

INFINITE TENSORS MEET MODEL THEORY

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Introduction

Infinite tensors arise from families of varieties defined independently of the dimension and stable under the general linear group – for instance secants of Segre–Veronese varieties, used in theoretical computer science to study the (border) rank of tensors.



Tensor spaces generalize polynomials, and their theory bridges algebraic geometry, representation theory and model theory.

Example

Field k of characteristic zero, V vector space, $\dim V = n$, and $\text{Form}^d(V)$ = forms of degree d in $x_1, \dots, x_n$.

$X_1(V) \subset \text{Form}^d(V)$: the reducible forms $f = gh$ with $\deg g, \deg h > 0$.

Key: $X_1(V)$ is defined independently of $\dim V$, and is $\text{GL}(V)$ -stable.



To treat all dimensions at once, set $\mathbf{V} = \bigcup_{i \geq 0} k^i$ and study $X_1(\mathbf{V}) \subset \text{Form}^d(\mathbf{V})$.

Strength: X_r = sums of r reducible tensors = tensors of *strength* (or *Smith rank*) at most r ; widely used in algebraic combinatorics and number theory.

New: forms with infinitely many monomials, e.g. $x_1^d + x_2^d + x_3^d + \dots$.

Object of investigation

A *tensor space* $(V, f_1, \dots, f_r)$ is a vector space V with homogeneous forms $f_1, \dots, f_r$ of degrees $d_1, \dots, d_r$; an *infinite tensor* is a multilinear map on $\mathbf{V}$. We treat symmetric tensors only, but the theory works for any Schur functor.

Structures on tensor spaces

Embeddability: an embedding $(V, f_1, \dots, f_r) \rightarrow (W, g_1, \dots, g_r)$ is a linear map $\phi : V \rightarrow W$ with $f_i = g_i \circ \phi$.

Tensor spaces are partially ordered by embeddability.



Universal tensor spaces exist: every tensor space supported on a finite dimensional subspace embeds in one.

Example

$(k^n, x_1^2 + \dots + x_n^2)$ embeds in $(\mathbf{V}, x_1^2 + x_2^2 + \dots)$, which is universal.

Isogeny: mutually embeddable tensor spaces form an *isogeny class*.

Embeddability well-quasi-orders the isogeny classes of universal tensor spaces; a maximal class exists.

Cubic spaces

The isogeny classes of $(\mathbf{V}, f)$ with $\deg f = 3$ are indexed by $r \in \{0, 1, 2, \dots\} \cup \{\infty\}$. For r finite a representative is

$$f_r = \sum_{i=1}^r x_i(x_{r+i}^2 + x_{2r+1+i}^2 + \dots) + \sum_{j>0} x_{j(r+1)+r}^3,$$

