FUNCTORIALITY AND THE THETA CORRESPONDENCE

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ABSTRACT. We study the functoriality of the local theta correspondence for classical p-adic groups. This is realized via the adaptation of the Adams conjecture to ABV-packets. We provide evidence for the conjecture, especially in the case of general linear groups.

1. Introduction

The theta correspondence was introduced by Howe ([36]) and has since proven to be a powerful tool within the Langlands program. However, despite its undeniable efficacy, it has historically been difficult to fit the theta correspondence into the theoretical framework of the Langlands program. Indeed, one early such attempt was by Langlands in a letter to Howe in 1975 ([40]), where Langlands speculated if the theta correspondence was an instance of what is now called Langlands functoriality.

However, history has shown that the theta correspondence is not an instance of Langlands functoriality. Indeed, there are examples where the local theta correspondence does not preserve L-packets. Nevertheless, it remains desirable to pin down the theta correspondence within the framework of the Langlands program. In 1989, Adams proposed what is now known as the Adams conjecture: the local theta correspondence should preserve local Arthur packets ([2]). Before stating the Adams conjecture, we introduce some notation.

Let F be a p-adic field and W_F denote the Weil group. Let G be a classical group which is quasi-split over F. We let $G = \mathrm{G}(F)$ and denote the L-group by ${}^LG = \widehat{\mathrm{G}}(\mathbb{C}) \rtimes W_F$. Roughly, a local Arthur parameter is a homomorphism (see §2.2)

$$\psi : W_F \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \to {}^LG.$$

To a local Arthur parameter ψ, Arthur and Mok attached a local Arthur packet denoted by Π_ψ ([5, 49]). This is a finite set of irreducible unitary representations of G. Furthermore, to ψ, one can associate an L-parameter ϕ_ψ such that the associated L-packet Π_{ϕ_ψ} is contained in Π_ψ.

Let H be a classical group which is quasi-split over F and forms a reductive dual pair with G. For an irreducible admissible representation π of G, we

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let $\theta(\pi)$ denote image of π under the local theta correspondence (see §2.1). Then $\theta(\pi)$ is an irreducible admissible representation (possibly vanishing) of $H = \mathrm{H}(F)$. We call $\theta(\pi)$ the theta lift of π.

With the above notation, we now state the Adams conjecture as follows (see Conjecture 2.4 for a precise statement).

Conjecture 1.1 (The Adams Conjecture ([2])). *Suppose that $\pi \in \Pi_\psi$ for some local Arthur parameter ψ of G. Then the $\theta(\pi) \in \Pi_{\psi'}$, where ψ' is an explicit local Arthur parameter of H which only depends on ψ.*

It has turned out that Adams was mostly correct. Mœglin showed that the Adams conjecture is largely true ([47, Theorem 6.1]). Recently, for symplectic-even orthogonal dual pairs (although analogous results are expected more generally, see Conjecture 2.6), Bakić and Hanzer developed a way to determine precisely the validity of the Adams conjecture for representations in a fixed local Arthur packet ([10]). In essence, these works completely determine when the Adams conjecture holds when the local Arthur packet Π_ψ is fixed.

However, Mœglin exhibited examples where the Adams conjecture fails ([47]). The failure can essentially be broken into two kinds.

(1) First, the theta lift of a representation could lie in a local Arthur packet, but not the one predicted by the Adams conjecture.
(2) Second, the theta lift of a representation may not lie in any local Arthur packet.

The first failure has a hope to be resolved. Indeed, this failure was studied by the author in [29] for symplectic-even orthogonal dual pairs (again, the results are expected to hold more generally, see Conjecture 2.6). Specifically, from [31, 32] (see Conjecture 2.3), there exists a specific local Arthur parameter $\psi^{\max}(\pi)$ such that $\pi \in \Pi_{\psi^{\max}(\pi)}$. For this local Arthur parameter, [29, Theorem 1.5] states that if $\theta(\pi) \in \Pi_{\psi'}$, then $\theta(\pi) \in \Pi_{\psi^{\max}(\pi)'}$, i.e., the Adams conjecture will hold in its greatest generality for $\psi^{\max}(\pi)$. Furthermore, [29, Conjecture 1.6] (see Conjecture 2.8) essentially says that if the the Adams conjecture fails for $\psi^{\max}(\pi)$, then it must be for the second reason: $\theta(\pi)$ does not lie in any local Arthur packet.

The second failure of the Adams conjecture is more critical. Indeed, the Adams conjecture only concerns local Arthur packets, while the second failure is when the theta lift does not lie in any local Arthur packet. This forces us to consider a (conjectural, see Conjecture 3.5) generalization of local Arthur packets known as ABV-packets. These packets were originally defined for real groups by Adams, Barbasch and Vogan ([3]). For connected reductive groups defined over a p-adic field, we follow a formulation given in [17]. We will only focus on the p-adic case in this article. ABV-packets are parameterized by L-parameters and consist of a certain finite set of irreducible admissible representations. To an L-parameter ϕ, we let Π_ϕ^{ABV}

denote the corresponding ABV-packet. We conjecture that the Adams conjecture has an analogue for ABV-packets (see Conjecture 3.6 for a precise statement).

Conjecture 1.2 (The Adams Conjecture for ABV-packets). *Suppose that $\pi \in \Pi_{\phi}^{\mathrm{ABV}}$ for some L-parameter ϕ of G. Then the $\theta(\pi) \in \Pi_{\phi'}^{\mathrm{ABV}}$, where ϕ' is an explicit L-parameter of H which only depends on ϕ.*

Given a local Arthur parameter ψ, we attach an L-parameter ϕ_ψ (see (2.5)). When $\phi = \phi_\psi$ for some local Arthur parameter ψ, it is conjectured that $\Pi_{\phi}^{\mathrm{ABV}} = \Pi_\psi$ (see Conjecture 3.5). Furthermore, we have that $\phi' = \phi_{\psi'}$. In this sense Conjecture 1.2 is expected to be the generalization of Conjecture 1.1.

Of course, since Conjecture 1.1 does fail, Conjecture 1.2 also does fail. However, every representation lies in some ABV-packet and so the only possible failure is of the first kind, i.e., $\theta(\pi)$ lies in some ABV-packet, but not $\Pi_{\phi'}^{\mathrm{ABV}}$. For symplectic-even-orthogonal dual pairs, the resolution for this failure was to consider a specific local Arthur parameter $\psi^{\max}(\pi)$. This has a natural analogue for ABV-packets, namely the L-parameter ϕ_π of π. Our first piece of evidence for Conjecture 1.2 is to verify it for ϕ_π (Lemma 3.12).

Lemma 1.3. *If H is the "going-down" tower (see §2.1) for π, then $\theta(\pi) \in \Pi_{(\phi_\pi)'}^{\mathrm{ABV}}$.*

This follows from the computation of the L-parameter of $\theta(\pi)$ ([6, 9]) and the fact that the L-packet Π_ϕ is contained in $\Pi_{\phi}^{\mathrm{ABV}}$ ([17, Proposition 7.13(b)], see Proposition 3.3).

Our second piece of evidence is that we establish the analogue of Mœglin's result ([47, Theorem 6.1]) for ABV-packets of general linear groups (which are dual pairs of type II). That is, let $G = \mathrm{GL}_n(F)$ and $H = \mathrm{GL}_m(F)$. For an irreducible admissible representation π of $\mathrm{GL}_n(F)$, we let $\theta(\pi)$ be the irreducible admissible representation of $\mathrm{GL}_m(F)$ which is the image of π under the local theta correspondence (see Theorem 4.1). We verify Conjecture 1.2 when $m \gg n$ (Theorem 4.34).

Theorem 1.4. *Suppose that $\pi \in \Pi_{\phi}^{\mathrm{ABV}}$ for some L-parameter ϕ of $\mathrm{GL}_n(F)$. If $m \gg n$, then $\theta(\pi) \in \Pi_{\phi'}^{\mathrm{ABV}}$.*

We remark that ABV-packets for $\mathrm{GL}_n(F)$ are not necessarily singletons. Indeed, it was demonstrated in [16] that there is an ABV-packet of $\mathrm{GL}_{16}(F)$ of size 2. The existence non-singleton ABV-packets presents the primary obstacle in the proof of Theorem 1.4. As an application of Theorem 1.4, we obtain that there are ABV-packets of $\mathrm{GL}_n(F)$ of size at least 2 where $n = 16, 18, 20$ or $n \geq 21$ (Corollary 4.35).

In [3, Theorem 25.8], a geometric analogue of endoscopy (for real groups) is given through the use of a fixed point formula. In [20, Proposition 3.2], an analogue of this fixed point formula is established for local Arthur parameters of $\mathrm{GL}_n(F)$. However, for our setting, we require an extension of this result to certain L-parameters, not necessarily of Arthur type. To establish a fixed point formula, one needs to relate the regular parts of certain conormal bundles (we defer to §4.3 for the terminology and precise meanings). This amounts to studying the intersections of the closures of conormal bundles which is generally a difficult problem. Instead, we relate these intersections to the intersections of closures of certain conormal bundles in a sub-Vogan variety (Corollary 4.22). From this result, we are able to extract the fixed point formula (Theorem 4.33) in our setting.

The fixed point formula then shows that $\theta(\pi) \in \Pi_{\phi'}^{\mathrm{ABV}}$ if and only if $\pi^\vee \in \Pi_{\phi^\vee}^{\mathrm{ABV}}$, where $\pi^\vee$ and $\phi^\vee$ are the contragredient of π and ϕ. In general, it is expected that ABV-packets are preserved by the contragredient. We verify this for $\mathrm{GL}_n(F)$ (Lemma 4.11) from which Theorem 1.4 follows.

Lemma 1.5. *Let π be an irreducible admissible representation of $\mathrm{GL}_n(F)$ and ϕ be an L-parameter of $\mathrm{GL}_n(F)$. We have $\pi \in \Pi_{\phi}^{\mathrm{ABV}}$ if and only if $\pi^\vee \in \Pi_{\phi^\vee}^{\mathrm{ABV}}$.*

For a general reductive dual pair (G, H), we expect a similar argument to yield an analogue of Theorem 1.4, but there are complications that need to be resolved. For example, we made use of the fact that L-packets of $\mathrm{GL}_n(F)$ are singletons. In general, this is not the case and so one needs to keep track of the enhanced L-parameter of the representations. This is done in [6, 9], but it will need to be translated into Vogan's perspective on the local Langlands correspondence ([55]; see also §3). Another complication is that we made use of the Mœglin-Waldspurger algorithm ([46, Theoreme II.13]) to compute the Pyatetskii dual of L-parameters of $\mathrm{GL}_n(F)$, (e.g., see Lemma 4.29). This algorithm needs to be generalized for other groups. This will be accomplished in a forthcoming joint work with Lo ([34]). A third problem arises if H is the "going-up" tower for π. In this case, the L-parameter of $\theta(\pi)$ is not necessarily ϕ'_π. This is a consequence of [6, Theorem 4.5] (and [10, Theorem 6.8]; see also [51, p. 558]). All of these issues (and more) would appear in adapting our approach to Theorem 1.4 to general reductive dual pairs.

Finally, we make some remarks on why it is desirable to have the Adams conjecture for ABV-packets.

(1) Langlands originally conjectured that the local theta correspondence was an instance of Langlands functoriality ([40]). This conjecture encompasses all irreducible admissible representations, not just those of Arthur type. By passing from local Arthur packets to ABV-packets, the Adams conjecture now applies to any irreducible admissible representation and is hence closer to Langlands' original conjecture.
(2) As remarked earlier, the Adams conjecture for local Arthur packets can and does fail. The critical failure was when the theta lift was not of Arthur type. By considering ABV-packets, we are able to resolve this failure.

(3) The Adams conjecture for local Arthur packets fits into the framework of the relative Langlands program ([11]). In this theory, the Adams conjecture is predicted to be the "dual problem" to the Gan-Gross-Prasad conjectures ([21, 22], see [27, Remark 7.12]). As conjectural generalizations of local Arthur packets, it is natural to ask if ABV-packets may play a role in the relative Langlands program. Having the Adams conjecture for ABV-packets would be suggestive of a positive answer to this question. It would be very interesting if there were an analogue of the Gan-Gross-Prasad conjectures for ABV-packets.

(4) It is an open problem to determine when the theta lift of a unitary representation is also unitary. In the stable range this is known to be true ([41]), but remains open more generally. The failure of the Adams conjecture for local Arthur packets at a specific local Arthur parameter conjecturally determines a lower bound for this problem (see Remark 2.9). Determining this bound currently remains mysterious; however, we suspect that the Adams conjecture for ABV-packets may play a role in this (see Example 3.14).

Here is the outline of this article. In §2, we recall the local theta correspondence, local Arthur packets, and the Adams conjecture for local Arthur packets precisely (Conjectures 2.4 and 2.6). In §3, we recall the definition of ABV-packets and discuss the Adams conjecture for ABV-packets for type I dual pairs (Conjectures 3.6 and 3.7). In §4, we discuss the Adams conjecture for ABV-packets for type II dual pairs (Conjectures 4.4 and 4.5). We provide more detail in this situation and carry out the above strategy to prove Theorem 1.4. Finally, in Appendix A, we carry out the proof of the fixed point formula, Theorem 4.33.

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2. Background

Let F be a non-Archimedean local field of characteristic 0 and $q = q_F$ be the cardinality of the residual field. We set $|\cdot|$ to be the normalized p-adic absolute value on F. By abuse of notation, we also set $|\cdot|$ to the composition of the p-adic absolute value with the determinant. For a set S acted upon by a group H, we let $Z_H(S)$ denote the centralizer of S in H. When $S = \{s\}$ is a singleton, we simply write $Z_H(S) = Z_H(s)$.

We let E be a field such that $[E : F] \leq 2$ and $c \in \mathrm{Gal}(E/F)$ be a generator. We fix a nontrivial additive character ψ of F and let ψ_E be the additive character of E defined by $\psi_E = \psi \circ \mathrm{tr}_{E/F}$.

Let $\epsilon \in \{\pm 1\}$, W_n be a ϵ-Hermitian space of dimension n over E, and V_m be an $-\epsilon$-Hermitian space of dimension m over E. We let $\langle \cdot, \cdot \rangle_W : W \times W \to$

E and $\langle\cdot,\cdot\rangle_V : V \times V \to E$ denote the ϵ-Hermitian and $-\epsilon$-Hermitian forms of W and V, respectively. We set

$$\epsilon_0 = \begin{cases} -\epsilon & \text{if } E = F, \\ 0 & \text{otherwise.} \end{cases}$$

The isometry group of W_n and V_m are denoted by $G = G(W_n)$ and $H = H(V_m)$ respectively (except in the below case). For example, when $\epsilon = -1$, $E = F$, and n and m are even, G is a symplectic group and H is an even orthogonal group. The exceptions are when $E = F$, $\epsilon = -1$, n is odd, and m is odd, we set $G = \mathrm{Mp}(W_n)$ and $H = \mathrm{SO}(V_m)$. Similarly, when $E = F$, $\epsilon = 1$, n is odd, and m is even, we set $G = \mathrm{SO}(W_n)$ and $H = \mathrm{Mp}(V_m)$.

Let $\mathbb{H}$ denote a hyperbolic plane. Any ϵ-Hermitian space W_n has a Witt decomposition

$$W_n = W_{n_0} + W_{r,r}, \tag{2.1}$$

where $n = n_0 + 2r$, W_{n_0} is anisotropic and $W_{r,r} \cong \mathbb{H}^r$. The isomorphism class of W_n uniquely determines the Witt index r and the space W_{n_0}. Fix an anisotropic ϵ-Hermitian space W_{n_0}. Then we associate a Witt tower to W_{n_0} as follows:

$$\mathcal{W} = \{W_{n_0} + W_{r,r} \mid r \geq 0\}. \tag{2.2}$$

Similarly, we associate a Witt tower to an anisotropic $-\epsilon$-Hermitian space V_{n_0} via

$$\mathcal{V} = \{V_{n_0} + V_{r,r} \mid r \geq 0\}. \tag{2.3}$$

For brevity, we often write $G' = G, H$. We let $\Pi(G')$ be the set of equivalence classes of irreducible admissible representations of G'.

2.1. **Theta Correspondence.** Recall that we fixed an additive character ψ on F. The pair (G, H) forms a reductive dual pair of a certain metaplectic group. We fix a pair of characters χ_W, χ_V of $E^\times$ as in [23, §3.2] and write $\chi = (\chi_W, \chi_V)$. This choice gives a splitting of the metaplectic group through which we may consider the Weil representation $\omega_{W_n,V_m,\psi}$ of $G \times H$ (see [23, §4.1]). Given $\pi \in \Pi(G)$, we denote the maximal π-isotypic quotient of the Weil representation by

$$\pi \boxtimes \Theta_{W_n,V_m,\chi,\psi}(\pi),$$

where $\Theta_{W_n,V_m,\chi,\psi}(\pi)$ is a smooth representation of H which is called the big theta lift of π. We let $\theta_{W_n,V_m,\chi,\psi}(\pi)$, the (little) theta lift of π, be the maximal semi-simple quotient of $\Theta_{W_n,V_m,\chi,\psi}(\pi)$. Originally conjectured by Howe ([36]), the following theorem was first proven by Waldspurger ([56]) when the residual characteristic of F is not 2 and then in full generality by Gan and Takeda ([26]) and Gan and Sun ([25]).

