

Well-order on infinite-strength cubic forms

Joint work with A. Bik and A. Snowden

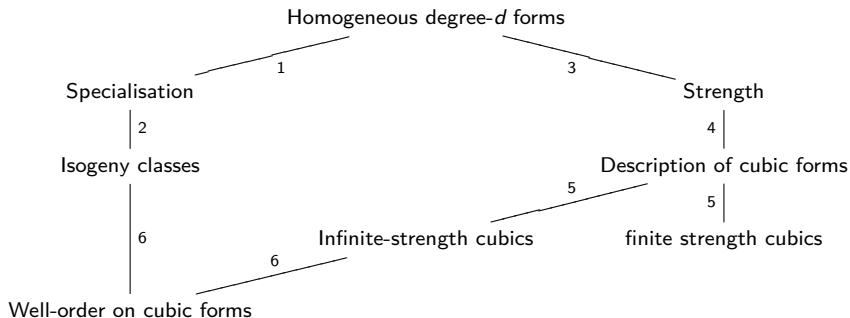
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Introduction



Take away:

Forms of infinite strength have good behaviour.

Players

We work over an algebraically closed field K of characteristic zero, say, over $\mathbb{C}$, the complex numbers.

Let $\mathbf{V} = \bigoplus_i K \cdot e_i$ be a countable dimensional vector space over a field K with basis $\{e_i\}_{i=1}^\infty$, and let x_i be the dual vector to e_i .

The d -th symmetric power $\text{Sym}^d(\mathbf{V})$ on $\mathbf{V}$ is

$$\bigoplus_{1 \leq i_1 \leq \dots \leq i_d} K \cdot e_{i_1} \cdots e_{i_d}.$$

Then, its dual $\text{Sym}^d(\mathbf{V})^*$ is given by formal (possibly infinite) K -linear combinations of degree- d monomials in the variables $x_1, x_2, \dots$.

Examples:

- *homogeneous degree- d polynomials,*
- $f = x_1^d + x_2^d + \dots,$
- $g = x_1 \cdots x_d + x_{d+1} \cdots x_{2d} + \dots.$

Specialisation

Any $\alpha \in \text{End}(\mathbf{V})$, induces a map $\alpha^t : \mathbf{V}^* \rightarrow \mathbf{V}^*$ mapping each x_i to a formal linear combination of x_j 's. Each x_i appears in finitely many $\alpha^t(x_j)$'s.

Let $g(x_1, x_2, \dots)$ be an element of $\text{Sym}^d(\mathbf{V})^*$. Then, the right action of α on $\text{Sym}^d(\mathbf{V})^*$ is given by:

$$\alpha \cdot g(x_1, x_2, \dots) = g(\alpha^t(x_1), \alpha^t(x_2), \dots).$$

Definition:

Let f and g be in $\text{Sym}^d(\mathbf{V})^$. Then, we say that g specialises to f , and write $f \leq g$, if there exists an $\alpha \in \text{End}(\mathbf{V})$ such that $\alpha \cdot g = f$. We say that f and g are isogenous if $f \leq g$ and $g \leq f$.*

Example:

$f = x_1^d + x_2^d + \dots$ and $g = x_1 \cdots x_d + x_{d+1} \cdots x_{2d} + \dots$ are isogenous.

Isogeny classes

Example: (Degree one, $d = 1$)

Any nonzero linear form in $\mathbf{V}^$ can specialise to any other linear form. Hence, there are two isogeny classes.*

Example: (Degree two, $d = 2$)

Any quadratic form is isogenous (via the Gram–Schmidt process) to

$$x_1^2 + x_2^2 + \cdots + x_k^2,$$

or

$$x_1^2 + x_2^2 + \cdots.$$

The set of isogeny classes is isomorphic to $\mathbb{N}_{\geq 0} \cup \{\infty\}$.

Example: (Degree three, $d = 3$)

Isogeny classes of polynomials correspond to isomorphism classes, and here the isomorphism relation doesn't say much about the structure.

Strength

Definition:

Let f be in $\text{Sym}^d(\mathbf{V})^*$. Suppose that for some positive integer r one has

$$f = \sum_{i=1}^r g_i h_i,$$

where g_i, h_i are homogeneous forms such that $\deg(g_i), \deg(h_i) < \deg(f)$.

Then, we say that f has strength r , and write $\text{str}(f) = r$, if r is the minimal integer for which such an expression exists. Otherwise, we say that f has infinite strength, and write $\text{str}(f) = \infty$.

