

PUBLIC SUMMARY OF THE THESIS

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Infinity is both a curiosity and a need. Zeno's paradox— *in a race, the quickest runner can never overtake the slowest, since the pursuer must first reach the point whence the pursued started, so that the slower must always hold a lead*— sounds like a curiosity, but its solution via finding criteria for the convergence of a series is a need of endless uses.

Infinite dimensional geometry. In our research, we address geometric objects that are defined by polynomials, named "*varieties*". Examples of varieties are curves, and surfaces (the line, the parabola, or the plane from high school). One associates to a variety its dimension: curves have dimension one, surfaces dimension two, and, leaving behind the comfort of three dimensions, we can study varieties of greater dimensions, where the mathematical formalism becomes our closest friend. Even if we cannot picture varieties of dimension, say, 15, we are still anchored in the bay of the finite dimensional world; but we can challenge our understanding further: can we think about a variety of infinite dimension? How does it look like? How do we study it?

Analogue. The analogue with numbers expressed in decimals will lead our intuition. Let's consider the numbers $1/7$ and $\sqrt{2}$. The former is rational while the latter isn't, and this is reflected in their decimal expressions. Indeed, $1/7 = 0,142857142857 \dots$ where 142857 keeps appearing infinitely many times, and it is called the period. On the other side,

$$\sqrt{2} = 1,4142135623730950488016887242096980785696718753769480731766797379 \dots$$

and there is no period: the numbers after the comma never repeat, they do not show any recurrence. The point here is that, even if both of these numbers have infinitely many nonzero digits, the first is somehow easier as it "*stabilises*": we can record a finite number of digits (142857, the period), and reconstruct the remainder from those (by translating the period over six digits again and again). We cannot do this with the second.

Jump to infinite dimensional varieties. We think about an infinite dimensional variety as a *limit* of finite dimensional ones: we construct, or *approximate* our variety step-by-step, first shaping it in dimension one, then in dimension two, then three, and so on, exactly how the numbers $1/7$ and $\sqrt{2}$ are approximated by considering at each step more and more digits.

In the same way as $1/7$ is intrinsically different that $\sqrt{2}$, not all the infinite dimensional varieties are equal, and we look at varieties that *stabilise* when the dimension tends to infinity— sort of what happens for the rational numbers with the period. However, in this case we need to exploit the geometric symmetries to let stabilisation appear. A geometric object can be moved around— in a mathematical sense— and the movements sending each point of our variety to other points in that same variety are called *symmetries*. For example, if we rotate a square clockwise over 90° around its centre (intersection of the diagonals), all points have changed their positions but the square occupies the same spot as before. We are interested in considering infinite dimensional varieties that stabilise *up to symmetry*, in the sense that the infinite data of the variety can be recovered from a finite amount of data *and* our knowledge of the symmetries. Stunningly enough, this can be done, and finding criteria for stabilisation depending on the varieties and the amount of symmetries is the first and main step in our research.

Cool, but what is it for? Our work is theoretical but it is rooted in applications, particularly in the fields of theoretical computer science and algebraic statistics, where many algebraic models come in infinite families. Stabilisation of the limit of such a family implies theorems that hold for all members of the family. In a nutshell, taking symmetries into account reduces the complexity of an object and makes it (more) manageable both by our minds and by a computer.