Theorem 2.1 (Howe Duality). *Let $\pi_1, \pi_2 \in \Pi(G)$.*

(1) *If $\theta_{W_n,V_m,\chi,\psi}(\pi_2) \neq 0$, then $\theta_{W_n,V_m,\chi,\psi}(\pi_2)$ is irreducible.*

(2) *If* $\pi_1 \not\cong \pi_2$ *and both* $\theta_{W_n,V_m,\chi,\psi}(\pi_1)$ *and* $\theta_{W_n,V_m,\chi,\psi}(\pi_2)$ *are nonzero, then*

$$\theta_{W_n,V_m,\chi,\psi}(\pi_1) \not\cong \theta_{W_n,V_m,\chi,\psi}(\pi_2).$$

For our purposes, we need to consider several towers of theta lifts at once. Recall that V_m lies in some Witt tower (see (2.3)). If $E = F$ and $\epsilon = -1$, then there is only once choice of anisotropic V_{n_0}. Otherwise, there are always two towers of the form (2.3), say $\mathcal{V}'$ and $\mathcal{V}''$. Fix a representation $\pi \in \Pi(G(W_n))$. We define the first occurrence of π in the tower $\mathcal{V}'$, denoted $m'(\pi)$, to be the minimal integer $m' = \dim V'$ such that $V' \in \mathcal{V}'$ and $\theta_{W_n,V',\chi,\psi}(\pi) \neq 0$. We define $m''(\pi)$ analogously for the tower $\mathcal{V}''$. We define

$$m^+(\pi) = \max\{m'(\pi), m''(\pi)\},$$
$$m^-(\pi) = \min\{m'(\pi), m''(\pi)\}.$$

When $E = F$ and $\epsilon = 1$, we have that $G(W_n) = \mathrm{O}_n(F)$. In this case, we have representations π and $\pi \otimes \det$ of $G(W_n)$. Thus for $V_m \in \mathcal{V}$, we have "towers"

$$\theta_{W_n,V_m,\chi,\psi}(\pi) \quad \text{and} \quad \theta_{W_n,V_m,\chi,\psi}(\pi \otimes \det).$$

We let $m(\pi)$ be the minimal integer $m = \dim V$ such that $V \in \mathcal{V}$ and $\theta_{W_n,V,\chi,\psi}(\pi) \neq 0$. We define

$$m^+(\pi) = \max\{m(\pi), m(\pi \otimes \det)\},$$
$$m^-(\pi) = \min\{m(\pi), m(\pi \otimes \det)\}.$$

In general, when $\pi \in \Pi(G(W_n))$, the conservation relation give a relation between $m^+(\pi)$ and $m^-(\pi)$. Note that $n = \dim(W_n)$.

Theorem 2.2 (Conservation relation, [54]). *Let* $\pi \in \Pi(G(W_n))$. *Then*

$$m^+(\pi) + m^-(\pi) = 2n + 2\epsilon_0 + 2.$$

As a consequence, we have that $m^+(\pi) \geq n + \epsilon_0 + 1 \geq m^-(\pi)$. Also, if one inequality is strict, then both inequalities are strict. In this situation, we call the tower whose first occurrence is $m^+(\pi)$ the "going-up" tower for π and denote it by $\mathcal{V}^+$. Similarly, we call the tower whose first occurrence is $m^-(\pi)$ the "going-down" tower for π and denote it by $\mathcal{V}^-$. When $m^+(\pi) = m^-(\pi)$, the designations of "going-up" or "going-down" will not matter (see Remarks 2.7 and 3.8).

Fix $\pi \in \Pi(G(W_n))$ and let $V_m^\pm \in \mathcal{V}^\pm$, where $n = \dim W_n$ and $m = \dim V_m^\pm$. We set $\alpha = M - N$, where M is the rank of the complex dual group of $G(W_n)$ and N is the rank of the complex dual group of $H(V_m^\pm)$ (note that $H(V_m^\pm)$ are pure inner forms of each other and hence have the same complex dual group). Given $\beta \in \mathbb{Z}_{\geq 0}$, we say β is suitable if $\beta = M - N$ for some suitable $G(W_n)$ and $H(V_m^\pm)$. For example, if $\epsilon = -1$, $E = F$, and n and m are even, then $G(W_n) = \mathrm{Sp}(W_n)$ and $H(V_m) = \mathrm{O}(V_m)$. The complex dual groups are $\mathrm{SO}_{n+1}(\mathbb{C})$ and $\mathrm{O}_m(\mathbb{C})$ and so $\alpha = m - n - 1$. In this case, $\beta \in \mathbb{Z}_{\geq 0}$ is suitable if and only if β is odd. In general, we write $\theta_{W_n,V_m^{\pm},\chi,\psi}(\pi)=\theta_{-\alpha}^{\pm}(\pi)$. When it is clear in context, we suppress the notation $\pm$. Furthermore, we let $m^{\pm,\alpha}(\pi)$ denote the value of α corresponding to $m^{\pm}(\pi)$. For example, if $\epsilon=-1$, $E=F$, and n and m are even, then $m^{\pm,\alpha}(\pi)=m^{\pm}(\pi)-n-1$.

We give a table for the dual groups and N below (M is determined similarly).

	G	$\widehat{G}(\mathbb{C})$	N
$E{=}F$, n odd $\epsilon{=}1$	$\mathrm{SO}(W_n)$	$\mathrm{Sp}_{n-1}(\mathbb{C})$	$n-1$
$E{=}F$, n,m even, $\epsilon{=}1$	$\mathrm{O}(W_n)$	$\mathrm{O}_n(\mathbb{C})$	n
$E{=}F$, n,m even, $\epsilon{=}-1$	$\mathrm{Sp}(W_n)$	$\mathrm{SO}_{n+1}(\mathbb{C})$	$n+1$
$E{\neq}F$, n arbitrary, $\epsilon{=}\pm1$	$\mathrm{U}(W_n)$	$\mathrm{GL}_n(\mathbb{C})$	n

Except for the following case, we set the L-group to tbe the L-group of the connected component, ${}^LG={}^LG^0$. We remark that when $E=F$, m is odd, and $\epsilon=-1$, we set $G=\mathrm{Mp}(W_n)$. In this case, n must be even and we set $\widehat{G}(\mathbb{C})=\mathrm{Sp}_n(\mathbb{C})={}^LG$ and $N=n$.

2.2. **Local Arthur packets.** In this subsection, we let $G=G(W_n)$ and discuss local Arthur parameters and local Arthur packets for G. We note that the analogous notions also make sense for $H=H(V_m)$. A local Arthur parameter may be considered as a direct sum of irreducible representations

$$\psi: W_F\times \mathrm{SL}_2(\mathbb{C})\times\mathrm{SL}_2(\mathbb{C})\to {}^LG,$$

$$\psi=\bigoplus_{i=1}^r \phi_i|\cdot|^{x_i}\otimes S_{a_i}\otimes S_{b_i}, \tag{2.4}$$

satisfying the following conditions:

(1) $\dim(\phi_i)=d_i$ and $\phi_i(W_F)$ is bounded and consists of semi-simple elements;
(2) $x_i\in\mathbb{R}$ and $|x_i|<\frac{1}{2}$;
(3) the restrictions of ψ to the two copies of $\mathrm{SL}_2(\mathbb{C})$ are analytic, S_k is the k-dimensional irreducible representation of $\mathrm{SL}_2(\mathbb{C})$, and
$$\sum_{i=1}^r d_ia_ib_i=N.$$

We consider local Arthur parameters up to $\widehat{G}(\mathbb{C})$-conjugacy, i.e., we say two local Arthur parameters are equivalent if they are conjugate under $\widehat{G}(\mathbb{C})$. We will not distinguish between ψ and its equivalence class. We define $\Psi^+(G)$ to be the set of equivalence classes of local Arthur parameters. Let $\Psi(G)$ be the subset of $\Psi^+(G)$ consisting of local Arthur parameters ψ whose restriction to W_F is bounded, i.e, if we decompose ψ as in in the decomposition (2.4), then $\psi\in\Psi(G)$ if and only if $x_i=0$ for $i=1,\dots,r$. In the literature, sometimes the set $\Psi(G)$ is considered in place of $\Psi^+(G)$. This is sufficient for global applications if one assumes the Ramanujan conjecture. We do not adopt this viewpoint and hence consider $\Psi^+(G)$.

Given $\psi \in \Psi^+(G)$, Arthur's conjectures ([4]) predict that there should exist a finite set Π_ψ consisting of equivalence classes of irreducible smooth representations which satisfy certain twisted endoscopic character identities. The set Π_ψ is called the local Arthur packet attached to ψ. Given a representation π of G, we say that π is of Arthur type if $\pi \in \Pi_\psi$ for some $\psi \in \Psi^+(G)$.

For the purposes of this article, we shall assume that local Arthur packets exist. We remark briefly on the current status of this assumption. For quasi-split classical groups, the existence of local Arthur packets is essentially proven by the works of Arthur and Mok ([5, 49]) when supplemented with the work of [7]. The only remaining step in this case is the verification of the twisted weighted fundamental lemma. There are also some partial extensions to the non-quasi-split cases in [37, 39]. For metaplectic groups, see [42].

Before we proceed to state the Adams conjecture for local Arthur packets, we recall a conjecture which will play a role in the refinement of the Adams conjecture. Let π be an irreducible admissible representation of G. We let

$$\Psi(\pi) = \{\psi \in \Psi^+(G) \mid \pi \in \Pi_\psi\}.$$

We will consider a partial ordering $\geq_C$ on $\Psi(\pi)$. Its existence is enough for us in this section and so we defer some of the following unexplained terminology to §3. We have an injection from $\Psi^+(G)$ to the set of L-parameters of G given by $\psi \mapsto \phi_\psi$ where

$$\phi_\psi(w, x) = \psi(w, x, \begin{pmatrix} |w|^{\frac{1}{2}} & 0 \\ 0 & |w|^{\frac{-1}{2}} \end{pmatrix}). \tag{2.5}$$

The representation π determines an infinitesimal parameter λ_{ϕ_π}, where ϕ_π is the L-parameter of π. On the set of L-parameters with fixed infinitesimal parameter λ_{ϕ_π} there exists a partial ordering $\geq_C$ which is known as the closure ordering (Definition 3.1). Given $\psi_1, \psi_2 \in \Psi^+(G)$, we define $\psi_1 \geq_C \psi_2$ if $\lambda_{\phi_{\psi_1}} = \lambda_{\phi_{\psi_2}}$ and $\phi_{\psi_1} \geq_C \phi_{\psi_2}$. It is conjectured that closure ordering gives a partial order on $\psi(\pi)$.

Conjecture 2.3 ([33, Conjecture 1.4])**.** *Let π be an irreducible admissible representation of G. Then for any $\psi_1, \psi_2 \in \Psi(\pi)$, we have that $\lambda_{\phi_{\psi_1}} = \lambda_{\phi_{\psi_2}}$. Furthermore, there exists elements in $\psi^{\max}(\pi)$ and $\psi^{\min}(\pi)$ in $\Psi(\pi)$ such that for any $\psi \in \Psi(\pi)$, we have*

$$\psi^{\max}(\pi) \geq_C \psi \geq_C \psi^{\min}(\pi).$$

When G is a quasi-split symplectic or orthogonal group, this conjecture has been verified in [31, 33]. We assume the above conjecture (which serves as our definition of $\psi^{\max}(\pi)$); however, it is only needed in a few places, e.g., the statement of Conjecture 2.6(4).

2.3. **The Adams Conjecture for local Arthur packets.** In this subsection, we explicate Conjecture 1.1 and also conjecture a refinement. Recall that we are considering the classical groups $G = G(W_n)$ and $H = H(V_m^{\pm})$ along with the theta lift $\theta_{-\alpha}^{\pm}$ between them. Let $\psi \in \Psi^+(G)$. We define

$$\psi_\alpha = (\chi_W \chi_V^{-1} \otimes \psi) \oplus \chi_W \otimes S_1 \otimes S_\alpha,$$

where we view the characters χ_V, χ_W as representations of W_F via local class field theory. We have that $\psi_\alpha \in \Psi^+(H)$. Note that our definition of ψ_α only makes sense if $\alpha > 0$. We shall assume that $\alpha > 0$ throughout the rest of this article as it will be implicit in the statements.

The Adams conjecture predicts that the theta lift sends Π_ψ to Π_{ψ_α}.

Conjecture 2.4 (The (naive) Adams Conjecture [2])**.** *Suppose that $\pi \in \Pi_\psi$. Then $\theta_{-\alpha}^{\pm}(\pi) \in \Pi_{\psi_\alpha}$ provided that $\theta_{-\alpha}^{\pm}(\pi) \neq 0$.*

As mentioned in §1, Conjecture 2.4 can and does fail. However, the works of [10, 29, 47] give a precise description of when the Adams conjecture holds for symplectic-even orthogonal dual pairs.

Theorem 2.5. *Suppose that the pair (G, H) is a quasi-split symplectic-even orthogonal dual pair and that $\pi \in \Pi_\psi$ for some $\psi \in \Psi^+(G)$.*

1. *For $\alpha \gg 0$, we have $\theta_{-\alpha}^{\pm}(\pi) \in \Pi_{\psi_\alpha}$* ([47, Theorem 6.1])*.*
2. *If $\theta_{-\alpha}^{+}(\pi) \neq 0$, then $\theta_{-\alpha}^{\pm}(\pi) \in \Pi_{\psi_\alpha}$* ([10, Theorem 2])*.*
3. *If $\theta_{-\alpha}^{\pm}(\pi) \in \Pi_{\psi_\alpha}$ for some α, then $\theta_{-(\alpha+2)}^{\pm}(\pi) \in \Pi_{\psi_{\alpha+2}}$* ([10, Theorem C])*.*
4. *If $\theta_{-\alpha}^{\pm}(\pi) \in \Pi_{\psi_\alpha}$, then $\theta_{-\alpha}^{\pm}(\pi) \in \Pi_{\psi^{\max}(\pi)_\alpha}$* ([29, Theorem 1.5])*.*

We conjecture that the above theorem should also hold in the general setting. We call this the refined Adams conjecture.

Conjecture 2.6. *Consider a dual pair (G, H) as above and let $\pi \in \Pi_\psi$ for some $\psi \in \Psi^+(G)$.*

1. *For $\alpha \gg 0$, we have $\theta_{-\alpha}^{\pm}(\pi) \in \Pi_{\psi_\alpha}$.*
2. *If $\theta_{-\alpha}^{+}(\pi) \neq 0$, then $\theta_{-\alpha}^{\pm}(\pi) \in \Pi_{\psi_\alpha}$.*
3. *If $\theta_{-\alpha}^{\pm}(\pi) \in \Pi_{\psi_\alpha}$ for some α, then $\theta_{-(\alpha+2)}^{\pm}(\pi) \in \Pi_{\psi_{\alpha+2}}$.*
4. *If $\theta_{-\alpha}^{\pm}(\pi) \in \Pi_{\psi_\alpha}$, then $\theta_{-\alpha}^{\pm}(\pi) \in \Pi_{(\psi^{\max}(\pi))_\alpha}$.*

Remark 2.7. *When $m^+(\pi) = m^-(\pi)$, it follows from the conservation relation that $m^+(\pi) = n + \epsilon_0 + 1$. Consequently, $m^{+,\alpha}(\pi) = \epsilon_0$ and so Conjecture 2.6(2) implies that the Adams conjecture is true for any suitable positive integer α and any $\psi \in \Psi(\pi)$. That is, the choice of "going-up" or "going-down" tower does not matter.*

We remark that Conjecture 2.6(4) implicitly assumes Conjecture 2.3. We also remark that for quasi-split symplectic-even orthogonal dual pairs, [29, Theorem 1.3] proves a stronger statement which implies [29, Theorem 1.5]. In general, we expect an analogue of [29, Theorem 1.3] for any reductive dual pairs of type II (which could be included in Conjecture 2.6 above); however, we opted not to include it in the above list as it does not seem to generalize to ABV-packets. We discuss this issue later (see Conjecture 3.10).

Conjecture 2.6 predicts how the failure of the Adams conjecture may occur. Indeed, Conjecture 2.6(1) states the Adams conjecture is always true when the difference in the ranks is large enough. Conjecture 2.6(2) states that the Adams conjecture is always true when considering the "going-up" tower for π. Conjecture 2.6(3) states that if the Adams conjecture holds at some level α, then it holds at any greater level which implies that if the Adams conjecture fails at some level, then it fails at every lower level. Together these conjectures determine the Adams conjecture's validity when the local Arthur parameter ψ is fixed. Conjecture 2.6(4) controls the validity of the Adams conjecture when $\psi \in \Psi(\pi)$ varies. In particular, Conjecture 2.6(4) says that, among all $\psi \in \Psi(\pi)$, the Adams conjecture holds in its greatest generality for $\psi^{\max}(\pi)$.

