

Ranks and classification of infinite tensors

joint work with Andrew Snowden

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Let $V = \bigoplus_{i \in \mathbb{N}} \mathbb{C} e_i$, and let $x_i = e_i^*$. The d -th symmetric power $\text{Sym}^d(V)$ on V is

$$\bigoplus_{1 \leq i_1 \leq \dots \leq i_d} \mathbb{C} \cdot e_{i_1} \cdots e_{i_d}.$$

Then, its dual $\text{Sym}^d(V)^*$ is given by formal (possibly infinite) K -linear combinations of degree- d monomials in the variables $x_1, x_2, \dots$. Consider a homogeneous symmetric d -form f on V . The couple (V, f) is a *tensor space* with $f \in S^d(V)^*$.

Examples:

- *homogeneous degree- d polynomials,*
- $f = x_1^d + x_2^d + \dots,$
- $g = x_1 \cdots x_d + x_{d+1} \cdots x_{2d} + \dots.$

Specialisation

An endomorphism α in $\text{End}(V)$ maps each basis vector to a linear combination of the e_i . The corresponding dual map is an infinite by infinite matrix with finitely supported columns, in other words, each x_i appears in finitely many $\alpha^t(x_j)$'s.

Let $g(x_1, x_2, \dots)$ be an element of $\text{Sym}^d(V)^*$. Then, the right action of α on $\text{Sym}^d(V)^*$ is given by:

$$\alpha \cdot g(x_1, x_2, \dots) = g(\alpha^t(x_1), \alpha^t(x_2), \dots).$$

Definition:

Let f and g be in $\text{Sym}^d(V)^*$. Then, we say that g specialises to f , and write $f \preceq g$, if there exists an $\alpha \in \text{End}(V)$ such that $\alpha \cdot g = f$. We say that f and g are isogenous if $f \preceq g$ and $g \preceq f$.

Example:

$f = x_1^d + x_2^d + \dots$ and $g = x_1 \cdots x_d + x_{d+1} \cdots x_{2d} + \dots$ are isogenous.

Isogeny classes

Example: (Degree one, $d = 1$)

Any nonzero linear form in $\mathbf{V}^$ can specialise to any other linear form. Hence, there are two isogeny classes.*

Example: (Degree two, $d = 2$)

Any quadratic form is isogenous (via the Gram–Schmidt process) to

$$x_1^2 + x_2^2 + \cdots + x_k^2,$$

or

$$x_1^2 + x_2^2 + \cdots.$$

The set of isogeny classes is isomorphic to $\mathbb{N}_{\geq 0} \cup \{\infty\}$.

Example: (Degree three, $d = 3$)

Isogeny classes of polynomials correspond to isomorphism classes. Not useful.

Strength

Definition:

The strength of a homogeneous degree- d polynomial f is the minimal number r such that there exists forms $g_1, h_1, \dots, g_r, h_r$ with degree strictly less than d such that $f = \sum_{i=1}^r g_i h_i$.

- $x_1^2 + \dots + x_k^2$ has strength $\lceil \frac{k}{2} \rceil$,
- any linear form has infinite strength.

Definition:

The collective strength of homogeneous degree- d forms $f_1, \dots, f_s$ is the minimal strength of any nontrivial linear combination.

- $x_1^2 + x_2^2 + \dots$ has infinite strength,
- $x_1^2 + x_2^2 + \dots$ and $x_2^2 + \dots$ have collective strength 1.

Proposition:

Let $f_1, \dots, f_r$ be forms of infinite collective strength, and consider scalars $c_1, \dots, c_r$. There there exists a $v \in V$ such that $f_i(v) = c_i$.

Well-order on infinite-strength cubic forms

Let $\mathcal{X}_3$ be the set of isogeny classes of infinite-strength cubic forms.

Theorem: (Bik-D.-Snowden)

There is an isomorphism of well-ordered sets between $\mathcal{X}_3$ and $\mathbb{N}_{\geq 0} \cup \{\infty\}$.

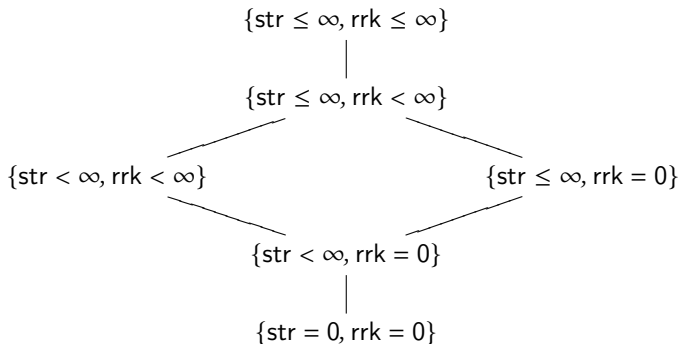
This classification depends on the dimension of the span of the derivatives in the quotient space $S^2(V)^*/S^2(V^*)$.

- ① $\sum_{i \geq 1}^{\infty} x_i(y_{i,1}^2 + y_{i,2}^2 + \cdots) + \sum_{i \geq 1} z_i^3,$
- ② $\sum_{i \geq 1}^k x_i(y_{i,1}^2 + y_{i,2}^2 + \cdots) + \sum_{i \geq 1} z_i^3,$
- ③ $\sum_{i \geq 1} z_i^3.$

For an infinite strength cubic use the previous proposition to specialize it coefficient by coefficient to one of the above three.

Subrepresentations

The $\text{End}(V)$ -subrepresentations of $S^3(V)^*$ are:



Take away:

The $\text{End}(V)$ -representations of $S^3(V)^$ are of finite length.*

For higher degrees:

- Description of the subrepresentations.
- Description of the isogeny classes in finite terms.

