{span}(X_1(q), X_2(q), [X_1, X_2](q), [X_1, [X_1, X_2]](q)) = T_qM, \quad \forall q \in M.$$

Next, for system (A.2) we will describe the construction of the nilpotent approximation—in a certain sense, the simplest system with the growth vector $(2, 3, 4)$—whose trajectories locally approximate the trajectories of the original system. Saying "the simplest," we mean the following property: the vector fields of the approximate system form a nilpotent Lie algebra in which all Lie brackets are zeros starting from the third order. Such an approximate system is the easiest to construct in special coordinates describing the motion of the system in the directions of the kernel vector fields and their commutators, the so-called privileged coordinates. Before describing the construction itself, we introduce some definitions; see [27] for details.

Definition 2. A change of coordinates for system (A.2) is a diffeomorphism $\sigma : M \to M : q \mapsto \sigma(q)$. The differential of this change will be denoted by $\sigma_* : T_qM \to T_{\sigma(q)}M : X_i \mapsto \sigma_*(X_i)$, $i = 1, \ldots, 4$.

Definition 3. For system (A.2), the *order of the differential operator* X at the point q^0 is the minimum number $s \in \mathbb{N}$ such that for any function σ having the order $p = \min\big\{p \in \mathbb{N} \mid$

$X_{k_1}\dots X_{k_p}(\sigma)(q^0)=0,\ k_j\in\{1,2\}\big\}$, all derivatives of order $s+p$ along the fields X_1, X_2 of $X(\sigma)$ are zero at this point,

$$X_{k_1}\dots X_{k_{s+p}}X(\sigma)(q^0)=0,\quad k_j\in\{1,2\}.$$

Definition 4. The system of local coordinates $\tilde q=(\tilde q_1,\dots,\tilde q_4)$ on M with the center at a point q^0, defined by the change $\big(\tilde q_1(q),\dots,\tilde q_4(q)\big)=\big(\sigma_1(q),\dots,\sigma_4(q)\big)$, is said to be *linearly adapted* at the point q^0 if the differentials $\mathrm{d}\,\tilde q_1,\dots,\mathrm{d}\,\tilde q_4$ form a basis of $T^*_{q^0}M$ adapted to the flag $\{0\}=\mathbf{L}^0(q^0)\subset\mathbf{L}^1(q^0)\subset\mathbf{L}^2(q^0)\subset\mathbf{L}^3(q^0)$; i.e., $\mathbf{L}^i(q^0)=\operatorname{span}(\frac{\partial}{\partial\tilde q_1}|_{q^0},\dots,\frac{\partial}{\partial\tilde q_i}|_{q^0})$, $i=1,2,3$. In this case, the *order of the coordinate* $\tilde q_i$ at the point q^0 is the minimum number $p\in\mathbb{N}$ such that all derivatives of order p along the fields X_{k_j} of σ_i are zero at this point, $X_{k_1}\dots X_{k_p}(\sigma_i)(q^0)=0$, $k_j\in\{1,2\}$, where $X_{k_j}(f)=\langle\nabla f,X_{k_j}\rangle$ denotes the derivative of the function f in the direction of the field X_{k_j}, the operation $\langle,\rangle$ is the inner product, and ∇ is the operation of taking the gradient.

Definition 5. For system (A.2) written in linearly adapted coordinates $\tilde q$, the *weight* of the coordinate $\tilde q_i$ is the least number $\omega_i\in\mathbb{N}$ such that $\mathbf{L}^{\omega_i}(q^0)$ does not vanish identically.

Definition 6. The system of local coordinates $\tilde q=(\tilde q_1,\dots,\tilde q_4)$ centered at a point q^0 is called *privileged* for system (A.2) if

– $(\tilde q_1,\dots,\tilde q_4)$ are linearly adapted at the point q^0.
– The order of the coordinate $\tilde q_i$ at the point q^0 equals the weight ω_i.