So if the Adams conjecture fails for some fixed ψ, it must happen on the going-down tower and for small α. The failure could be for one of two reasons, either the Adams conjecture failed for ψ but holds for some $\psi' \in \Psi(\pi)$ (with $\psi' \geq_C \psi$) or the Adams conjecture fails for $\psi^{\max}(\pi)$ (and hence any $\psi' \in \Psi(\pi)$). The following conjecture predicts if we our failure is in the latter case, then it cannot be fixed using local Arthur packets.

Conjecture 2.8 ([29, Conjecture 1.6])**.** *Consider a dual pair (G, H) as above and let π be a representation of G of Arthur type. Let α_0 be the minimum among all positive suitable integers α such that $\theta^-_{-\alpha}(\pi) \in \Pi_{(\psi^{\max}(\pi))_\alpha}$. If $\alpha_0 \geq 3$, then $\theta^-_{-(\alpha_0-2)}(\pi)$ is not of Arthur type.*

Therefore, in order to remedy the failure of the Adams conjecture for $\psi^{\max}(\pi)$, we are forced to consider a conjectural generalization of local Arthur packets known as ABV-packets.

Remark 2.9. *We also note that the Adams conjecture for π is expected to hold for some $\psi \in \Psi(\pi)$ if the theta lift is unitary. This expectation is conjecturally equivalent to Conjecture 2.8. Indeed, first note that for any suitable $\alpha \geq \alpha_0$, we have $\theta^-_{-\alpha}(\pi) \in \Pi_{(\psi^{\max}(\pi))_\alpha}$ and hence $\theta^-_{-\alpha}(\pi)$ is unitary. Next, the Adams conjecture can be reduced to the "good parity" case (e.g.,* [29, Lemma 2.33]*). It follows from* [6, 9] *that π is of good parity if and only if $\theta^-_{-\alpha}(\pi)$ is of good parity. It is conjectured in* [30, Conjecture 1.2] *that an irreducible admissible representation of good parity is of Arthur type if and only if it is unitary. Consequently, Conjecture 2.8 and* [30, Conjecture 1.2] *imply that if $\alpha_0 \geq 3$, then $\theta^-_{-(\alpha_0-2)}(\pi)$ is not unitary. We also note that* [30, Conjecture 1.2] *is known for split symplectic and odd special orthogonal groups by the work of Atobe and Mínguez (*[8, Theorem 1.1]*) and quasi-split even orthogonal groups by* [31].

3. ABV-packets

In this section, we recall the construction of p-adic ABV-packets following [17]. The nomenclature "ABV" stands for Adams-Barbasch-Vogan and is an hommage to [3] where ABV-packets were defined for real groups. We remark that [17] only treats connected reductive groups; however, for the local theta correspondence we must also consider metaplectic and split even orthogonal groups. Analogous definitions and results are still expected to hold in these cases and we shall assume them. In particular, even orthogonal groups will be treated in a forthcoming work ([18]).

We continue to restrict ourselves to the setting that $G = G(W_n)$, $H = H(V_m)$, and we let $G' \in \{G, H\}$. We also allow for $G' = \mathrm{GL}_n(F)$. An L-parameter of G' may be regarded as a $\widehat{G}'(\mathbb{C})$-conjugacy class of an admissible homomorphism $\phi : W_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G'$ ([12, §8]). We do not require that ϕ is relevant for G'. Let $\Phi(G')$ denote the set of L-parameters of G'. We do not distinguish a representative ϕ from its conjugacy class. We assume that there is a local Langlands correspondence for G'. Specifically, the local Langlands correspondence defines a map $rec : \Pi(G') \to \Phi(G')$. The L-packet attached to ϕ is $\Pi_\phi := rec^{-1}(\phi)$. For $\pi \in \Pi(G_n)$, we let $\phi_\pi := rec(\pi)$ denote the L-parameter of π. The local Langlands correspondence for G' is understood (subject to the twisted weighted fundamental lemma) through the works ([5, 7, 13, 14, 24, 28, 35, 37, 39, 48, 49, 52]).

An infinitesimal character of G' is a continuous homomorphism $\lambda : W_F \to {}^L G'$ which is a section of ${}^L G' \to W_F$ and whose image consists only of semi-simple elements. Given $\phi \in \Phi(G')$, we denote the infinitesimal character associated to ϕ by

$$\lambda_\phi(w) := \phi\left(w, \begin{pmatrix} |w|^{\frac{1}{2}} & 0 \\ 0 & |w|^{\frac{-1}{2}} \end{pmatrix}\right),$$

where $|\cdot|$ denotes the norm on W_F which is trivial on the inertia subgroup and sends the Frobenius element to q.

Let λ be an infinitesimal character of G'. We let

$$\Phi_\lambda(G') = \{\phi \in \Phi(G') \mid \lambda_\phi = \lambda\}.$$

Similarly, we let

$$\Pi_\lambda(G') = \{\pi \in \Pi(G') \mid \lambda_{\phi_\pi} = \lambda\}.$$

Both of these sets are finite.

Let $\mathfrak{g}'(\mathbb{C})$ be the Lie algebra of $\widehat{G}'(\mathbb{C})$. We let

$$H_\lambda = \{g \in \widehat{G}'(\mathbb{C}) \mid \lambda(w) g \lambda(w)^{-1} = g, \forall w \in W_F\},$$
$$K_\lambda = \{g \in \widehat{G}'(\mathbb{C}) \mid \lambda(w) g \lambda(w)^{-1} = g, \forall w \in I_F\},$$

where I_F is the inertia subset of the absolute Galois group of F. Note that $H_\lambda = Z_{\widehat{G}'(\mathbb{C})}(\lambda)$ is the centralizer of the image of λ. We consider the Vogan variety

$$V_\lambda = \{x \in \mathrm{Lie}(K_\lambda) \mid \mathrm{Ad}(\lambda(w))x = |w|x, \forall w \in W_F\}.$$

The group H_λ acts on V_λ via conjugation. The action stratifies V_λ into finitely many orbits and we let $C_\lambda(G')$ denote the collection of these orbits. These orbits are in bijection with $\Phi_\lambda(G')$ ([17, Proposition 4.2.2]). The bijection is given by identifying $\phi \in \Phi_\lambda(G')$ with the H_λ-orbit of x_ϕ where

$$x_\phi = d(\phi|_{\mathrm{SL}_2(\mathbb{C})}) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

We let C_ϕ be the H_λ-orbit of x_ϕ. The geometry of V_λ induces a partial ordering $\geq_C$ on $\Phi(G)_\lambda$. We call this partial ordering, the closure ordering.

Definition 3.1. *Let $\phi_1, \phi_2 \in \Phi_\lambda(G')$. We define a partial ordering $\geq_C$ on $\Phi_\lambda(G')$ by $\phi_1 \geq_C \phi_2$ if $\overline{C_{\phi_1}} \supseteq C_{\phi_2}$.*

Now, G' always has a quasi-split pure inner form which we denote by G'_0. When G' is symplectic, it is split and so $G' = G'_0$. Otherwise, G' always has exactly one nontrivial pure inner form corresponding to the "other" tower (see §2.1). It is possible that both pure inner forms are quasi-split in which case we fix a choice of G'_0. The pure inner forms of G' are indexed by $H^1(F, G'_0)$ and G'_0 corresponds to the trivial class. We let $\Pi^{\mathrm{pure}}(G')$ denote the set of equivalence classes of respresentations of G' and its pure inner forms.

Let $\mathrm{Per}_{H_\lambda}(V_\lambda)$ denote the category of H_λ-equivariant perverse sheaves on V_λ. Vogan's perspective on the local Langlands correspondence ([55]) gives a bijection between $\Pi^{\mathrm{pure}}_\lambda(G')$ and the simple objects in $\mathrm{Per}_{H_\lambda}(V_\lambda)$ (up to isomorphism). For $\pi \in \Pi_\lambda(G')$, we write $\mathcal{P}(\pi)$ for the corresponding simple perverse sheaf.

Let $\phi \in \Phi_\lambda(G')$. Cunningham et al. defined the ABV-packet attached to ϕ ([17, §8.1]). It is denoted by defined by Π^{ABV}_ϕ and in our setting, we have

$$\Pi^{\mathrm{ABV}}_\phi := \{\pi \in \Pi^{\mathrm{pure}}_\lambda(G') \mid \mathrm{Evs}_{C_\phi}(\mathcal{P}(\pi)) \neq 0\}.$$

Here Evs is the microlocal vanishing cycles functor defined in [17, §7.9]. This functor is essential to compute ABV-packets; however, for our purposes, it suffices to give several properties of ABV-packets instead. We set

$$\Pi^{\mathrm{ABV}}_\phi(G') = \Pi^{\mathrm{ABV}}_\phi \cap \Pi(G').$$

First, we have that ABV-packets respect the closure ordering.

Proposition 3.2 ([17, Proposition 7.10])**.** *If $\pi \in \Pi^{\mathrm{ABV}}_\phi$, then $\phi_\pi \geq_C \phi$.*

Second, we have that ABV-packets contain their L-packets.

Proposition 3.3 ([17, Proposition 7.13(b)])**.** *We have that $\Pi_\phi \subseteq \Pi^{\mathrm{ABV}}_\phi$.*

The following proposition follows directly from Propositions 3.2 and 3.3 (see also [17, §10.2.6]).

Proposition 3.4. *If C_ϕ is the unique open orbit in $C_\lambda(G')$, then $\Pi^{\mathrm{ABV}}_\phi = \Pi_\phi$.*

Finally, we mention that ABV-packets are expected to generalize local Arthur packets through the following conjecture. However, aside from relating our two main conjectures (see Proposition 3.9), we note that none of our results or conjectures rely on the below conjecture. Furthermore, the below conjecture is known for $\mathrm{GL}_n(F)$ by the independent works of [19, 20, 43, 50].

Conjecture 3.5 (([17, Conjecture 8.3.1])). *Suppose $\phi = \phi_\psi$ for some $\psi \in \Psi(G')$. Then*

$$\Pi_\psi^{\mathrm{pure}} = \Pi_{\phi_\psi}^{\mathrm{ABV}}.$$

Here Π_ψ^{pure} is the union of the local Arthur packets attached to ψ for all of the pure inner forms of G'.

The above conjecture will also be verified more generally (assuming a theory of local Arthur packets) in [18].

In many ways, results about local Arthur packets are expected to be generalized to ABV-packets, e.g., the Adams conjecture. However, this is not always the case. Indeed, L-packets and local Arthur packets of $\mathrm{GL}_n(F)$ are singletons, but ABV-packets of $\mathrm{GL}_n(F)$ may not be singletons (see [16]). In fact, this is the main reason that the Adams conjecture for ABV-packets of $\mathrm{GL}_n(F)$ (Conjecture 4.5) will not follow trivially from Theorem 4.1 below.

3.1. **The Adams conjecture for ABV-packets.** We continue with the case that $G = G(W_n)$. We briefly recall the notation from §2.1. We consider towers $\mathcal{V}^\pm$ and let $H^\pm = H(V_m^\pm)$ for $V_m^\pm \in \mathcal{V}^\pm$. We let $\theta_{-\alpha}^\pm$ denote the local theta correspondence from G to $H^\pm$. Recall that our choice of $\pm$ depends on the first occurrence of $\pi \in \Pi(G_n)$.

Let $\phi \in \Phi(G)$. We define $\phi_\alpha \in \Phi(H^\pm)$ by

$$\phi_\alpha = (\chi_W \chi_V^{-1} \otimes {}^c\phi^\vee) \oplus \left(\bigoplus_{i=0}^{\alpha-1} \chi_W |\cdot|^{\frac{\alpha-1}{2}-i} \otimes S_1 \right).$$

Here, we recall that c is the generator of $\mathrm{Gal}(E/F)$. Note that since G is a classical group, we have ${}^c\phi^\vee = \phi$. However, in the next section we consider $G = \mathrm{GL}_n(F)$ and $H = \mathrm{GL}_m(F)$, where we take c to be trivial, but we do not necessarily have that $\phi^\vee = \phi$ and the contragredient will be needed. Thus, for uniformity, we define ϕ_α as above.

The map $\phi \mapsto \phi_\alpha$ is understood as a generalization of the map $\psi \mapsto \psi_\alpha$. Indeed, if $\phi = \phi_\psi$ for some local Arthur parameter ψ of G, then $\phi_{\psi_\alpha} = \phi_\alpha$. Motivated by Conjecture 3.5, we now formulate the analogues of Conjectures 2.4 and 2.6.

Conjecture 3.6 (The (naive) Adams Conjecture for ABV-packets). *If $\pi \in \Pi_\phi^{\mathrm{ABV}}$, then $\theta_{-\alpha}^\pm(\pi) \in \Pi_{\phi_\alpha}^{\mathrm{ABV}}$ provided that $\theta_{-\alpha}^\pm(\pi) \neq 0$.*

Assuming Conjecture 3.5, it follows that Conjecture 3.6 can and does fail since its analogue for local Arthur packets, Conjecture 2.4, also does fail. Analogous to Conjecture 2.6, we conjecture the following refinement which we call the refined Adams conjecture for ABV-packets.

Conjecture 3.7. *Let* $\pi \in \Pi_{\phi}^{\mathrm{ABV}}$ *for some* $\phi \in \Phi(G)$.

(1) *For* $\alpha \gg 0$, *we have* $\theta_{-\alpha}^{\pm}(\pi) \in \Pi_{\phi_\alpha}^{\mathrm{ABV}}$.
(2) *If* $\theta_{-\alpha}^{+}(\pi) \neq 0$, *then* $\theta_{-\alpha}^{\pm}(\pi) \in \Pi_{\phi_\alpha}^{\mathrm{ABV}}$.
(3) *If* $\theta_{-\alpha}^{\pm}(\pi) \in \Pi_{\phi_\alpha}^{\mathrm{ABV}}$ *for some* α, *then* $\theta_{-(\alpha+2)}^{\pm}(\pi) \in \Pi_{\phi_{\alpha+2}}^{\mathrm{ABV}}$.
(4) *If* $\theta_{-\alpha}^{\pm}(\pi) \in \Pi_{\phi_\alpha}^{\mathrm{ABV}}$, *then* $\theta_{-\alpha}^{\pm}(\pi) \in \Pi_{(\phi_\pi)_\alpha}^{\mathrm{ABV}}$.
(5) *Assume that* $\pi \in \Pi_{\phi}^{\mathrm{ABV}} \cap \Pi_{\phi'}^{\mathrm{ABV}}$ *with* $\phi \geq_C \phi'$. *If* $\theta_{-\alpha}^{\pm}(\pi) \in \Pi_{(\phi')_\alpha}^{\mathrm{ABV}}$, *then* $\theta_{-\alpha}^{\pm}(\pi) \in \Pi_{\phi_\alpha}^{\mathrm{ABV}}$.

Remark 3.8. *When* $m^+(\pi) = m^-(\pi)$, *it follows from the conservation relation that* $m^+(\pi) = n+\epsilon_0+1$. *Consequently,* $m^{+,\alpha}(\pi) = \epsilon_0$ *and so Conjecture 3.7(2) implies that the Adams conjecture for ABV packets (Conjecture 3.6) is true for any suitable positive integer* α *and any* $\phi \in \Phi(\pi)$ *(see below for notation). That is, the choice of "going-up" or "going-down" tower does not matter.*

We remark that while Conjecture 2.6(4) implicitly assumes Conjecture 2.3, its analogue here, Conjecture 3.7(4), does not. This is because the analogue of $\psi^{\max}(\pi)$ for ABV-packets is well-understood. Indeed, Conjecture 2.3 says that $\psi^{\max}(\pi)$ is the unique maximal element in $\Psi(\pi)$ with respect to $\geq_C$. This is not the case for ABV-packets. Here, the analogue of $\Psi(\pi)$ is

$$\Phi(\pi) = \{\phi \in \Phi(G) \mid \pi \in \Pi_{\phi}^{\mathrm{ABV}}\}.$$

There is a unique maximal element of $\Phi(\pi)$ with respect to $\geq_C$, namely ϕ_π. Indeed, this is an immediate consequence of Propositions 3.2 and 3.3. However, if ϕ_π is not of Arthur type, then $\phi_\pi \neq \phi_{\psi^{\max}(\pi)}$. Indeed, [33, Conjecture 1.4] predicts that $\phi_\pi \geq_C \phi_{\psi^{\max}(\pi)}$ and so by passing to ABV-packets, it sometimes necessary to go beyond $\psi^{\max}(\pi)$. Note that Conjecture 3.7(5) implies Conjecture 3.7(4) based on the above discussion.