Basic representations for higher degrees, I

Let S_k^d be the collection of forms in $S^d(V)^*$ using only variables whose index is bigger than k . For a subset E of $S^d(V)^*$ let $\Pi(E)$ be the set of elements of the form $\sum_i f_i$ with $f_i \in E \cap S_i^d$.

Example:

If E are the cubic polynomials, then $x_1^3 + x_2^3 + \dots$ is an element of $\Pi(E)$.

Let Λ be the set of formal expressions with

- 1 $1 = [1] \in \Lambda$,
- 2 if $\lambda \in \Lambda$ then $[\lambda] \in \Lambda$ and $[[\lambda]] = [\lambda]$,
- 3 if $\lambda_1, \lambda_2 \in \Lambda$ then $\lambda_1 \cdot \lambda_2 \in \Lambda$ (with $n + m = n \cdot m$).

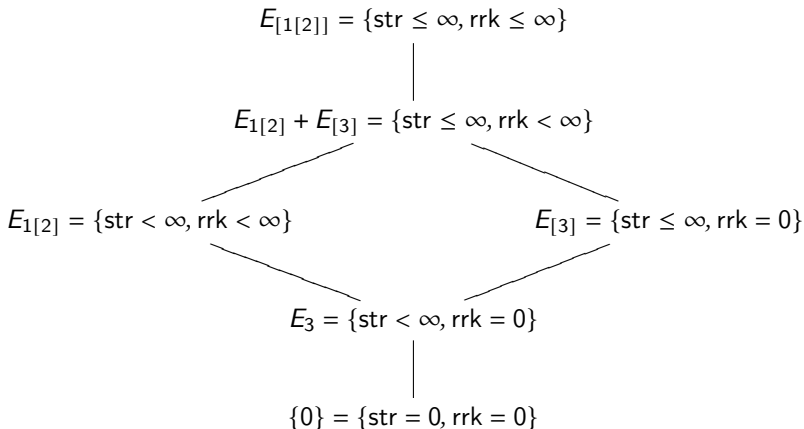
We do not consider the expressions of shape $[[\cdot] \cdots [\cdot]]$.

$$\Lambda = \{1, 2, [2],$$

$$1 = \bullet, \quad 2 = \bullet \sqcup \bullet, \quad [2] = \begin{array}{c} \circ \\ \swarrow \quad \searrow \\ \bullet \quad \bullet \end{array}, \quad 1[2] = \bullet \sqcup \begin{array}{c} \circ \\ \swarrow \quad \searrow \\ \bullet \quad \bullet \end{array}, \quad [1[2]] = \begin{array}{c} \circ \\ \swarrow \quad \searrow \\ \bullet \quad \circ \\ \quad \swarrow \quad \searrow \\ \quad \bullet \quad \bullet \end{array}.$$

Basic representations for higher degrees, II

Set $E_1 = S^1$, For $d = 3$, these *basic* representations are $E_3, E_{[3]}, E_{1[2]}, E_{[1[2]]}$ as follows:



The λ -rank

Consider a form f in E_λ with $\lambda = [1[\nu]]$. Consider the quotient of $E_{[\nu]}$ by all its proper subrepresentations $E_{[\nu]}^\circ$.

Definition:

The λ -rank of f is the dimension of the span of the derivatives in the quotient space $E_{[\nu]}/E_{[\nu]}^\circ$. For other λ , the λ -rank is the supremum of the strength of $\partial_\nu f$ modulo proper subrepresentations.

An element of infinite λ -rank generates E_λ , and any homogeneous form belongs to the sum of basic representations.

- An infinite rank element in $E_{[1[1[\dots 1[2]]]]}$ generates S^d .
- The sum of cubes generates $E_{[3]}$.

Definition:

The collective λ -rank is the natural generalization of the collective strength.

Having high λ -rank means that the forms and all their derivatives behave as independent variables.

Description of isogeny classes for higher degree

A theorem by Erman, Sam, and Snowden states that $R = \bigoplus_d S^d(V)^*$ is abstractly a polynomial ring, and it is generated by elements of infinite collective strength elements in $S^d(V)^*$.

A refinement states the elements $g_{\lambda,i}$ of infinite collective λ -rank in E_λ for any $\lambda \in \Lambda^\circ = \{[\square] \mid \square \in \Lambda\}$ generate R as a $\mathbb{C}$ -algebra and they are algebraically independent.

We then show that isogeny classes are equivalent to isomorphisms classes in the graded polynomial ring $S = \mathbb{C}[x_{\lambda,i} \mid \lambda \in \Lambda^\circ, i \in I] = \bigoplus_d S_d$ with respect to $\mathbb{C}$ -algebra morphisms $\phi : S \rightarrow S$ such that any monomial $x_{\lambda_1,i_1} \cdots x_{\lambda_k,i_k}$ appearing in $\phi(x_{\lambda,i})$ satisfies $E_{\lambda_1 \cdots \lambda_k} \subset E_\lambda$.

Indeed, we can write a form f as a polynomial in the $g_{\lambda,i}$, and there is a natural surjective map from R to S mapping $g_{\lambda,i}$ to $x_{\lambda,i}$

Finiteness for infinite tensor

In this talk we focused on symmetric forms and used

- Strength: finite and universal forms, independence of forms,
- λ -rank: classification of isogeny classes, independence of derivatives,
- $\text{End}(V)$ action: representation theory for infinite forms.

Moreover,

- generalization of the results of this talk to arbitrary Schur functors (work in progress),
- $\text{Aut}(V, f)$ representation theory: a λ -space V is linearly ω -categorical if and only if $\text{Aut}(V)$ is linearly oligomorphic. True for biquadratic spaces length two.

Thank you for the attention!