Examples:

- ① Linear forms have infinite strength;
- ② $\text{str}(x_1^2 + x_2^2) = 1$ as $x_1^2 + x_2^2 = (x_1 + ix_2)(x_1 - ix_2)$,
- ③ and one can prove that $\text{str}(x_1^2 + x_2^2 + \cdots + x_k^2) = \lceil \frac{k}{2} \rceil$,
- ④ hence, if $f = x_1^2 + x_2^2 + \cdots$, then $\text{str}(f) = \infty$.

Informal description of $\text{Sym}^3(\mathbf{V})^*$

Residual rank

Let f be a cubic form in $\text{Sym}^3(\mathbf{V})^*$ in the variables $x_1, x_2, \dots$. Then, $\frac{\partial f}{\partial x_i}$ is a quadratic form.

Denote by F_2 the set of finite strength quadratic forms, and consider the space $\text{Sym}^2(\mathbf{V})^*/F_2$.

Definition:

The residual rank of f , denoted by $\text{rrk}(f)$, is the dimension of the span of the partial derivatives $\frac{\partial f}{\partial x_i}$ in $\text{Sym}^2(\mathbf{V})^/F_2$.*

Examples:

- $f = x_1^3 + x_2^3 + \dots$ has residual rank zero, $\text{rrk}(f) = 0$.
- $g = x_1(x_{r+1}^2 + x_{2(r+1)}^2 + \dots) + \dots + x_r(x_{(r+1)+(r-1)}^2 + x_{2(r+1)+(r-1)}^2 + \dots) + (x_{(r+1)+r}^3 + x_{2(r+1)+r}^3 + \dots)$ has residual rank r , $\text{rrk}(g) = r$.

Well-order on infinite-strength cubic forms

Denote with $\mathcal{X}_3$ the set of isogeny classes of infinite-strength cubic forms, and denote with $[f]$ the isogeny class of the cubic form f .

Theorem: (Bik-D.-Snowden)

The map:

$$\mathrm{rrk} : \mathcal{X}_3 \rightarrow \mathbb{N}_{\geq 0} \cup \{\infty\},$$

sending a class $[f]$ to $\mathrm{rrk}(f)$ is an isomorphism of well-ordered sets.

The idea is the following. Let f be an infinite strength cubic. We have the following cases:

- if $\mathrm{rrk}(f) = r$, then $f \sim f_r$ where

$$f_r := \sum_{i=1}^r x_i (x_{r+1+(i-1)}^2 + x_{2(r+1)+(i-1)}^2 + \cdots) + \left(\sum_{j>0} x_{j(r+1)+r}^3 \right),$$

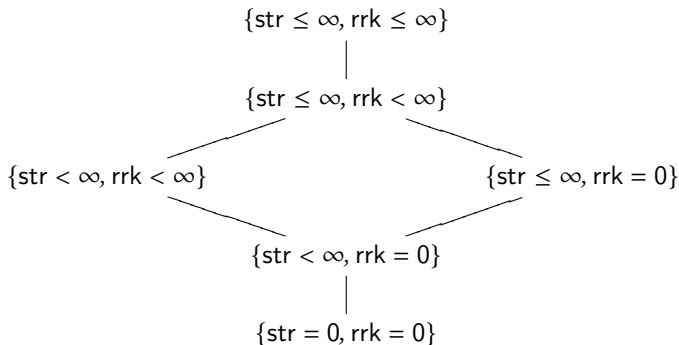
- if $\mathrm{rrk}(f) = \infty$, then $f \sim f_\infty$ where, after re-indexing variables,

$$f_\infty := \sum_{i=1}^{\infty} x_{i0} (x_{i1}^2 + x_{i2}^2 + \cdots).$$

Informal description of $\text{Sym}^3(\mathbf{V})^*$

Subrepresentations

The $\text{End}(\mathbf{V})$ -subrepresentations of $\text{Sym}^3(\mathbf{V})^*$ are:



Take away:

The $\text{End}(\mathbf{V})$ -representations of $\text{Sym}^3(\mathbf{V})^$ are of finite length.*

What we know:

- Draisma's Theorem: GL -stable closed subsets of $\mathrm{Sym}^d(\mathbf{V})^*$ satisfy the descending chain condition.
- Blatter-Draisma-Rupniewski: homogeneous polynomials over finite fields are well-quasi-ordered under specialisation.

Let $\mathcal{X}_d$ be the set of isogeny classes of homogeneous degree- d forms of infinite strength. What we can hope for is:

- (Conjecture I) $\mathcal{X}_d$ satisfies the descending chain condition.
- (Conjecture II) $\mathcal{X}_d$ over finite fields is well-quasi-ordered.
- (Conjecture III) $\mathrm{End}(\mathbf{V})$ -representations of $\mathrm{Sym}^d(\mathbf{V})^*$ are of finite length.

The end

Thanks for the attention!