Now that we have all the definitions needed, let us describe the construction of the nilpotent approximation. The nilpotent approximation for system (A.2) is constructed in the space $\mathbb{R}^4$ in the following manner:

1. System (A.2) is written in the privileged coordinates $\tilde q$,

$$\dot{\tilde q}=\mathbf{u}_1X_1(\tilde q)+\mathbf{u}_2X_2(\tilde q),\quad \tilde q\in M,\ (\mathbf{u}_1,\mathbf{u}_2)\in\mathbb{R}^2.\tag{A.3}$$

2. The vector fields $X_i(\tilde q)$ are expanded in a Maclaurin series with the subsequent grouping of terms of the same order,

$$X_i(\tilde q)=X_i^{(-1)}(\tilde q)+X_i^{(0)}(\tilde q)+X_i^{(1)}(\tilde q)+X_i^{(2)}(\tilde q)+\dots.$$

3. Terms starting with the zero order are dropped, and the remaining terms of order -1 form the kernel vector fields $\widehat X_i(\tilde q)=X_i^{(-1)}(\tilde q)$ of the approximate system—the nilpotent approximation

$$\dot{\hat q}=\mathbf{u}_1\widehat X_1(\hat q)+\mathbf{u}_2\widehat X_2(\hat q),\quad \hat q\in\mathbb{R}^4,\ (\mathbf{u}_1,\mathbf{u}_2)\in\mathbb{R}^2.\tag{A.4}$$

The nilpotent approximation (A.4) for the original system (A.3) possesses the following key properties:

1. All commutators of order ≥ 3 of the vector fields $\widehat X_1$, $\widehat X_2$ are zero.
2. The growth vector of system (A.4) is $(2,3,4)$.
3. Under the controls $\mathbf{u}_1(t)$ and $\mathbf{u}_2(t)$, the trajectory $\hat q(t)$ of system (A.4) locally (for small $t>0$) approximates the trajectory $\tilde q(t)$ of system (A.3).

Chapter 8 of Montgomery's book [21] explains the relation between the original system and its nilpotentization (nilpotent approximation): it is given by the Gromov–Mitchell theorem (8.4.1). An estimate of the proximity of paths is given in Sec. 8.7. More details on the construction of the nilpotent approximation can be found in Bellaïche's monograph [25]. Section 7 deals with estimates on distances; in particular, see Assertion 7.29 on the closeness of the trajectories of the original and approximating systems.

Remark 7. In the general case of $\dim M = n$, for the original $\tilde{q}(t) = (\tilde{q}_1(t), ..., \tilde{q}_n(t))$ and the approximating trajectory $\hat{q}(t) = (\hat{q}_1(t), ..., \hat{q}_n(t))$ in privileged coordinates issuing from one point, one has the local estimate $|\tilde{q}_i(t) - \hat{q}_i(t)| \leq Ct^{w_i+1}$, where C is a constant and w_i is the weight of the coordinate $\tilde{q}_i$ (the degree of nonholonomity in the direction $\tilde{q}_i$, which is calculated as the least depth of the flag of the distribution (A.1) that does not set to zero the ith direction).

In the present paper, for the trajectories $\tilde{q}(t)$ and $\hat{q}(t)$ of systems (A.3) and (A.4), one has the estimate

$$|\tilde{q}_i(t) - \hat{q}_i(t)| \leq Ct^{w_i+1}, \quad w = (1, 1, 2, 3),$$

where C is a constant defined by the form of the vector fields X_i and the initial point q^0.

ACKNOWLEDGMENTS

The authors are grateful to Prof. Yu.L. Sachkov for valuable remarks and support for the entire duration of this study of the problem.

FUNDING

This work was supported by the Russian Science Foundation, project no. 17-11-01387-P, and performed at the Ailamazyan Program Systems Institute of the Russian Academy of Sciences.