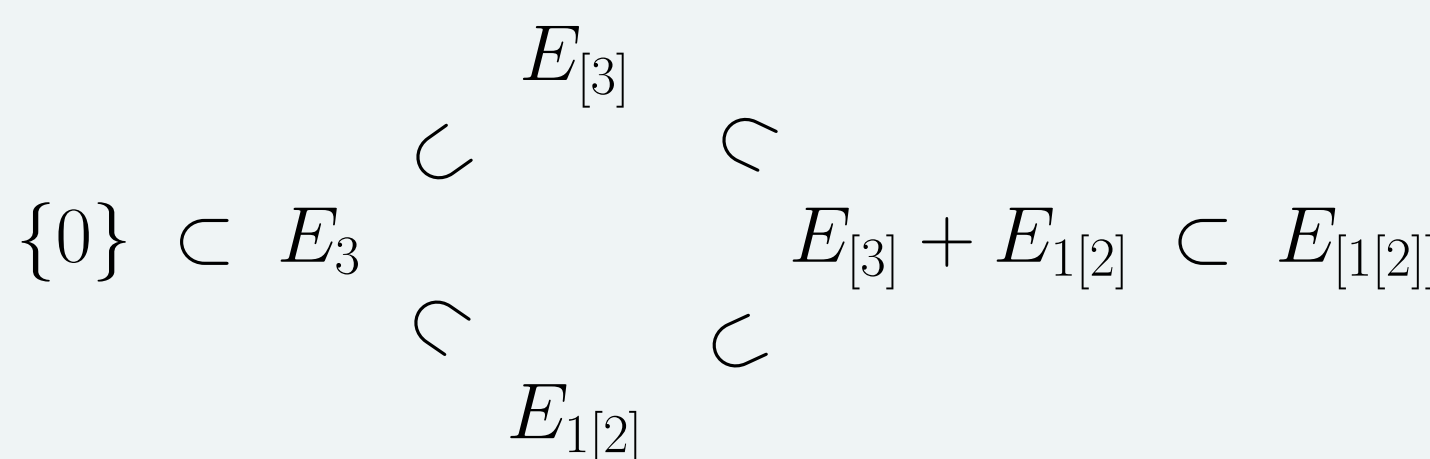
and the maximal class is generated by

$$f_\infty = \sum_{i \geq 1} x_{i,0}(x_{i,1}^2 + x_{i,2}^2 + \dots).$$

Representation theory I: $(\mathbf{V}, f_1, \dots, f_r)$ spans an $\text{End}(\mathbf{V})$ -representation inside $\text{Form}^{d_1}(\mathbf{V}) \oplus \dots \oplus \text{Form}^{d_r}(\mathbf{V})$.

These $\text{End}(\mathbf{V})$ -representations have finite length.

$\text{End}(\mathbf{V})$ -representations of cubic forms



Poset of cubic $\text{End}(\mathbf{V})$ -subrepresentations

E_3 = the polynomials

$E_{[3]}$ = $\langle x_1^3 + x_2^3 + \dots \rangle$

$E_{[1[2]]}$ = $\langle x(y_1^2 + y_2^2 + \dots) \rangle$

$E_{[1[2]]}$ = $\langle f_\infty \rangle$

$\langle \cdot \rangle$ = generated as an $\text{End}(\mathbf{V})$ -representation.

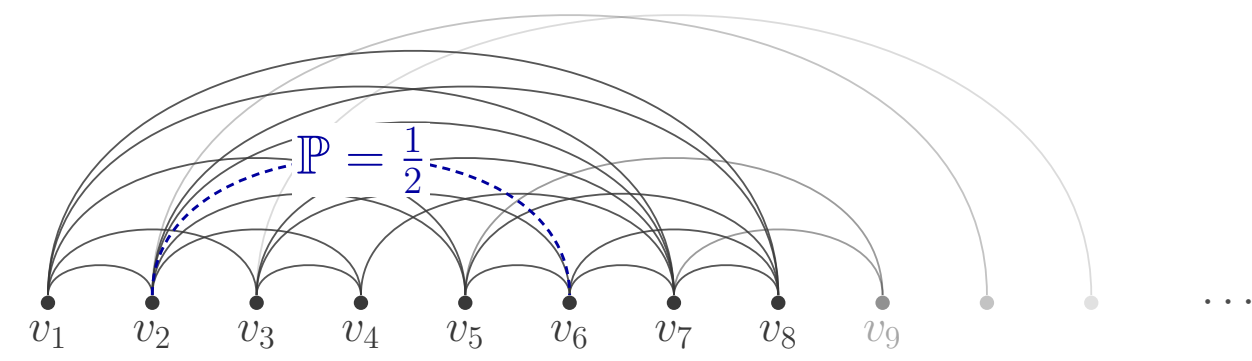
Parallel notions

Homogeneity = highly symmetric

Universality = finite structures embed into an infinite one

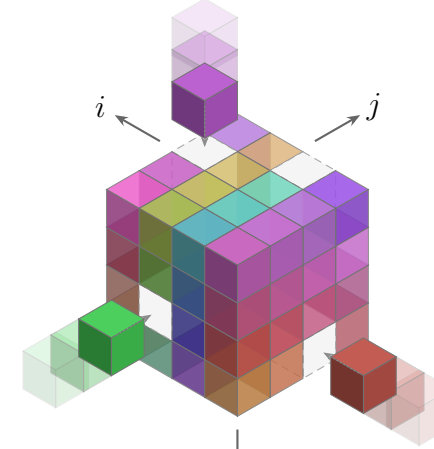
Fraïssé categories = categories admitting a universal homogeneous limiting object

Model theory



The Rado graph Γ above satisfies: 1) every finite graph embeds into it (*universality*), and 2) every isomorphism between two finite subgraphs lifts to an automorphism of Γ (*homogeneity*).