We show that Conjecture 3.7 generalizes Conjecture 2.6 assuming Conjecture 3.5.

Proposition 3.9. *Assume Conjecture 3.5. Then the following holds.*

(1) *Conjecture 3.7(1) implies Conjecture 2.6(1).*
(2) *Conjecture 3.7(2) implies Conjecture 2.6(2).*
(3) *Conjecture 3.7(3) implies Conjecture 2.6(3).*
(4) *Conjecture 3.7(5) implies Conjecture 2.6(4).*

Proof. The first three statements are immediate since $\Pi_{\psi}^{\mathrm{pure}} = \Pi_{\phi_\psi}^{\mathrm{ABV}}$ and $\Pi_\psi \subseteq \Pi_{\psi}^{\mathrm{pure}}$. We give the details for the last statement. Assume that $\pi \in \Pi_\psi$ and $\theta_{-\alpha}^{\pm}(\pi) \in \Pi_{\psi_\alpha}$. By Conjecture 3.5, we have that $\theta_{-\alpha}^{\pm}(\pi) \in \Pi_{\phi_{\psi_\alpha}}^{\mathrm{ABV}}$. Note that $\phi_{\psi^{\max}(\pi)} \geq \phi_\psi$ by Conjecture 2.3. Also, by definition of $\psi^{\max}(\pi)$, we have that $\pi \in \Pi_{\psi^{\max}(\pi)}$ and hence $\pi \in \Pi_{\phi_{\psi^{\max}(\pi)}}^{\mathrm{ABV}}$ by Conjecture 3.5. Therefore, Conjecture 3.7(5) implies that $\theta_{-\alpha}^{\pm}(\pi) \in \Pi_{\phi_{\psi^{\max}(\pi)_\alpha}}^{\mathrm{ABV}}$. Finally, by

Conjecture 3.5, it follows that $\theta^{\pm}_{-\alpha}(\pi) \in \Pi_{(\psi^{\max}(\pi))_\alpha}$. That is, Conjecture 3.7(5) implies Conjecture 2.6(4). □

Note that the analogue of Conjecture 3.7(5) is not conjectured in Conjecture 2.6. This is because the closure order $\geq_C$ is natural in the setting of ABV-packets but less so for local Arthur packets. However, Conjectures 3.7(5) and 3.5 imply the following analogous conjecture for local Arthur packets.

Conjecture 3.10. *Suppose that $\pi \in \Pi_\psi \cap \Pi_{\psi'}$ for some $\psi, \psi' \in \Psi^+(G_n)$ with $\psi \geq_C \psi'$. If $\theta^{\pm}_{-\alpha}(\pi) \in \Pi_{\psi'_\alpha}$, then $\theta^{\pm}_{-\alpha}(\pi) \in \Pi_{\psi_\alpha}$.*

For symplectic-even orthogonal dual pairs, Theorem 2.5 states that Conjecture 2.6 is true. However, Conjecture 3.10 remains open in all cases including symplectic-even orthogonal dual pairs. Indeed, the argument in [29] uses an ordering $\geq_O$ on $\Psi(\pi)$ which implies $\geq_C$ ([33, Theorem 4.5(1)]) but the reverse is not true (see [33, Example 5.9(2)]). Therefore, [29, Theorem 1.3] only provides partial evidence for Conjecture 3.10.

Remark 3.11. *With the above discussion in mind, it is possible that Conjecture 3.7(5) is false. In this case, one would expect that there exists a stronger partial order, analogous to $\geq_O$, on $\Phi(\pi)$ which would replace $\geq_C$ in Conjecture 3.7(5). This order should replace $\geq_C$ in Conjecture 3.10 as well. However, $\geq_C$ is the most natural partial ordering on $\Phi(\pi)$ and at the time of writing, no counter-example to Conjecture 3.7(5) or Conjecture 3.10 is known to the author.*

We now turn towards verifying an implication of Conjecture 3.7. Specifically, we verify that Conjecture 3.7(4) holds on the going-down tower. This follows immediately from the following result.

Lemma 3.12. *For any suitable positive integer α, we have that $\theta^{-}_{-\alpha}(\pi) \in \Pi^{\mathrm{ABV}}_{(\phi_\pi)_\alpha}$.*

Proof. By [6, 9], we have $\phi_{\theta^{-}_{-\alpha}(\pi)} = (\phi_\pi)_\alpha$. Thus we have that $\theta^{-}_{-\alpha}(\pi) \in \Pi_{\phi_{\theta^{-}_{-\alpha}(\pi)}}$. From Proposition 3.3, it follows that $\theta^{-}_{-\alpha}(\pi) \in \Pi^{\mathrm{ABV}}_{\phi_{\theta^{-}_{-\alpha}(\pi)}} = \Pi^{\mathrm{ABV}}_{(\phi_\pi)_\alpha}$. □

The above lemma entirely resolves the failure of the Adams conjecture for local Arthur packets. Indeed, recall Conjecture 2.8. Let α_0 be the minimum among all positive integers α such that $\theta^{-}_{-\alpha}(\pi) \in \Pi_{(\psi^{\max}(\pi))_\alpha}$. If $\alpha \geq 3$, then Conjecture 2.8 predicts that $\theta^{-}_{-(\alpha-2)}(\pi)$ is not of Arthur type. In this case, we must consider ABV-packets instead of local Arthur packets. Conjecture 3.7(5) suggests that we should move to some $\phi \in \Phi(\pi)$ for which $\phi \geq_C \phi_{\psi^{\max}(\pi)}$ and check if the Adams conjecture for ABV-packets holds for ϕ. Lemma 3.12 says that we may always do this. Indeed, one may take $\phi = \phi_\pi$.

Curiously, Conjecture 3.7(2) predicts that the analogue of Lemma 3.12 should also hold on the going-up tower. However, the above proof does not verify this. Indeed, this is primarily a consequence of [6, Theorem 4.5] (and [10, Theorem 6.8]), see also [51, p. 558]. More specifically, it is possible that $\phi_{\theta^+_{-\alpha}(\pi)} \neq (\phi_\pi)_\alpha$.

We end this section by giving some examples which hint at a possible relationship between Conjecture 2.8 and Conjecture 3.7. First, we fix some notation.

Definition 3.13. *Let $\pi \in \Pi_\psi$ for some local Arthur parameter $\psi \in \Psi(G_n)$. We define*

$$d^\pm(\pi,\psi) := \min\{\text{suitable } \alpha > 0 \mid \theta^\pm_{-\alpha}(\pi) \in \Pi_{\psi_\alpha}\}.$$

Similarly, let $\pi \in \Pi_\phi^{\mathrm{ABV}}$ for some $\phi \in \Phi(G_n)$. We define

$$d^\pm(\pi,\phi) := \min\{\text{suitable } \alpha > 0 \mid \theta^\pm_{-\alpha}(\pi) \in \Pi_{\phi_\alpha}^{\mathrm{ABV}}\}.$$

If $\phi = \phi_\psi$ for some $\psi \in \Psi(G_n)$, then Conjecture 3.5 predicts that $d^\pm(\pi,\psi) = d^\pm(\pi,\phi_\psi)$. Conjectures 2.6(2) and 3.7(2) essentially become $d^\pm(\pi,\psi) = m^{+,\alpha}(\pi)$ and $d^\pm(\pi,\phi) = m^{+,\alpha}(\pi)$, respectively. Assuming Conjectures 2.6(3) and 3.7(3) would make the previous sentence precise. However, we wish to pay particular attention to Conjecture 2.6(4). It states that for any $\psi \in \Psi(\pi)$, we have

$$d^-(\pi,\psi^{\max}(\pi)) \leq d^-(\pi,\psi),$$

i.e., the Adams conjecture for local Arthur packets holds in its greatest generality for $\psi^{\max}(\pi)$. Consequently, it is desirable to understand how to compute $d^-(\pi,\psi^{\max}(\pi))$ (especially as it is the conjectural lower bound for determining when the theta lift is unitary, see Remark 2.9). We suspect that it is related to determining when ϕ_α is of Arthur type for some $\phi \in \Phi(\pi)$ (this is not entirely correct as written; one would need to focus on the "χ_W-part" of ϕ_α).

Let $\phi \in \Phi(G_n)$ and $\pi \in \Pi_\phi^{\mathrm{ABV}}$. By [6, 9], for any suitable positive integer α, we have that $\phi_{\theta^-_{-\alpha}(\pi)} = (\phi_\pi)_\alpha$. This observation is useful in the following examples.

Example 3.14. *This example is* [29, Example 6.1]. *There is a representation π of $\mathrm{Sp}_{10}(F)$ of Arthur type with L-parameter*

$$\phi_\pi = \chi_V|\cdot|^3 \otimes S_1 + \chi_V|\cdot|^{-3} \otimes S_1 + \chi_V \otimes S_1 + \chi_V \otimes S_3 + \chi_V \otimes S_5$$

and $d^-(\pi,\psi^{\max}(\pi)) = 5$, where

$$\psi^{\max}(\pi) = \chi_V \otimes S_1 \otimes S_7 + \chi_V \otimes S_3 \otimes S_1 + \chi_V \otimes S_1 \otimes S_1.$$

Note that ϕ_π is not of Arthur type, but

$$\begin{aligned}(\phi_\pi)_5 = {} & |\cdot|^3 \otimes S_1 + |\cdot|^2 \otimes S_1 + |\cdot|^1 \otimes S_1 + |\cdot|^{-1} \otimes S_1 + |\cdot|^{-2} \otimes S_1 \\ & + |\cdot|^{-3} \otimes S_1 + \mathbb{1}_{W_F} \otimes S_1 + \mathbb{1}_{W_F} \otimes S_1 + \mathbb{1}_{W_F} \otimes S_3 + \mathbb{1}_{W_F} \otimes S_5\end{aligned}$$

is of Arthur type. Indeed, we have $\phi = \phi_\psi$ where

$$\psi = \mathbb{1}_{W_F} \otimes S_1 \otimes S_7 + \mathbb{1}_{W_F} \otimes S_1 \otimes S_1 + \mathbb{1}_{W_F} \otimes S_3 \otimes S_1 + \mathbb{1}_{W_F} \otimes S_5 \otimes S_1.$$

Furthermore, $(\phi_\pi)_\alpha$ is not of Arthur type for any $\alpha \neq 5$. In other words, we have $(\phi_\pi)_\alpha$ is of Arthur type if and only if $\alpha = d^-(\pi, \psi^{\max}(\pi))$. Equivalently, from [31] (see Remark 2.9), it follows that $\theta^-_{-\alpha}(\pi)$ is unitary for $\alpha > 0$ if and only if $\alpha \geq 5 = d^-(\pi, \psi^{\max}(\pi))$. In particular, $\theta^-_{-3}(\pi)$ and $\theta^-_{-1}(\pi)$ are not unitary.

We remark that computing $d^-(\pi, \psi^{\max}(\pi))$ is done algorithmically which limits its theoretical use. On the other hand, computing whether $(\phi_\pi)_\alpha$ (or generally ϕ_α) is of Arthur type is incredibly simple. Having a relation between the two would be desirable in light of Conjecture 2.8 and Remark 2.9. Here is another example.

Example 3.15. *There is a unique representation π of $\mathrm{Sp}_{10}(F)$ of Arthur type with L-parameter*

$$\phi_\pi = \chi_V|\cdot|^{\frac{3}{2}} \otimes S_2 + \chi_V|\cdot|^{\frac{-3}{2}} \otimes S_2 + \chi_V \otimes S_1 + \chi_V \otimes S_3 + \chi_V \otimes S_3$$

and satisfying

$$\psi^{\max}(\pi) = \chi_V \otimes S_2 \otimes S_4 + \chi_V \otimes S_3 \otimes S_1.$$

Note that $(\phi_\pi)_\alpha$ is never of Arthur type. However, $d^-(\pi, \psi^{\max}(\pi)) = 1$ and so we would not expect a relationship with $(\phi_\pi)_\alpha$ being of Arthur type.

4. The Adams conjecture for general linear groups

For this section, we focus on the case that $G = G_n = \mathrm{GL}_n(F)$ and $H = H_m = \mathrm{GL}_m(F)$, where $\alpha = m - n \geq 0$. In this setting, c is trivial and the analogue of the characters χ_W and χ_V are the trivial characters. We remark that the local Langlands correspondence in known for general linear groups ([28, 35, 52]). The pair $(\mathrm{GL}_n(F), \mathrm{GL}_m(F))$ forms a reductive dual pair of type II. Consequently, there is a local theta correspondence $\theta_{-\alpha}(\pi)$ from G to H. Since L-packets of $\mathrm{GL}_n(F)$ and $\mathrm{GL}_m(F)$ are singletons, we may take the following theorem of Mínguez to be our definition of the local theta correspondence in this setting.

Theorem 4.1 ([45, Theorem 1])**.** *Suppose that $n \leq m$ and $\pi \in \Pi(\mathrm{GL}_n(F))$ is the unique irreducible quotient of*

$$M(\pi) = \tau_1 \times \cdots \times \tau_r.$$

Then $\theta_{-\alpha}(\pi)$ is the unique irreducible quotient of

$$|\cdot|^{-\frac{m-n-1}{2}} \times \cdots \times |\cdot|^{\frac{m-n-1}{2}} \times \tau_1^\vee \times \cdots \times \tau_r^\vee.$$

Theorem 4.1 gives following corollary immediately.

Corollary 4.2. *Let $\pi \in \Pi(\mathrm{GL}_n(F))$. Then*

$$\phi_{\theta_{-\alpha}(\pi)} = \phi_\pi^\vee \oplus \left(\oplus_{i=0}^{\alpha-1} |\cdot|^{\frac{\alpha-1}{2}-i} \otimes S_1 \right).$$

In particular, if $\alpha = 0$, then $\phi_{\theta_0(\pi)} = \phi_\pi^\vee$ and $\theta_0(\pi) = \pi^\vee$ is the contragredient of π.

One difference between the local theta correspondence for dual pairs of classical groups and the local theta correspondence for general linear groups is that there is only one tower. That is, we do not have the concept of a going-up or going-down tower in this setting. Also if ψ is a local Arthur parameter of $\mathrm{GL}_n(F)$, then $\Pi_\psi = \Pi_{\phi_\psi}$. We immediately obtain the Adams conjecture for $\mathrm{GL}_n(F)$. We remark that this result is already well-understood (see also [2, Theorem 6.7]), but we include a proof for completeness.

Lemma 4.3. *Let $\pi \in \Pi_\psi$ for some local Arthur parameter ψ of $\mathrm{GL}_n(F)$. Then $\theta_{-\alpha}(\pi) \in \Pi_{\psi_\alpha}$ for any $\alpha \in \mathbb{Z}_{\geq 0}$.*

Proof. For any local Arthur parameter ψ of $\mathrm{GL}_n(F)$, we have that $\Pi_\psi \cap \Pi_{\psi'} \neq \emptyset$ if and only if $\Pi_{\phi_\psi} \cap \Pi_{\phi_{\psi'}} \neq \emptyset$. Since L-packets are disjoint, we obtain that $\Pi_\psi \cap \Pi_{\psi'} \neq \emptyset$ if and only if $\psi = \psi'$. Since $\pi \in \Pi_\psi = \Pi_{\phi_\psi}$, it follows that $\Psi(\pi) = \{\psi\}$. Also, we have $\phi_\pi = \phi_\psi$. We obtain from Corollary 4.2 that $(\phi_\pi)_\alpha = (\phi_\psi)_\alpha = \phi_{\theta_{-\alpha}(\pi)}$. In the last step, we used that since ψ is of Arthur type and hence self-dual, we have that $\phi_\pi^\vee = \phi_\pi$. We obtain $\theta_{-\alpha}(\pi) \in \Pi_{\phi_{\theta_{-\alpha}(\pi)}} = \Pi_{\psi_\alpha}$ which proves the lemma. □

Next we state the (naive?) Adams conjecture for ABV-packets of $\mathrm{GL}_n(F)$.

Conjecture 4.4 (The (naive?) Adams Conjecture for ABV-packets of $\mathrm{GL}_n(F)$)**.** *If $\pi \in \Pi_\phi^{\mathrm{ABV}}$ for some $\phi \in \Phi(\mathrm{GL}_n(F))$, then $\theta_{-\alpha}(\pi) \in \Pi_{\phi_\alpha}^{\mathrm{ABV}}$.*

We remark on why we wrote "naive?" here. Recall that we do not have the concept of a going-up or going-down tower for general linear groups. If general linear groups behave like a going-up tower, then we should expect Conjecture 4.4 to hold as written. This is in contrast with Conjecture 3.6 which we know does fail. On the other hand, if general linear groups behave like a going-down tower, then we should expect Conjecture 4.4 to possibly fail. It is unclear which is the correct expectation currently.

Regardless of the situation, we make the following refined Adams conjecture for ABV-packets of $\mathrm{GL}_n(F)$.