REFERENCES

1. Laumond, J.-P., *Nonholonomic Motion Planning for Mobile Robots. Tutorial Notes*, Toulouse: LAAS-CNRS, 1998.
2. Ardentov, A.A., Controlling of a mobile robot with a trailer and its nilpotent approximation, *Regular Chaot. Dyn.*, 2016, vol. 21, no. 7–8, pp. 775–791.
3. Mashtakov, A.P., Algorithmic and software tools for solving constructive problem of control of nonholonomic five-dimensional systems, *Program. Sist: Teoriya Prilozh.*, 2012, vol. 3, no. 1(10), pp. 3–29.
4. Krasovskii, N.N., *Teoriya upravleniya dvizheniem* (Motion Control Theory), Moscow: Nauka, 1968.
5. Chitour, Y., Jean, F., and Long, R., A global steering method for nonholonomic systems, *J. Differ. Equat.*, 2013, vol. 254, pp. 1903–1956.
6. Kushner, A.G., Lychagin, V.V., and Rubtsov, V.N., *Contact Geometry and Nonlinear Differential Equations*, Cambridge: Cambridge Univ. Press, 2007.
7. Murray, R.M. and Sastry, S., Steering nonholonomic systems using sinusoids, *IEEE Int. Conf. Decis. Control.* (1990), pp. 2097–2101.
8. Murray, R.M., Robotic control and nonholonomic motion planning, *PhD Thesis,* Memo. no. UCB/ERL M90/117, Berkeley: Univ. California, 1990.
9. Tilbury, D., Murray, R., and Sastry, S., Trajectory generation for the n-trailer problem using Goursat normal form, *IEEE TAC*, 1995, vol. 40, no. 5, pp. 802–819.
10. Monaco, S. and Norman-Cyrot, D., On Carnot–Caratheodory metrics, *J. Differ. Geom.*, 1985, vol. 21, pp. 35–45.
11. Murray, R.M., Nilpotent bases for a class on nonintegrable distributions with applications to trajectory generation for nonholonomic systems, in *Math. Control Signal Syst.*, Berkeley: Univ. California, 1990.
12. Venditelli, M., Oriolo, G., Jea, F., and Laumond, J.P., Nonhomogeneous nilpotent approximations for nonholonomic systems with singularities, *Trans. Autom. Control*, 2004, vol. 49, no. 2, pp. 261–266.
13. Fernandes, C., Gurvits, L., and Li, Z.X., A variational approach to optimal nonholonomic motion planning, *IEEE ICRA* (Sacramento, 1991), pp. 680–685.

14. Agrachev, A.A. and Sachkov, Yu.L., *Geometricheskaya teoriya upravleniya* (Geometric Control Theory), Moscow: Fizmatlit, 2005.

15. Duits, R., Meesters, S.P.L., Mirebeau, J.M., and Portegies, J.M., Optimal paths for variants of the 2D and 3D Reeds-Shepp car with applications in image analysis, *J. Math. Imaging Vision*, 2018, vol. 60, no. 6, pp. 816–848.

16. Lafferriere, G. and Sussmann, H.J., A differential geometric approach to motion planning, in *Nonholonomic Motion Planing*, Zexiang Li and Canny, J.F., Eds., 1992.

17. Bellaïche, A., Laumond, J.P., and Chyba, M., Canonical nilpotent approximation of control systems: application to nonholonomic motion planning, *32nd IEEE CDC* (1993).

18. Bellaïche, A., Laumond, J.P., and Riser, J.J., Nilpotent infinitesimal approximations to a control Lie algebra, *IFAC NCSDS* (Bordeaux, 1992), pp. 174–181.

19. Ardentov, A.A. and Sachkov, Yu.L., Extremal trajectories in a nilpotent sub-Riemannian problem on the Engel group, *Sb. Math.*, 2011, vol. 202, no. 11, pp. 1593–1615.

20. Sachkov, Yu.L., *Upravlyaemost' i simmetrii invariantnykh sistem na gruppakh Li i odnorodnykh prostranstvakh* (Controllability and Symmetries of Invariant Systems on Lie Groups and Homogeneous Spaces), Moscow: Fizmatlit, 2007.

21. Montgomery, R., *A Tour of Sub-Riemannian Geometries, Their Geodesics and Applications. Vol. 91 of Math. Surv. Monogr.*, Providence: Am. Math. Soc., 2002.

22. Stefani, G., Polynomial approximations to control systems and local controllability, *Proc. 24th. IEEE Conf. Decis. Control* (Ft. Lauderdale. Fla., 1985), pp. 33–38.