Tensor spaces



distinct colors = coefficients of $F = \sum_{i,j,l \geq 1} c_{ijk} x_i y_j z_l$ are algebraically independent over k

There exists, unique up to isomorphism, a universal tensor space that is *homogeneous*: any isomorphism of tensor spaces supported on finite dimensional vector spaces lifts to an automorphism of $\mathbf{V}$.

Oligomorphicity = finiteness up to symmetry

ω -categoricity = determined uniquely by simple statements

A permutation group G acting on an infinite set X is *oligomorphic* if there are finitely many orbits on X^n for every n (G acts diagonally).

A group $G \subset \text{GL}(\mathbf{V})$ acts *linearly oligomorphically* on $\mathbf{V}$ if for any n there is a subspace E of dimension $d(n)$ such that any n -dimensional $W \subset \mathbf{V}$ embeds into E via G .

A countable model X is *ω -categorical* if every other countable model with the same theory is isomorphic to X .

A countable tensor space is *linearly ω -categorical* if any other countable tensor space with the same theory is isomorphic to it.

Linear Ryll-Nardzewski Statement

The group $\text{Aut}(\mathbf{V})$ of invertible linear maps $\phi : \mathbf{V} \rightarrow \mathbf{V}$ such that $f_i = f_i \circ \phi$ turns $\mathbf{V}$ into an $\text{Aut}(\mathbf{V})$ -representation.

Ryll-Nardzewski Theorem. *Let X be a countable model with automorphism group G . Then G is oligomorphic if and only if X is ω -categorical.*

Conjecture. *Let $(\mathbf{V}, f_1, \dots, f_r)$ be a tensor space. Then $G = \text{Aut}(\mathbf{V})$ is linearly oligomorphic if and only if $(\mathbf{V}, f_1, \dots, f_r)$ is linearly ω -categorical.*

Harman–Snowden: the universal homogeneous space is linearly oligomorphic and linearly ω -categorical.

The biquadratic case

Theorem. *Universal biquadratic space $(\mathbf{V}, q_1, q_2)$ with $G = \text{Aut}(\mathbf{V}, q_1, q_2)$, and $\mathbf{V}$ a G -representation of length 2*



- $(\mathbf{V}, q_1, q_2)$ is isomorphic to one of the seven cases below, and linearly ω -categorical,
- G is linearly oligomorphic.

	q_1	q_2		q_1	q_2		q_1	q_2
Ia	$\begin{pmatrix} \mathbb{I} & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & \mathbb{I} \end{pmatrix}$		IIa	$\begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}$		$\begin{pmatrix} 0 & 0 \\ 0 & \mathbb{I} \end{pmatrix}$	
Ib	$\begin{pmatrix} \mathbb{I} & 0 \\ 0 & A \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & \mathbb{I} \end{pmatrix}$		IIb	$\begin{pmatrix} \mathbb{I} & A \\ A & 0 \end{pmatrix}$		$\begin{pmatrix} 0 & 0 \\ 0 & \mathbb{I} \end{pmatrix}$	III $\begin{pmatrix} A & B \\ B & D \end{pmatrix} \begin{pmatrix} C & A \\ A & B \end{pmatrix}$
Ic	$\begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & \mathbb{I} \end{pmatrix}$		IIc	$\begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix}$		$\begin{pmatrix} 0 & 0 \\ 0 & \mathbb{I} \end{pmatrix}$	

A, B, C, D infinite symmetric; the columns of $\mathbb{I}, A, B, C, D$ linearly independent.

Conclusion

Tensor spaces have a tame behaviour once the symmetry is taken into account.

Bibliography

- A. Bik, A. Danelon, J. Draisma, R. H. Eggermont, *Universality of high-strength tensors*, Vietnam J. Math. **50** (2022), 557–580.
- A. Bik, A. Danelon, A. Snowden, *Isogeny classes of cubic spaces*, Selecta Math. (N.S.) **31** (2025), no. 4.
- A. Danelon, A. Snowden, *Biquadratic spaces of length two*, arXiv:2412.20681.
- N. Harman, A. Snowden, *Ultrahomogeneous tensor spaces*, Adv. Math. **443** (2024).

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