Conjecture 4.5. *Let $\pi \in \Pi_\phi^{\mathrm{ABV}}$ for some $\phi \in \Phi(\mathrm{GL}_n(F))$.*

1. *For $\alpha \gg 0$, we have $\theta_{-\alpha}(\pi) \in \Pi_{\phi_\alpha}^{\mathrm{ABV}}$.*
2. *If $\theta_{-\alpha}(\pi) \in \Pi_{\phi_\alpha}^{\mathrm{ABV}}$ for some $\alpha \in \mathbb{Z}_{\geq 0}$, then $\theta_{-(\alpha+1)}(\pi) \in \Pi_{\phi_{\alpha+1}}^{\mathrm{ABV}}$.*
3. *If $\theta_{-\alpha}(\pi) \in \Pi_{\phi_\alpha}^{\mathrm{ABV}}$, then $\theta_{-\alpha}(\pi) \in \Pi_{(\phi_\pi)_\alpha}^{\mathrm{ABV}}$.*
4. *Assume that $\pi \in \Pi_\phi^{\mathrm{ABV}} \cap \Pi_{\phi'}^{\mathrm{ABV}}$ with $\phi \geq_C \phi'$. If $\theta_{-\alpha}(\pi) \in \Pi_{(\phi')_\alpha}^{\mathrm{ABV}}$, then $\theta_{-\alpha}(\pi) \in \Pi_{\phi_\alpha}^{\mathrm{ABV}}$.*

Remark 4.6. *While we stated the above as a conjecture, we will prove both parts (1) and (3) in this article (see the below discussion).*

We remark that Conjecture 4.5 is the analogue of Conjecture 3.7. Indeed, Conjecture 4.5(1, 2, 3, 4) is the analogue of Conjecture 3.7(1, 3, 4, 5), respectively. The omission of the analogue of Conjecture 3.7(2) is because general linear groups only have one tower (see the above discussion). Again, we have that Conjecture 4.5(4) implies Conjecture 4.5(3).

We remark on why the proof of Lemma 4.3 does not generalize to the ABV-packets. The argument requires that $\Pi_\psi = \Pi_{\phi_\psi}$ and hence is a singleton. Recall that by Conjecture 3.5 (which is a theorem for $\mathrm{GL}_n(F)$ by [19, 20, 43, 50]), we can view ABV-packets as generalizations of local Arthur packets. However, for ABV-packets of $\mathrm{GL}_n(F)$, it is not true that $\Pi_\phi^{\mathrm{ABV}} = \Pi_\phi$ generally. Indeed, there is a counter-example for $\mathrm{GL}_{16}(F)$ ([16]). This makes Conjecture 4.5 nontrivial.

We have two pieces of evidence for Conjecture 4.5. The first piece of evidence is that the analogue of Lemma 3.12 holds. Indeed, Corollary 4.2 and Proposition 3.3 imply that $\theta_{-\alpha}(\pi) \in \Pi_{\phi_\alpha}^{\mathrm{ABV}}$ for any nonnegative integer α. This proves Conjecture 4.5(3) in full generality.

The second piece of evidence is more substantial. We confirm Conjecture 4.5(1) in full generality (Theorem 4.34). The majority of the remainder of this article is devoted to this verification.

4.1. **Representation theory.** In this subsection, we continue to focus on the case $G_n = \mathrm{GL}_n(F)$. We fix B_n to be the Borel subgroup of G_n consisting of upper triangular matrices. Consider a parabolic subgroup P of G_n with Levi decomposition $P = MN$ where M is a Levi subgroup isomorphic to $G_{n_1} \times \cdots \times G_{n_r}$, where $n_1 + \cdots + n_r = n$. For $\pi_i \in \Pi(G_{n_i})$, we denote the normalized parabolic induction by

$$\mathrm{Ind}_P^{G_n}(\pi_1 \otimes \cdots \otimes \pi_r) = \pi_1 \times \cdots \times \pi_r.$$

Given $\pi \in \Pi(G_n)$, we let $\pi^\vee$ denote its contragredient.

The Langlands classification for G_n was established by Zelevinsky using segments ([57]); however, we do not need such a precise form. Instead we give the Langlands classification in terms of essentially square-integrable representations following [45, §6]. For $i = 1, \ldots, r$, let $\pi_i \in \Pi(G_{n_i})$ be essentially square-integrable. Then there exists $\alpha_i \in \mathbb{R}$ such that $\pi_i |\cdot|^{\alpha_i}$ is square-integrable. Let σ be a permutation of $\{1, \ldots, r\}$ such that $\alpha_{\sigma(i)} \geq \alpha_{\sigma(j)}$ if $i < j$. Then the induced representation

$$\pi_{\sigma(1)} \times \cdots \times \pi_{\sigma(r)}$$

has a unique irreducible quotient π known as the Langlands quotient. In this setting, we write

$$\pi = L(\pi_1, \ldots, \pi_r).$$

Moreover, any $\pi \in \Pi(G_n)$ can be realized as such a Langlands quotient. In the above situation, we write

$$M(\pi) = \pi_{\sigma(1)} \times \cdots \times \pi_{\sigma(r)}$$

and say that $M(\pi)$ is the standard module of π.

Let W_F be the Weil group associated to F and $\widehat{G}_n(\mathbb{C}) = \mathrm{GL}_n(\mathbb{C})$ be the complex dual group of G_n. Since G_n is split, an L-parameter of G_n may be regarded as a $\widehat{G}_n(\mathbb{C})$-conjugacy class of an admissible homomorphism $\phi : W_F \times \mathrm{SL}_2(\mathbb{C}) \to \widehat{G}_n(\mathbb{C})$ ([12, §8]). Let $\Phi(G_n)$ denote the set of L-parameters of G_n. We do not distinguish a representative ϕ from its conjugacy class. The local Langlands correspondence for G_n is well-understood ([28, 35, 52]). One consequence is that there is a bijection $rec : \Pi(G_n) \to \Phi(G_n)$. The L-packet attached to ϕ is $\Pi_\phi := rec^{-1}(\phi)$. Since the map is a bijection, the L-packet is a singleton. We write $\Pi_\phi = \{\pi_\phi\}$. Conversely, for $\pi \in \Pi(G_n)$, we let $\phi_\pi := rec(\pi)$ denote the L-parameter of π.

Let λ be an infinitesimal character of G_n. Recall that we have

$$\Phi_\lambda(G_n) = \{\phi \in \Phi(G_n) \mid \lambda_\phi = \lambda\}$$

and

$$\Pi_\lambda(G_n) = \{\pi \in \Pi(G_n) \mid \lambda_{\phi_\pi} = \lambda\}$$

and that both of these sets are finite. The Grothendieck group of finite length representations of G_n with infinitesimal parameter λ is denoted by $K\Pi_\lambda(G_n)$. Given $\pi \in \Pi_\lambda(G_n)$, we let $[\pi]$ denote its image in $K\Pi_\lambda(G_n)$. The set $B_{\Pi_\lambda} = \{[\pi] \mid \pi \in \Pi_\lambda(G_n)\}$ forms a $\mathbb{Z}$-basis for $K\Pi_\lambda(G_n)$.

Another basis for $K\Pi_\lambda(G_n)$ is given by $B_{\Pi_\lambda^{std}} = \{[M(\pi)] \mid \pi \in \Pi_\lambda(G_n)\}$. This is a consequence of the Langlands classification above. Suppose that $\Pi_\lambda(G_n) = \{\pi_1, \ldots, \pi_r\}$. For each $j = 1, \ldots, r$, we write

$$[M(\pi_j)] = \sum_{i=1}^{r} m_{ij}[\pi_i],$$

where $m_{ij} \in \mathbb{Z}$. The matrix $m_\lambda = (m_{ij})_{i,j=1}^r$ then defines the change of basis matrix of $K\Pi_\lambda(G_n)$ from $\{[\pi_1], \ldots, [\pi_r]\}$ to $\{[M(\pi_1)], \ldots, [M(\pi_r)]\}$. Based on this observation, given an arbitrary $[\pi] \in K\Pi_\lambda(G_n)$, we define $M([\pi]) := m_\lambda^{-1}[\pi]$. We remark that the ordered bases may be chosen so that m is lower triangular; however, we do not necessarily require this.

4.2. **Perverse Sheaves.** We continue to assume that $G_n = \mathrm{GL}_n(F)$. We recall some notation from §3. Let λ be an infinitesimal character of G_n. The group H_λ acts on the Vogan variety V_λ with finitely many orbits and we let $C_\lambda(G_n)$ denote the collection of these orbits. These orbits are in bijection with $\Phi_\lambda(G_n)$ ([17, Proposition 4.2.2]). For $\phi \in \Phi_\lambda(G_n)$, we let $C_\phi \in C_\lambda(G_n)$ denote the corresponding orbit. Through the orbit closure, we defined a partial order $\geq_C$ on $\Phi_\lambda(G_n)$ (Definition 3.1).

We let $D_{H_\lambda}(V_\lambda)$ denote the H_λ-equivariant derived category of ℓ-adic sheaves on V_λ and $\mathrm{Per}_{H_\lambda}(V_\lambda)$ denote the category of H_λ-equivariant perverse sheaves on V_λ (see [1]). Vogan's perspective on the local Langlands correspondence ([55]) gives a bijection between $\Pi_\lambda(G_n)$ and the simple objects in $\mathrm{Per}_{H_\lambda}(V_\lambda)$ (up to isomorphism). For $\pi \in \Pi_\lambda(G_n)$, we write $\mathcal{P}(\pi)$ for the corresponding simple perverse sheaf. For G_n, it is simple to describe these objects. Namely, $\mathcal{P}(\pi) = \mathcal{IC}(\mathbb{1}_{C_{\phi_\pi}})$, where $\mathbb{1}_{C_{\phi_\pi}}$ denotes the trivial local system on C_{ϕ_π} and $\mathcal{IC}(\cdot)$ denotes the intersection cohomology complex.

Let $K\mathrm{Per}_\lambda(G_n)$ denote the Grothendieck group of $\mathrm{Per}_{H_\lambda}(V_\lambda)$. Given $\mathcal{F} \in \mathrm{Per}_{H_\lambda}(V_\lambda)$, we let $[\mathcal{F}]$ denotes its image in $K\mathrm{Per}_\lambda(G_n)$. Vogan's perspective on the Langlands classification shows that $K\mathrm{Per}_\lambda(G_n)$ has a $\mathbb{Z}$-basis given by $B_{\mathrm{Per}_\lambda} = \{\mathcal{IC}(\mathbb{1}_C) \mid C \in C_\lambda(G_n)\}$.

Let $C \in C_\lambda(G_n)$ and consider the trivial local system $\mathbb{1}_C$. The standard sheaf associated to $\mathbb{1}_C$ is the H_λ-equivariant perverse sheaf $\mathbb{1}_C^\natural$ defined by the property that for $C' \in C_\lambda(G_n)$, we have

$$(\mathbb{1}_C^\natural)|_{C'} = \begin{cases} \mathbb{1}_{C'} & \text{if } C' = C, \\ 0 & \text{otherwise.} \end{cases}$$

The set $B_{\mathrm{Per}_\lambda}^\natural = \{\mathbb{1}_C^\natural \mid C \in C_\lambda(G_n)\}$ forms a $\mathbb{Z}$-basis for $K\mathrm{Per}_\lambda(G_n)$.

Let $C_\lambda(G_n) = \{C_1, \ldots, C_r\}$. For $C \in C_\lambda(G_n)$, let $d(C) := \dim C$. Write

$$[\mathcal{IC}(\mathbb{1}_{C_j})] = (-1)^{d(C_j)} \sum_{i=1}^{r} c_{ij} [\mathbb{1}_{C_i}^\natural].$$

The matrix $c_\lambda = (c_{ij})_{i,j=1}^r$ gives the change of basis matrix of $K\mathrm{Per}_\lambda(G_n)$ from the ordered basis $\{[\mathbb{1}_{C_1}^\natural], \ldots, [\mathbb{1}_{C_r}^\natural]\}$ to

$$\{[(-1)^{d(C_1)}\mathcal{IC}(\mathbb{1}_{C_1})], \ldots, [(-1)^{d(C_r)}\mathcal{IC}(\mathbb{1}_{C_r})]\}.$$

Based on this observation, given an arbitrary $[\mathcal{F}] \in K\mathrm{Per}_\lambda(G_n)$, we define $[\mathcal{F}^\natural] := c_\lambda[\mathcal{F}]$.

The p-adic analogue of the Kazhdan-Lusztig hypothesis relates the change of basis matrices m_λ and c_λ. For $G_n = \mathrm{GL}_n(F)$, the Kazhdan-Lusztig hypothesis is known (see [15, 44, 53]).

Theorem 4.7 (The Kazhdan-Lusztig hypothesis)**.** *We have* $m_\lambda = {}^t c_\lambda$.

Now, we introduce a perfect pairing between the Grothendieck groups above. We define

$$\langle \cdot, \cdot \rangle : K\Pi_\lambda(G_n) \times K\mathrm{Per}_\lambda(G_n) \to \mathbb{Z},$$

by defining it on the basis $B_{\Pi_\lambda} \times B_{\mathrm{Per}_\lambda}$ via

$$\langle [\pi], [\mathcal{F}] \rangle = \begin{cases} (-1)^{d(\pi)} & \text{if } \mathcal{F} = \mathcal{P}(\pi), \\ 0 & \text{otherwise,} \end{cases} \tag{4.1}$$

where $d(\pi) := \dim C_{\phi_\pi}$, and extending linearly. The Kazhdan-Lusztig hypothesis (Theorem 4.7) gives the pairing on the dual basis $B_{\Pi_\lambda}^{std} \times B_{\mathrm{Per}_\lambda}^\natural$.

Lemma 4.8 ([20, Lemma 1.2])**.** *For* $[M(\pi)] \in B^{std}_{\Pi_\lambda}$ *and* $[\mathbb{1}^\natural_C] \in B^\natural_{\mathrm{Per}_\lambda}$*, we have*

$$\langle [M(\pi)], [\mathbb{1}^\natural_C]\rangle = \begin{cases} 1 & \text{if } \mathcal{IC}(\mathbb{1}_C) = \mathcal{P}(\pi), \\ 0 & \text{otherwise.} \end{cases}$$

For $C \in C_\lambda(G_n)$, Cunningham et al. attach an element $\eta^{\mathrm{Evs}}_C \in K\Pi_\lambda(G_n)$ ([17, §8.4]). In our setting, we have

$$\eta_C := \eta^{\mathrm{Evs}}_C = (-1)^{d(C)} \sum_{\pi \in \Pi_\lambda(G_n)} (-1)^{d(\pi)} \mathrm{rank}(\mathrm{Evs}_C(\mathcal{P}(\pi)))[\pi],$$

where $d(C) := \dim C$ and Evs is the functor on perverse sheaves defined in [17, §7.9]. See also Equation (A.1). Let $\phi \in \Phi_\lambda(G_n)$. We set $\eta_\phi = \eta^{\mathrm{Evs}}_{C_\phi}$. Recall that

$$\Pi^{\mathrm{ABV}}_\phi = \{\pi \in \Pi_\lambda(G_n) \mid \mathrm{Evs}_{C_\phi}(\mathcal{P}(\pi)) \neq 0\}.$$

Thus, it follows that we may use the pairing of the Grothendieck groups and η_ϕ to determine Π^{ABV}_ϕ.

Lemma 4.9 ([20, Proposition 1.6])**.** *We have that* $\pi \in \Pi^{\mathrm{ABV}}_\phi$ *if and only if*

$$\langle \eta_\phi, [\mathcal{P}(\pi)]\rangle \neq 0.$$

We remark the above lemma follows simply from observing that [20, Proposition 1.6] holds for a general L-parameter, rather than an Arthur parameter.

We also remark that the Kazhdan-Lusztig hypothesis provides a way to pass compute the above pairing using the different bases.

Lemma 4.10. *For any* $[\mathcal{F}] \in K\mathrm{Per}_\lambda(G_n)$*, we have*

$$\langle \eta_\phi, [\mathcal{F}]\rangle_\lambda = \langle M(\eta_\phi), [\mathcal{F}^\natural]\rangle_\lambda$$

Proof. Recall that for any $[\pi] \in K\Pi_\lambda(G)$, we have $[M(\pi)] := m_\lambda^{-1}[\pi]$. Similarly, for $[\mathcal{F}] \in K\mathrm{Per}_\lambda(G_n)$, we have $[\mathcal{F}^\natural] = c_\lambda[\mathcal{F}]$. Furthermore, by the Kazhdan-Lusztig hypothesis (Theorem 4.7), we have $m_\lambda = {}^t c_\lambda$. Thus, we obtain

$$\begin{aligned} \langle \eta_\phi, [\mathcal{F}]\rangle_\lambda &= \langle {}^t c_\lambda m_\lambda^{-1} \eta_\phi, [\mathcal{F}]\rangle_\lambda \\ &= \langle m_\lambda^{-1}\eta_\phi, c_\lambda[\mathcal{F}]\rangle_\lambda \\ &= \langle M(\eta_\phi), [\mathcal{F}^\natural]\rangle_\lambda \end{aligned}$$

which proves the lemma. □

Next, we show that the contragredient preserves ABV-packets for $\mathrm{GL}_n(F)$. In general, we have that $\phi^\vee(w,x) = {}^t\phi(w,x)^{-1}$ for $w \in W_F$ and $x \in \mathrm{SL}_2(\mathbb{C})$. Let $\lambda = \lambda_\phi$ and $\lambda^\vee = \lambda_{\phi^\vee}$. We relate V_λ and $V_{\lambda^\vee}$ using the framework introduced in [17, Section 10.2.1].