23. Agrachev, A.A. and Sarychev, A.V., Filtration of the Lie algebra of vector fields and nilpotent approximation to control systems, *Dokl. Akad. Nauk SSSR*, 1987, vol. 295, pp. 777–781.

24. Hermes, H., Nilpotent and high-order approximations of vector fields systems, *SIAM*, 1991, vol. 33, pp. 238–264.

25. Bellaïche, A., The tangent space in sub-Riemannian geometry, in *Sub-Riemannian Geometry*, Basel: Birkhäuser, 1996, pp. 1–78.

26. Gromov, M., Lafontaine, J., and Pansu, P., Structures métriques pour les variétés riemanniennes, in *Textes Mathématiques*, Paris: CEDIC/Fernand Nathan, 1981.

27. Jean, F., *Control of Nonholonomic Systems: from Sub-Riemannian Geometry to Motion Planning*, Berlin–Heidelberg: Springer, 2014.

28. Sachkov, Yu.L., Symmetries of flat rank two distributions and sub-Riemannian structures, *Trans. Am. Math. Soc.*, 2004, vol. 356, pp. 457–494.

29. Ardentov, A.A. and Sachkov, Yu.L., Conjugate points in nilpotent sub-Riemannian problem on the Engel group, *JMS*, 2013, vol. 195, no. 3, pp. 369–390.

30. Ardentov, A.A. and Sachkov, Yu.L., Cut time in sub-Riemannian problem on Engel group, *ESAIM: COCV.*, 2015, vol. 21, no. 4, pp. 958–988.

31. Ardentov, A.A. and Sachkov, Yu.L., Maxwell strata and cut locus in sub-Riemannian problem on Engel group, *RCD*, 2017, vol. 22, no. 8, pp. 909–936.

32. Ardentov, A.A. and Sachkov, Yu.L., Cut locus in the sub-Riemannian problem on Engel group, *Dokl. Math.*, 2018, vol. 97, no. 1, pp. 82–85.

33. Whittacker, E.T. and Watson, J.N., *A Course of Modern Analysis*, Cambridge: Cambridge Univ. Press, 1996. Translated under the title: *Kurs sovremennogo analiza*, Moscow: URSS, 2002.

34. Moiseev, I. and Sachkov, Yu.L., Maxwell strata in sub-Riemannian problem on the group of motions of a plane, *ESAIM: Control Optim. Calculus Var.*, 2010, vol. 16, pp. 380–399.

35. Sachkov, Yu.L., Conjugate and cut time in the sub-Riemannian problem on the group of motions of a plane, *ESAIM: Control Optim. Calculus Var.*, 2010, vol. 16, pp. 1018–1039.

36. Sachkov, Yu.L., Cut locus and optimal synthesis in the sub-Riemannian problem on the group of motions of a plane, *ESAIM: Control Optim. Calculus Var.*, 2011, vol. 17, pp. 293–321.

37. David, J. and Manivannan, P.V., Control of truck-trailer mobile robots: a survey, *Intell. Serv. Rob.*, 2014, vol. 7, no. 4, pp. 245–258.

38. Lamiraux, F., Sekhavat, S., and Laumond, J.-P., Motion planning and control for Hilare pulling a trailer, *IEEE Trans. Rob. Autom.*, 1999, vol. 15, no. 4, pp. 640–652.

39. Dubins, L.E., On curves of minimal length with a constraint on average curvature, and with prescribed initial and terminal positions and tangents, *Am. J. Math.*, 1957, vol. 79, no. 3, pp. 497–516.

40. Ardentov, A.A., Karavaev, Y.L., and Yefremov, K.S., Euler elasticas for optimal control of the motion of mobile wheeled robots: the problem of experimental realization, *RCD*, 2019, vol. 24, no. 3, pp. 312–328.

41. Lokutsievskii, L.V., Convex trigonometry with applications to sub-Finsler geometry, *Sb. Math.*, 2019, vol. 210, no. 8, pp. 1179–1205.

This paper was recommended for publication by L.B. Rapoport, a member of the Editorial Board