First, by [17, Theorem 5.1.1], we may assume that λ is unramified, i.e., trivial on I_F, and $\chi(\lambda(\mathrm{Fr})) \in \mathbb{R}_{>0}$ for any character $\chi : \widehat{T} \to \mathrm{GL}_1(\mathbb{C})$, where $\widehat{T}$ is any torus in $\mathrm{GL}_n(\mathbb{C})$ containing $\lambda(\mathrm{Fr})$. Consequently, we may write

$$\lambda = m_1|\cdot|^{x_1} + m_2|\cdot|^{x_2} + \cdots + m_r|\cdot|^{x_r}$$

where $m_i \in \mathbb{Z}_{\geq 1}$ denotes the multiplicity and $x_i \in \mathbb{R}$ with $x_i > x_{i+1}$ for $i = 1, \ldots, r-1$. Since $y \in V_\lambda$ if and only if $\mathrm{Ad}(\lambda(\mathrm{Fr}))y = q_F y$, we may assume that $x_i - 1 = x_{i+1}$ for $i = 1, 2, \ldots, r-1$ (otherwise the Vogan variety decomposes as a product of such Vogan varieties). For $i = 1, \ldots, r$, let E_i denote the $q_F^{x_i}$-eigenspace of $\lambda(\mathrm{Fr})$. We have $m_i = \dim(E_i)$. Furthermore, we have that

$$V_\lambda \cong \mathrm{Hom}(E_1, E_2) \times \mathrm{Hom}(E_2, E_3) \times \cdots \times \mathrm{Hom}(E_{r-1}, E_r). \tag{4.2}$$

In this setting, we have

$$H_\lambda \cong \mathrm{GL}(E_1) \times \mathrm{GL}(E_2) \times \cdots \times \mathrm{GL}(E_r).$$

Let

$$y = (y_1, \ldots, y_{r-1}) \in \mathrm{Hom}(E_1, E_2) \times \cdots \times \mathrm{Hom}(E_{r-1}, E_r) \cong V_\lambda.$$

The H_λ-orbit of y is the set of

$$z = (z_1, \ldots, z_{r-1}) \in \mathrm{Hom}(E_1, E_2) \times \cdots \times \mathrm{Hom}(E_{r-1}, E_r)$$

such that $\mathrm{rank}(y_i \circ y_{i+1} \circ \cdots \circ y_j) = \mathrm{rank}(z_i \circ z_{i+1} \circ \cdots \circ z_j)$ for any $1 \leq i \leq j \leq r-1$.

Now, note that $\lambda^\vee = m_1|\cdot|^{-x_1} + m_2|\cdot|^{-x_2} + \cdots + m_r|\cdot|^{-x_r}$. For $i = 1, \ldots, r$, let $w_i = -x_{r-i+1}$ and F_i denote the $q_F^{w_i}$-eigenspace of $\lambda^\vee(\mathrm{Fr})$. Analogously to the above discussion, we have

$$V_{\lambda^\vee} \cong \mathrm{Hom}(F_1, F_2) \times \mathrm{Hom}(F_2, F_3) \times \cdots \times \mathrm{Hom}(F_{r-1}, F_r).$$

Of course, for $i = 1, \ldots, r$, we also have $F_i \cong E_{r-i+1}$ and thus

$$V_{\lambda^\vee} \cong \mathrm{Hom}(E_r, E_{r-1}) \times \mathrm{Hom}(E_{r-1}, E_{r-2}) \times \cdots \times \mathrm{Hom}(E_2, E_1).$$

Similarly, we have

$$H_{\lambda^\vee} \cong \mathrm{Hom}(E_r) \times \mathrm{Hom}(E_{r-1}) \times \cdots \times \mathrm{Hom}(E_1).$$

This is a reflection of $\phi^\vee = {}^t\phi^\vee$ on the Vogan variety. Indeed, let

$$y = (y_1, \ldots, y_{r-1}) \in \mathrm{Hom}(E_1, E_2) \times \cdots \times \mathrm{Hom}(E_{r-1}, E_r) \cong V_\lambda.$$

Define

$$y^\vee := ({}^t y_{r-1}, \ldots, {}^t y_1) \in \mathrm{Hom}(E_r, E_{r-1}) \times \cdots \times \mathrm{Hom}(E_2, E_1).$$

The map $y \mapsto y^\vee$ induces an isomorphism $V_\lambda \cong V_{\lambda^\vee}$. Furthermore, it sends the orbit C_ϕ to $C_{\phi^\vee}$. Indeed, this follows from simply computing the ranks.

In summary, we have an isomorphism $V_\lambda \cong V_{\lambda^\vee}$ which sends C_ϕ to $C_{\phi^\vee}$. Since the underlying geometry is the same up to isomorphism, we obtain the following lemma.

Lemma 4.11. *Suppose that $\pi \in \Pi_\phi^{\mathrm{ABV}}$. Then $\pi^\vee \in \Pi_{\phi^\vee}^{\mathrm{ABV}}$.*

We note that this is expected more generally; however, the contragredient may permute the elements in an L-packet (see [38]). For $\mathrm{GL}_n(F)$, we avoided this issue as the L-packets are all singletons. Here is a simple example illustrating the above ideas.

Example 4.12. *Suppose that $\phi = |\cdot|^{\frac{3}{2}} + |\cdot|^{\frac{1}{2}}$. Then $\phi^\vee = |\cdot|^{-\frac{1}{2}} + |\cdot|^{-\frac{3}{2}}$. We have that*

$$V_{\lambda_\phi} = \left\{ \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} \mid y \in \mathbb{C} \right\} = V_{\lambda_{\phi^\vee}}.$$

The eigenvalues of $\lambda(\mathrm{Fr})$ are $q_F^{x_i}$ where $x_1 = \frac{3}{2}$ and $x_2 = \frac{1}{2}$. We have that the eigenspaces of both eigenvalues are 1-dimensional and hence

$$V_\lambda \cong \mathrm{Hom}(\mathbb{C}, \mathbb{C})$$
$$\begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} \mapsto y,$$

where we consider $y \in \mathbb{C}$ as the linear transformation defined by $y(z) = yz$ for any $z \in \mathbb{C}$.

Similarly, we have that

$$V_{\lambda_{\phi^\vee}} = \left\{ \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} \mid y \in \mathbb{C} \right\}.$$

The eigenvalues of $\lambda^\vee(\mathrm{Fr})$ are are $q_F^{w_i}$ where $w_1 = -\frac{1}{2} = -x_2$ and $w_2 = -\frac{3}{2} = -x_1$. We have that the eigenspaces of both eigenvalues are 1-dimensional and hence

$$V_{\lambda^\vee} \cong \mathrm{Hom}(\mathbb{C}, \mathbb{C})$$
$$\begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} \mapsto y,$$

where we consider $y \in \mathbb{C}$ as the linear transformation defined by $y(z) = yz$ for any $z \in \mathbb{C}$. The map $y \mapsto y^\vee$ is simply the identity map and hence $V_{\lambda_\phi} \cong V_{\lambda_{\phi^\vee}}$ is also the identity map.

Furthermore, the L-parameter ϕ corresponds to the 0-orbit in V_{λ_ϕ}. This corresponds to $y = 0$. We have $0^\vee = 0$ which corresponds to the L-parameter $\phi^\vee$.

Our next goal is to state a fixed point formula (Theorem 4.33) which will be the key step in our proof of Theorem 4.34.

For $i = 1, \dots, r$, let $\phi_i \in \Phi(G_{n_i})$, and $n = n_1 + \cdots + n_r$. We set $G^\times := G_{n_1} \times \cdots \times G_{n_r}$ and $\phi^\times = \phi_1 \times \cdots \times \phi_r$. Note that $\widehat{G}^\times(\mathbb{C}) = \widehat{G}_{n_1}(C) \times \cdots \times \widehat{G}_{n_r}(\mathbb{C})$ and $\Pi_{\phi^\times} = \Pi_{\phi_1} \times \cdots \times \Pi_{\phi_r}$. We also let $\lambda^\times = \lambda_1 \times \cdots \times \lambda_r$ be the corresponding infinitesimal parameter, where $\lambda_i = \lambda_{\phi_i}$. Furthermore, we have that the Vogan variety is $V_{\lambda^\times} = V_{\lambda_1} \times \cdots \times V_{\lambda_r}$ and $H_{\lambda^\times} = H_{\lambda_1} \times \cdots \times H_{\lambda_r}$. There is an action of $H_{\lambda^\times}$ on $V_{\lambda^\times}$ in the obvious manner. Alternatively, these could be directly computed from the definitions in [17, §4].

We let $\phi = \phi_1 + \cdots + \phi_r \in \Phi(G_n)$ and $\lambda = \lambda_\phi$. Let $s \in \widehat{G}_n(\mathbb{C})$ be of finite order (and hence semi-simple) such that $Z_{\widehat{G}_n(\mathbb{C})}(s) \cong \widehat{G}^\times$. The resulting inclusion $\widehat{G}^\times \hookrightarrow \widehat{G}_n(\mathbb{C})$ induces inclusions $H_{\lambda^\times} \hookrightarrow H_\lambda$ and

$$\varepsilon : V_{\lambda^\times} \hookrightarrow V_\lambda$$

which is equivariant for the action by $H_{\lambda^\times}$. Indeed, we have that

$$V_{\lambda^\times} = V_{\lambda_\alpha}^s := \{x \in V_\lambda \mid \mathrm{Ad}(s)x = x\}.$$

Let $\varepsilon^* : \mathrm{D}_{H_{\lambda_\alpha}}(V_{\lambda_\alpha}) \to \mathrm{D}_{H_{\lambda^\times}}(V_{\lambda^\times})$ denote the equivariant restriction functor for the equivariant derived categories. As a shorthand, we write

$$\mathcal{F}|_{V_{\lambda^\times}} := \varepsilon^*\mathcal{F}.$$

We note that ε^* is an exact functor, but does not preserve perverse sheaves.

We define a special case of endoscopic lifting (see [3, Definition 26.18] or [20, §4]) to be the linear transformation

$$\mathrm{Lift}_{G^\times}^{G_n} : K\Pi_{\lambda^\times}(G^\times) \to K\Pi_{\lambda_\alpha}(G_n)$$

defined by

$$\langle \mathrm{Lift}_{G^\times}^{G_n}[\pi], [\mathcal{F}]\rangle_\lambda = \langle [\pi], [\varepsilon^*\mathcal{F}]\rangle_{\lambda^\times}.$$

In this setting, the endoscopic lifting is simple to describe.

Proposition 4.13 ([20, Proposition 4.5])**.** *We continue with the above notation. Let $[\pi] \in K\Pi_{\lambda^\times}(G^\times)$ and P be the standard parabolic subgroup of G_n whose Levi subgroup is isomorphic to $G^\times$. Then*

$$\mathrm{Lift}_{G^\times}^{G_n}[\pi] = [\mathrm{Ind}_P^{G_n}\pi].$$

An equation of the form $\langle \eta_\phi, [\mathcal{F}]\rangle_\lambda = \langle \eta_{\phi^\times}, [\mathcal{F}|_{V_{\lambda^\times}}]\rangle_{\lambda^\times}$ is called a fixed point formula as it is usually obtained from a Lefschetz fixed point formula, e.g., [3, Theorem 25.8]. Note that it is equivalent to $\mathrm{Lift}_{G^\times}^{G_n}(\eta_{\phi^\times}) = \eta_\phi$. The Kazhdan-Lusztig hypothesis provides an equivalent formulation.

Corollary 4.14. *We continue with the above notation. That is, we let $\phi^\times = \phi_1 \times \cdots \times \phi_r \in \Phi(G^\times)$, $\phi = \phi_1 + \cdots + \phi_r \in \Phi(G_n)$ and $\lambda^\times$, resp. λ, be the infinitesimal parameter of $\phi^\times$, resp. ϕ. Then, for any $[\mathcal{F}] \in K\mathrm{Per}_\lambda(G_n)$, we have*

$$\langle \eta_\phi, [\mathcal{F}]\rangle_\lambda = \langle \eta_{\phi^\times}, [\mathcal{F}|_{V_{\lambda^\times}}]\rangle_{\lambda^\times}$$

if and only if

$$\langle M(\eta_\phi), [\mathcal{F}^\natural]\rangle_\lambda = \langle M(\eta_{\phi^\times}), [\mathcal{F}^\natural|_{V_{\lambda^\times}}]\rangle_{\lambda^\times}.$$

Proof. By Lemma 4.10, we obtain

$$\langle \eta_\phi, [\mathcal{F}]\rangle_\lambda = \langle M(\eta_\phi), [\mathcal{F}^\natural]\rangle_\lambda$$

and

$$\langle \eta_{\phi^\times}, [\mathcal{F}|_{V_{\lambda^\times}}]\rangle_{\lambda^\times} = \langle M(\eta_{\phi^\times}), [\mathcal{F}^\natural|_{V_{\lambda^\times}}]\rangle_{\lambda^\times}.$$

The corollary follows directly. □

Let $\phi \in \Phi(G_n)$. Recall that

$$\phi_\alpha = \phi^\vee \oplus \left(\bigoplus_{i=0}^{\alpha-1} |\cdot|^{\frac{\alpha-1}{2}-i} \otimes S_1 \right).$$

Let $\phi^\alpha = \bigoplus_{i=0}^{\alpha-1} |\cdot|^{\frac{\alpha-1}{2}-i} \otimes S_1$. Then $\phi_\alpha = \phi^\vee + \phi^\alpha$ and we let $\phi^\times = \phi^\vee \times \phi^\alpha$. Note that $\phi^\vee \in \Phi(G_n)$, $\phi^\alpha \in \Phi(G_\alpha)$, and $\phi_\alpha \in \Phi(G_m)$. Let $\lambda_\alpha = \lambda_{\phi_\alpha}$ and $\lambda^\times = \lambda_{\phi^\vee} \times \lambda_{\phi^\alpha}$. Per the above discussion, we have inclusions $H_{\lambda^\times} \hookrightarrow H_{\lambda_\alpha}$ and

$$\varepsilon : V_{\lambda^\times} \hookrightarrow V_{\lambda_\alpha}$$

which is equivariant for the action by $H_{\lambda^\times}$. We work towards showing that for any $[\mathcal{F}] \in K\mathrm{Per}_\lambda(G_m)$, we have (Theorem 4.33)

$$\langle \eta_{\phi_\alpha}, [\mathcal{F}] \rangle_{\lambda_\alpha} = \langle \eta_{\phi^\times}, [\mathcal{F}|_{V_{\lambda^\times}}] \rangle_{\lambda^\times}. \tag{4.3}$$

4.3. **Conormal bundles.** The results of this subsection hold more generally than just for general linear groups. Consequently, in this subsection, we allow for $G = G_n = G(W_n)$ to be any of the classical groups in §2 or $G_n = \mathrm{GL}_n(F)$. We also remark that if G_n is disconnected, e.g., metaplectic or even orthogonal, these results should be taken with a grain of salt as [17] only considers connected groups.

Let $\phi \in \Phi(G_n)$ and $\lambda = \lambda_\phi$. Let $\phi \in \Phi(G_n)$ and $\lambda = \lambda_\phi$. We let V_λ^* denote the dual Vogan variety to V_λ. By considering

$${}^tV_\lambda := \{x \in \mathrm{Lie}(K_\lambda) \mid \mathrm{Ad}(\lambda(\mathrm{Fr}))x = q_F^{-1}x\}$$

we identify the dual Vogan variety $V_\lambda^* \cong {}^tV_\lambda$ hereinafter ([17, Proposition 6.2.1]).

The conormal bundle is denoted by

$$\Lambda_\lambda := \{(x,y) \in V_\lambda \times V_\lambda^* \mid [x,y] = 0\},$$

where $[\cdot,\cdot]$ denotes the Lie bracket ([17, Proposition 6.2]).

We define

$$\Lambda_{C_\phi} := \{(x,y) \in C_\phi \times V_\lambda^* \mid [x,y] = 0\}.$$

For an H_λ-orbit B of V_λ^*, we consider

$$\Lambda_B := \{(y,x) \in B \times V_\lambda \mid [y,x] = 0\}.$$

By [17, Lemma 6.5], there exists a unique H_λ-orbit of V_λ^*, denoted $(C_\phi)^*$ such that

$$\overline{\Lambda}_{C_\phi} = \overline{\Lambda}_{(C_\phi)^*}.$$

We say that $(C_\phi)^*$ is the dual orbit to C_ϕ. The H_λ orbits of V_λ^* are also in bijection with Φ_λ. We let $\hat{\phi}$ be the L-parameter (called the Pyatetskii dual) corresponding to $(C_\phi)^*$. We define the regular part of the conormal bundle of C_ϕ to be

$$\Lambda_{C_\phi}^{reg} := \Lambda_{C_\phi} \setminus \bigcup_{\substack{C' \\ C_\phi \subsetneq \overline{C'}}} \overline{\Lambda}_{C'}.$$

Consider the L-parameter $\phi^\alpha = \oplus_{i=0}^{\alpha-1}\chi_W|\cdot|^{\frac{\alpha-1}{2}-i}\otimes S_1$ which corresponds to the 0-orbit in $V_{\lambda_{\phi^\alpha}}$. Let $x_{\phi^\alpha}=0$ and so $\Lambda_{x_{\phi^\alpha}} = \Lambda_{C_{\phi^\alpha}} \cong V^*_{\lambda_{\phi^\alpha}}$. It follows that $\Lambda^{reg}_{C_{\phi^\alpha}}$ is the set of $(0,y)\in\Lambda$ where $y\in {}^tC_{\chi_W\otimes S_\alpha}$ (this is the unique open orbit in $V^*_{\lambda_{\phi^\alpha}}$). In particular, this set is nonempty. Let $(x_{\phi^\alpha}, y_{\phi^\alpha})\in\Lambda^{reg}_{C_{\phi^\alpha}}$ be arbitrary. For $\phi^\vee$, we let $(x_{\phi^\vee}, y_{\phi^\vee})\in\Lambda^{reg}_{C^\vee_\phi}$ also be arbitrary.

Recall that $\phi_\alpha = \phi^\vee + \phi^\alpha$. Let $\phi^\times = \phi^\vee\times\phi^\alpha$ and $\lambda^\times = \phi^\times$. We have that $V_{\lambda^\times} = V_{\lambda_{\phi^\vee}}\times V_{\lambda_{\phi_2}}$ Consider the embeddings $\varepsilon : V_{\lambda^\times}\hookrightarrow V_{\lambda_{\phi_\alpha}}$ and ${}^t\varepsilon : V^*_{\lambda^\times}\hookrightarrow V^*_{\lambda_{\phi_\alpha}}$. We also consider $\varepsilon' = \varepsilon\times{}^t\varepsilon$.

To establish the fixed point formula (4.3), we must find $(x,y)\in\Lambda^{reg}_{C_{\phi^\times}}$ such that $\varepsilon'(x,y)\in\Lambda^{reg}_{C_{\phi_\alpha}}$. Next, we provide two running examples which will explain why we require $\alpha\gg 0$ later. First is the example where our strategy will succeed.

Example 4.15. *Let $G=\mathrm{GL}_2(F)$, $\alpha=2$, Let $\phi=\phi^\vee = |\cdot|^{\frac{1}{2}} + |\cdot|^{\frac{-1}{2}}$, and $\phi^\alpha=\phi$ (so our theta lift is from $\mathrm{GL}_2(F)$ to $\mathrm{GL}_4(F)$). Let $\lambda=\lambda_\phi$. The Vogan varieties for $V_\lambda = V_{\lambda_{\phi^\vee}} = V_{\lambda_{\phi^\alpha}}$ are given by*

$$V_\lambda = \begin{pmatrix} 0 & x\\ 0 & 0\end{pmatrix}.$$

We have $H_\lambda = \mathrm{GL}_1(\mathbb{C})\times\mathrm{GL}_1(\mathbb{C})$ with action given by $(h_1,h_2)\cdot x\mapsto \frac{h_1}{h_2}x$. There are 2 orbits, the 0-orbit and the open orbit ($x\neq 0$). We let C denote the 0-orbit which corresponds to $\phi^\vee=\phi^\alpha$. Also, we have

$$V^*_\lambda = \begin{pmatrix} 0 & 0\\ y & 0\end{pmatrix}$$

with the action of H_λ given by $(h_1,h_2)\mapsto\frac{h_2}{h_1}y$. A choice of $(x,y)\in\Lambda^{reg}_C$ is given by $x=0, y=\begin{pmatrix} 0 & 0\\ 1 & 0\end{pmatrix}$. Now $\phi_\alpha = 2\phi$. (note that $\alpha=2$). It corresponds to the 0-orbit C_α in

$$V_{\lambda_{\phi_\alpha}} = \left\{\begin{pmatrix} 0 & x\\ 0 & 0\end{pmatrix} \mid x\in\mathrm{Mat}_{2\times 2}(\mathbb{C})\right\}.$$

We have $H_{\lambda_{\phi_\alpha}} = \mathrm{GL}_2(\mathbb{C})\times\mathrm{GL}_2(\mathbb{C})$ with action given by

$$(a,b)\cdot\begin{pmatrix} 0 & x\\ 0 & 0\end{pmatrix}\mapsto\begin{pmatrix} 0 & axb^{-1}\\ 0 & 0\end{pmatrix}.$$

We have that

$$V^*_{\lambda_{\phi_\alpha}} = \left\{\begin{pmatrix} 0 & 0\\ y & 0\end{pmatrix} \mid y\in\mathrm{Mat}_{2\times 2}(\mathbb{C})\right\}.$$

We have $(x_{\phi_\alpha}, y_{\phi_\alpha})\in\Lambda^{reg}_{C_\alpha}$ where $x_{\phi_\alpha}=0$ and

$$y_{\phi_\alpha} = \begin{pmatrix} 0 & 0\\ \begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix} & 0\end{pmatrix}.$$

The embedding of $V_{\lambda_{\phi^\vee}} \times V_{\lambda_{\phi^\alpha}}$ *into* $V_{\lambda_{\phi_\alpha}}$ *is given by*

$$\left(\begin{pmatrix}0 & x_1\\0 & 0\end{pmatrix}, \begin{pmatrix}0 & x_2\\0 & 0\end{pmatrix}\right) \mapsto \begin{pmatrix}0 & x\\0 & 0\end{pmatrix},$$

where

$$x = \begin{pmatrix}x_1 & 0\\0 & x_2\end{pmatrix}.$$

The embedding of $V^*_{\lambda_\phi} \times V^*_{\lambda_{\phi^\alpha}}$ *into* $V^*_{\lambda_{\phi_\alpha}}$ *is given similarly by taking the transpose. Thus we see that the image of* $((x_\phi, y_\phi), (x_\phi, y_\phi)) \in \Lambda^{reg}_C \times \Lambda^{reg}_C$ *is precisely* $(x_{\phi_\alpha}, y_{\phi_\alpha}) \in \Lambda^{reg}_{C_\alpha}$ *as desired.*

The next example is where we see that the condition $\alpha \gg 0$ will be needed in our strategy.

Example 4.16. *Let* $G = \mathrm{GL}_2(F)$, $\alpha = 2$, $\phi = \phi^\vee = |\cdot|^{\frac{3}{2}} + |\cdot|^{\frac{-3}{2}}$, *and* $\phi^\alpha = |\cdot|^{\frac{1}{2}} + |\cdot|^{\frac{-1}{2}}$. *The geometry for the Vogan variety of* ϕ^α *is the same as in Example 4.15. Let* $\lambda = \lambda_\phi = \lambda_{\phi^\vee}$. *Then* $V_\lambda = \left\{\begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}\right\}$. *The group* H_λ *is isomorphic to* $\mathrm{GL}_1(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C})$, *but the action is trivial. Consequently, we have that* $\Lambda_{C_{\phi^\vee}} = \Lambda^{reg}_{C_{\phi^\vee}}$ *is the singleton* $(x_{\phi^\vee}, y_{\phi^\vee})$ *where* $x_{\phi^\vee} = y_{\phi^\vee} = \begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}$.

On the other hand $\phi_\alpha = \phi^\vee + \phi^\alpha = |\cdot|^{\frac{3}{2}} + |\cdot|^{\frac{1}{2}} + |\cdot|^{\frac{-1}{2}} + |\cdot|^{\frac{-3}{2}}$. *Let* $\lambda_\alpha = \lambda_{\phi_\alpha}$. *The Vogan variety is*

$$V_{\lambda_\alpha} = \left\{\begin{pmatrix}0 & a & & \\ & 0 & b & \\ & & 0 & c\\ & & & 0\end{pmatrix} \mid a, b, c \in \mathbb{C}\right\}.$$

The group H_λ *is the standard torus of* $\mathrm{GL}_4(\mathbb{C})$, *i.e. it is isomorphic to* $\mathrm{GL}_1(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C})$, *and its action is given by the usual simple roots. The* 0*-orbit in* V_{λ_α} *corresponds to* ϕ_α.

However, the dual orbit of C_{ϕ_α} *is the unique open orbit corresponding to the tempered parameter* S_4. *That is,*

$$C^*_{\lambda_\alpha} = \left\{\begin{pmatrix}0 & & & \\ a & 0 & & \\ & b & 0 & \\ & & c & 0\end{pmatrix} \mid a, b, c \in \mathbb{C}^\times\right\}.$$

Note that by [17, Lemma 6.4.2], *we have* $\Lambda^{reg}_{C_{\phi_\alpha}} \subseteq C_{\phi_\alpha} \times C^*_{\phi_\alpha}$. *The embedding* $\varepsilon : V_\lambda \times V_{\lambda_{\phi^\alpha}} \hookrightarrow V_{\lambda_\alpha}$ *is given by*

$$\left(\begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}, \begin{pmatrix}0 & x\\0 & 0\end{pmatrix}\right) \mapsto \begin{pmatrix}0 & & & \\ & 0 & x & \\ & & 0 & \\ & & & 0\end{pmatrix}.$$

The map ${}^t\varepsilon : V^*_\lambda \times V^*_{\lambda_{\phi^\alpha}} \hookrightarrow V^*_{\lambda_\alpha}$ *is given by taking the transpose of these matrices. Consequently, the image of* ${}^t\varepsilon$ *does not intersect with* $C^*_{\lambda_\alpha}$*. That is, in contrast with Example 4.15, there does not exist an element* $(x,y) \in \Lambda^{reg}_{C_{\phi^\times}}$ *such that* $\varepsilon'(x,y) \in \Lambda^{reg}_{C_{\phi_\alpha}}$*. The reason for this is because* $\alpha = 2$ *is too small. We will discuss this example more in Example 4.27.*

We return to the general setting and relate the conormal bundles of $V_{\lambda_{\phi_\alpha}}$ with those of $V_{\lambda^\times}$.

Lemma 4.17. *Let* C *be an orbit of* $V_{\lambda_{\phi_\alpha}}$*. Then*

$$\Lambda_C \cap (V_{\lambda^\times} \times V^*_{\lambda^\times}) = \cup_{C'} \Lambda_{C'},$$

where the union is over all orbits C' *of* $V_{\lambda^\times}$ *such that* $C' \subseteq C \cap V_{\lambda^\times}$*.*

Proof. Let $(x,y) \in \Lambda_C \cap (V_{\lambda^\times} \times V^*_{\lambda^\times})$. Then there exists $(x_1,x_2) \in V_{\lambda^\times}$ and $(y_1,y_2) \in V^*_{\lambda^\times}$ such that $\varepsilon(x_1,x_2) = x$ and ${}^t\varepsilon(y_1,y_2) = y$. It follows that $[x,y] = 0$ if and only if $[x_1,y_1] = 0$ and $[x_2,y_2] = 0$. Also, $x \in C \cap V_{\lambda^\times}$ if and only if $(x_1,x_2) \in C'$ for some orbit C' of $V_{\lambda^\times}$ such that $C' \subseteq C \cap V_{\lambda^\times}$. The lemma follows directly from these observations. □

To study the relations between $\Lambda^{reg}_{C_{\phi^\times}}$ and $\Lambda^{reg}_{C_{\phi_\alpha}}$, it is necessary to study how the closures of conormal bundles behave with respect to restriction. We begin by recalling the notion of a H_{λ_α}-component in Λ_{λ_α}.

Definition 4.18. *A subset* $X \subseteq \Lambda_{\lambda_\alpha}$ *is called an* H_{λ_α}*-component if* X *is a minimal* H_{λ_α}*-invariant union of irreducible components.*

We also define a relative version as follows.

Definition 4.19. *A subset* $X \subseteq \Lambda_{\lambda^\times}$ *is called an* H_{λ_α}*-component if* X *is a minimal union of irreducible components such that* $(H_{\lambda_\alpha} X) \cap \Lambda_{\lambda^\times} = X$*. Here, we are identify these sets inside* Λ_{λ_α} *via* ε'*.*

We will connect these notions later. First, we recall a lemma of [3].

Lemma 4.20 ([3, Lemma 19.2(b)])**.** *Let* λ *be an infinitesimal parameter of* G *and* C *be an orbit of the Vogan variety* V_λ*. Then*

(1) $\dim \Lambda_C = \dim V_\lambda$,
(2) Λ_C *is* H_λ*-irreducible, i.e.,* H_λ *permutes the irreducible components of* Λ_C *transitively, and*
(3) *the* H_λ*-components of* Λ_λ *are the closures* $\overline{\Lambda}_{C'}$ *where* $C' \in C_\lambda(G)$*.*

Note that Parts (1) and (2) of the above lemma imply Part (3). Our next goal is to classify the H_{λ_α}-components of $\Lambda_{\lambda^\times}$.

Lemma 4.21. *The* H_{λ_α}*-components of* $\Lambda_{\lambda^\times}$ *are* $\overline{\Lambda_C \cap \Lambda_{\lambda^\times}}$ *where* C *is an orbit of* V_{λ_α} *for which* $C \cap V_{\lambda^\times} \neq \emptyset$*. Note that by Lemma 4.17, these sets are described by*

$$\overline{\Lambda_C \cap \Lambda_{\lambda^\times}} = \overline{\Lambda}_C \cap (V_{\lambda^\times} \times V^*_{\lambda^\times}) = \cup_{C'} \overline{\Lambda}_{C'}.$$

Proof. Suppose that C is an orbit of V_{λ_α} for which $C \cap V_{\lambda^\times} \neq \emptyset$. By Lemma 4.17, we have

$$\Lambda_C \cap \Lambda_{\lambda^\times} = \Lambda_C \cap (V_{\lambda^\times} \times V^*_{\lambda^\times}) = \cup_{C'} \Lambda_{C'},$$

where the union runs through $C' \subseteq C \cap V_{\lambda^\times}$. By assumption, there is at least one such orbit and so this set is nonempty. Fix such an orbit C'. By Lemma 4.20(1), $\dim \Lambda_{C'} = \dim V_{\lambda^\times}$ and hence $\dim \Lambda_C \cap \Lambda_{\lambda^\times} = \dim V_{\lambda^\times}$. Now, $H_{\lambda^\times}$ permutes the irreducible components of $\Lambda_{C'}$ transitively by Lemma 4.20(2). Furthermore, $\Lambda_{C''} \subseteq H_{\lambda_\alpha} \Lambda_{C'}$ for any $C'' \subseteq C \cap V_{\lambda^\times}$ and hence H_{λ_α} also permutes the irreducible components of $\Lambda_C \cap \Lambda_{\lambda^\times}$ transitively. Therefore, $\overline{\Lambda_C \cap \Lambda_{\lambda^\times}}$ is an H_{λ_α}-component of $\Lambda_{\lambda^\times}$.

The fact that all H_{λ_α}-components are of this form follows from the fact that

$$\Lambda_{\lambda^\times} = \cup_{C'} \overline{\Lambda}_{C'}.$$

Indeed, each $\overline{\Lambda}_{C'}$ lies in $\overline{\Lambda_C \cap \Lambda_{\lambda^\times}}$ where $\varepsilon(C') \subseteq C$ and the claim follows from the minimality of H_{λ_α}-components. □

We have an immediate corollary on the restrictions of closures of conormal bundles.

Corollary 4.22. *Let C be an orbit of $V_{\lambda_{\phi_\alpha}}$. Suppose that $C \cap V_{\lambda^\times} \neq \emptyset$. Then,*

$$\overline{\Lambda}_C \cap (V_{\lambda^\times} \times V^*_{\lambda^\times}) = \cup_{C'} \overline{\Lambda}_{C'},$$

where the union is over all orbits C' of $V_{\lambda^\times}$ such that $C' \subseteq C \cap V_{\lambda^\times}$.

Proof. It follows from Lemma 4.17 that the dimensions match and hence (similar to the proof of Lemma 4.21) we have that $\overline{\Lambda}_C \cap (V_{\lambda^\times} \times V^*_{\lambda^\times})$ is an H_{λ_α}-component of $\Lambda_{\lambda^\times}$. From Lemmas 4.17 and 4.21, it is clear that the component is

$$\overline{\Lambda}_C \cap (V_{\lambda^\times} \times V^*_{\lambda^\times}) = \cup_{C'} \overline{\Lambda}_{C'},$$

where the union is over all orbits C' of $V_{\lambda^\times}$ such that $C' \subseteq C \cap V_{\lambda^\times}$. □

Remark 4.23. *Based on many examples, we suspect that if C is an orbit of $V_{\lambda_{\phi_\alpha}}$, then there exists an orbit $\tilde{C}$ of $V_{\lambda_{\phi_\alpha}}$ such that*

$$\overline{\Lambda}_C \cap (V_{\lambda^\times} \times V^*_{\lambda^\times}) = \cup_{C'} \overline{\Lambda}_{C'},$$

where the union is over all orbits C' of $V_{\lambda^\times}$ such that $C' \subseteq \tilde{C} \cap V_{\lambda^\times}$. This would also imply Corollary 4.22.

4.4. **The fixed point formula.** We continue with the notation of the previous subsection, except we restrict ourselves to the case that $G = G_n = \mathrm{GL}_n(F)$. Recall that $\phi \in \Phi(\mathrm{GL}_n(F))$, $\phi^\times = \phi^\vee \times \phi^\alpha$, and $\phi_\alpha = \phi^\vee + \phi^\alpha$, where

$$\phi^\alpha = \oplus_{i=0}^{\alpha-1} |\cdot|^{\frac{\alpha-1}{2}-i} \otimes S_1$$

which corresponds to the 0-orbit in $V_{\lambda_{\phi^\alpha}}$.

Definition 4.24. *Write*

$$\phi^\vee = \bigoplus_{i=1}^{r} |\cdot|^{x_i} \otimes S_{a_i} \oplus \left(\bigoplus_{i=r+1}^{k} \rho_i \otimes S_{a_i} \right),$$

where $\rho_i \neq |\cdot|^x$ for any $x \in \mathbb{R}$. We define the set of trivial exponents of $\phi^\vee$ to be

$$\exp_{\mathbb{1}_{W_F}}(\phi^\vee) = \bigcup_{i=1}^{r} \{\frac{a_i-1}{2} + x_i, \frac{a_i-1}{2} - 1 + x_i, \ldots, \frac{1-a_i}{2} + x_i\}.$$

Let $\beta \in \frac{1}{2}\mathbb{Z}$, (later we take $\beta = \frac{\alpha-1}{2}$). We define the set of trivial β-exponents of $\phi^\vee$ to be $\exp^{\beta}_{\mathbb{1}_{W_F}}(\phi^\vee) = \exp_{\mathbb{1}_{W_F}}(\phi^\vee) \cap (\beta + \mathbb{Z})$. Finally, we let $m^{\beta}_{\mathbb{1}_{W_F}}(\phi^\vee) = \max\{|x| \mid x \in \exp^{\beta}_{\mathbb{1}_{W_F}}(\phi^\vee)\}$.

Remark 4.25. *Let $\beta = \frac{\alpha-1}{2}$. It is possible that $\exp^{\beta}_{\mathbb{1}_{W_F}}$ is empty. In this case, $\exp_{\mathbb{1}_{W_F}}(\phi^\vee) \cap (\beta+\mathbb{Z}) = \emptyset$ and, by convention, we write $\beta \geq m^{\beta}_{\mathbb{1}_{W_F}}(\phi^\vee)$.*

We recall the previous examples.

Example 4.26. *We continue Example 4.15. In this case, $\alpha = 2$ and so we have $\beta = \frac{\alpha-1}{2} = \frac{1}{2} \in \frac{1}{2} + \mathbb{Z}$. We have*

$$\exp_{\mathbb{1}_{W_F}}(\phi^\vee) = \exp^{\beta}_{\mathbb{1}_{W_F}}(\phi^\vee) = \{\frac{1}{2}, -\frac{1}{2}\}$$

and $m^{\beta}_{\mathbb{1}_{W_F}}(\phi^\vee) = \frac{1}{2}$. In particular $\frac{\alpha-1}{2} \geq m^{\beta}_{\mathbb{1}_{W_F}}(\phi^\vee)$.

Example 4.27. *We continue Example 4.16. In this case, $\alpha = 2$ and so we have $\beta = \frac{\alpha-1}{2} = \frac{1}{2} \in \frac{1}{2} + \mathbb{Z}$. We have*

$$\exp_{\mathbb{1}_{W_F}}(\phi^\vee) = \exp^{\beta}_{\mathbb{1}_{W_F}}(\phi^\vee) = \{\frac{3}{2}, -\frac{3}{2}\}$$

and $m^{\beta}_{\mathbb{1}_{W_F}}(\phi^\vee) = \frac{3}{2}$. In contrast with Example 4.26, we have $\frac{\alpha-1}{2} = \frac{1}{2} < m^{\beta}_{\mathbb{1}_{W_F}}(\phi^\vee)$.

Recall that given an L-parameter ϕ corresponding to the orbit C_ϕ, we attach a dual L-parameter $\hat{\phi}$ corresponding to the dual orbit $(C_\phi)^*$. In general, the computation of $\hat{\phi}$ is determined by the Mœglin-Waldspurger algorithm ([46, Theoreme II.13]).

Lemma 4.28. *We have $\widehat{\phi^\alpha} = \mathbb{1}_{W_F} \otimes S_\alpha$.*

Proof. This follows simply from the Mœglin-Waldspurger algorithm ([46, Theoreme II.13]). Alternatively, ϕ^α corresponds to the 0-orbit in V_{λ^α}. Its dual orbit is the unique open orbit in V_{λ^α}. This orbit corresponds to $\mathbb{1}_{W_F} \otimes S_\alpha$ from which the lemma follows. □

Let $\beta = \frac{\alpha-1}{2}$. We show that if $\beta \geq m^{\beta}_{\mathbb{1}_{W_F}}(\phi^\vee)$, then $\widehat{\phi}_\alpha = \widehat{\phi^\vee} + \widehat{\phi^\alpha}$.

Lemma 4.29. *Assume that* $\beta = \frac{\alpha-1}{2} \geq m^{\beta}_{\mathbb{1}_{W_F}}(\phi^{\vee})$. *Then* $\widehat{\phi}_{\alpha} = \widehat{\phi^{\vee}} + \widehat{\phi^{\alpha}}$.

Proof. The proof is a direct consequence of the Mœglin-Waldspurger algorithm ([46, Theoreme II.13]). Indeed, since $\beta \geq m^{\beta}_{\mathbb{1}_{W_F}}(\phi^{\vee})$, the first iteration of the algorithm groups the "segments" $|\cdot|^{\frac{\alpha-1}{2}}, \ldots, |\cdot|^{\frac{1-\alpha}{2}}$ into a segment (which corresponds to $\widehat{\phi^{\alpha}} = \mathbb{1}_{W_F} \otimes S_{\alpha}$; see Lemma 4.28). The algorithm then repeats on the rest of the segments and hence computes $\widehat{\phi^{\vee}}$. Therefore, $\widehat{\phi}_{\alpha} = \widehat{\phi^{\vee}} + \widehat{\phi^{\alpha}}$. $\square$

We recall the current situation for our examples.

Example 4.30. *We continue Example 4.26. In this case,* $\widehat{\phi^{\vee}} = \mathbb{1}_{W_F} \otimes S_2 = \widehat{\phi^{\alpha}}$ *and*

$$\widehat{\phi_{\alpha}} = \mathbb{1}_{W_F} \otimes S_2 + \mathbb{1}_{W_F} \otimes S_2 = \widehat{\phi^{\vee}} + \widehat{\phi^{\alpha}}$$

as stated by Lemma 4.29.

Example 4.31. *We continue Example 4.27. In this case,* $\widehat{\phi^{\vee}} = \phi^{\vee} = |\cdot|^{\frac{3}{2}} + |\cdot|^{-\frac{3}{2}}$ *and* $\mathbb{1}_{W_F} \otimes S_2 = \widehat{\phi^{\alpha}}$. *By the Mœglin-Waldspurger algorithm (*[46, Theoreme II.13]*), we obtain that*

$$\widehat{\phi_{\alpha}} = \mathbb{1}_{W_F} \otimes S_4 \neq \widehat{\phi^{\vee}} + \widehat{\phi^{\alpha}}.$$

Indeed, $\phi_{\alpha} = |\cdot|^{\frac{3}{2}} + |\cdot|^{\frac{1}{2}} + |\cdot|^{-\frac{1}{2}} + |\cdot|^{-\frac{3}{2}}$ *and the Mœglin-Waldspurger algorithm (*[46, Theoreme II.13]*) groups the "segments"* $|\cdot|^{\frac{3}{2}}, |\cdot|^{\frac{1}{2}}, |\cdot|^{-\frac{1}{2}}$ $|\cdot|^{-\frac{3}{2}}$ *into one segment which corresponds to* $\mathbb{1}_{W_F} \otimes S_4$.

In other words, Lemma 4.29 fails for this example. The reason is that a trivial exponent of ϕ^{α} *interacted (meaning it can form a segment) with a trivial exponent of* $\phi^{\vee}$ *in the Mœglin-Waldspurger algorithm (*[46, Theoreme II.13]*). This is why we require* $\beta \geq m^{\beta}_{\mathbb{1}_{W_F}}(\phi^{\vee})$ *in Lemma 4.29, so that there is no interaction between the trivial exponents of* ϕ^{α} *and* $\phi^{\vee}$.

Note that despite the failure of Lemma 4.29, the Adams conjecture (Conjecture 4.4) still holds for this example. Indeed, it is simple to check that the ABV-packets are singletons and hence agree with their L-packets. The Adams conjecture then follows for this example from Theorem 4.1.

With Lemma 4.29 in hand, we can now relate $\Lambda^{reg}_{C_{\phi^{\times}}}$ and $\Lambda^{reg}_{C_{\phi_{\alpha}}}$.

Proposition 4.32. *Let* $\beta = \frac{\alpha-1}{2} \geq m^{\beta}_{\mathbb{1}_{W_F}}(\phi^{\vee})$ *and*

$$((x_{\phi^{\vee}}, y_{\phi^{\vee}}), (x_{\phi^{\alpha}}, y_{\phi^{\alpha}})) \in \Lambda^{reg}_{C_{\phi^{\times}}}.$$

Then $\varepsilon'((x_{\phi^{\vee}}, y_{\phi^{\vee}}), (x_{\phi^{\alpha}}, y_{\phi^{\alpha}})) \in \Lambda^{reg}_{C_{\phi_{\alpha}}}$.

Proof. Let $\varepsilon'((x_{\phi^{\vee}}, y_{\phi^{\vee}}), (x_{\phi^{\alpha}}, y_{\phi^{\alpha}})) = (x, y) \in \overline{\Lambda}_C$ for some $C \geq C_{\phi_{\alpha}}$. To show that $\varepsilon'((x_{\phi^{\vee}}, y_{\phi^{\vee}}), (x_{\phi^{\alpha}}, y_{\phi^{\alpha}})) \in \Lambda^{reg}_{C_{\phi_{\alpha}}}$, we must show that $C = C_{\phi_{\alpha}}$. Let ϕ' be the L-parameter corresponding to C.

By [17, Lemma 6.4.2], we have $\Lambda^{reg}_{C_{\phi^\times}} \subseteq C_{\phi^\times} \times C^*_{\phi^\times}$. From Lemma 4.29, we obtain that $y \in C^*_{\phi_\alpha}$. Since $(x,y) \in \overline{\Lambda}_C \subseteq \overline{C} \times \overline{C^*}$, it follows that $C^* \geq C^*_{\phi_\alpha} = C_{\widehat{\phi_\alpha}}$. By Lemma 4.28 and Lemma 4.29, we have $\widehat{\phi_\alpha} = \widehat{\phi^\vee} + \mathbb{1}_{W_F} \otimes S_\alpha$. Since $m^\beta_{\mathbb{1}_{W_F}}(\widehat{\phi_\alpha}) = \beta$ and $\widehat{\phi}' \geq_C \widehat{\phi_\alpha}$, it follows that $\widehat{\phi}' = \widehat{\phi''} + \mathbb{1}_{W_F} \otimes S_\alpha$ for some L-parameter ϕ''. By the Mœglin-Waldspurger algorithm ([46, Theoreme II.13]), we obtain $\phi' = \phi'' + \phi^\alpha$.

Recall that $C \geq C_{\phi_\alpha}$ and $\phi_\alpha = \phi^\vee + \phi^\alpha$. Since $\phi' \geq \phi_\alpha$ and $\phi' = \phi'' + \phi^\alpha$, it follows that $\phi'' \geq_C \phi^\vee$. By Corollary 4.22,

$$\overline{\Lambda}_C \cap (V_{\lambda^\times} \times V^*_{\lambda^\times}) = \cup_{C'} \overline{\Lambda}_{C'},$$

where the union is over all orbits C' of $V_{\lambda^\times}$ such that $C' \subseteq C \cap V_{\lambda^\times}$. Now, we have that $((x_{\phi^\vee}, y_{\phi^\vee}), (x_{\phi^\alpha}, y_{\phi^\alpha}))$ must lie in $\overline{\Lambda}_{C'}$ where $C' \geq C_{\phi^\vee} \times C_{\phi^\alpha}$ (note that some C' in the union may be incomparable, but our element cannot lie in those conormal bundles). But, by regularity of $((x_{\phi^\vee}, y_{\phi^\vee}), (x_{\phi^\alpha}, y_{\phi^\alpha}))$, it follows that $C' = C_{\phi^\vee} \times C_{\phi^\alpha}$ and hence $C = C_{\phi_\alpha}$. □

Proposition 4.32 gives the following fixed point formula.

Theorem 4.33. *Suppose that $\beta = \frac{\alpha-1}{2} \geq m^\beta_{\mathbb{1}_{W_F}}(\phi^\vee)$. Then for any $[\mathcal{F}] \in K\mathrm{Per}_\lambda(G_m)$, we have*

$$\langle \eta_{\phi_\alpha}, [\mathcal{F}] \rangle_{\lambda_\alpha} = \langle \eta_{\phi^\times}, [\mathcal{F}|_{V_{\lambda^\times}}] \rangle_{\lambda^\times}.$$

Proof. We defer the proof of the above theorem to Theorem A.5. □

We remark that the above theorem is a generalization of the fixed point formula established by Cunningham and Ray in [20, Proposition 3.2]. This generalization is nontrivial though and requires significant further technical discussion. We defer this discussion to Appendix A in order to not distract from our goal of investigating the Adams conjecture for ABV-packets.

With the fixed point formula in hand, we can now prove our main result, i.e., we verify Conjecture 4.5(1) for $\mathrm{GL}_n(F)$.

Theorem 4.34. *Suppose $\pi \in \Pi^{\mathrm{ABV}}_\phi$ and $\beta = \frac{\alpha-1}{2} \geq m^\beta_{\mathbb{1}_{W_F}}(\phi^\vee)$. Then $\theta_{-\alpha}(\pi) \in \Pi^{\mathrm{ABV}}_{\phi_\alpha}$.*

Proof. By Vogan's perspective on the local Langlands correspondence, we have that π corresponds to the perverse sheaf $\mathcal{IC}(\mathbb{1}_{C_{\phi_\pi}})$. Similarly, from Theorem 4.1, we have that $\theta_{-\alpha}(\pi)$ corresponds to the perverse sheaf $\mathcal{IC}(\mathbb{1}_{C_{(\phi_\pi)_\alpha}})$. Let $\lambda_\alpha = \lambda_{\phi_\alpha}$. By Lemma 4.9, it is sufficient to show that

$$\langle \eta_{\phi_\alpha}, \mathcal{IC}(\mathbb{1}_{C_{(\phi_\pi)_\alpha}}) \rangle_{\lambda_\alpha} \neq 0.$$

By Lemma 4.10, it suffices to show that

$$\langle M(\eta_{\phi_\alpha}), \mathbb{1}^\natural_{C_{(\phi_\pi)_\alpha}} \rangle_{\lambda_\alpha} \neq 0.$$

Let $\phi^\times = \phi^\vee + \phi^\alpha$ and $\lambda^\times = \lambda_{\phi^\times}$. By the fixed point formula (Theorem 4.33) and Corollary 4.14, it is enough to show that

$$\langle M(\eta_{\phi^\times}), \mathbb{1}^\natural_{C_{\phi_\pi^\vee \times \phi^\alpha}}\rangle_{\lambda^\times} \neq 0.$$

Since $V_{\lambda^\times} = V_{\lambda_\phi} \times V_{\lambda_{\phi^\alpha}}$, it follows that $m_{\lambda^\times} = \mathrm{diag}(m_{\lambda_\phi}, m_{\lambda^\alpha})$. Consequently, we have that

$$\langle M(\eta_{\phi^\times}), \mathbb{1}^\natural_{C_{\phi_\pi^\vee \times \phi^\alpha}}\rangle_{\lambda^\times} = \langle M(\eta_{\phi^\vee}), \mathbb{1}^\natural_{C_{\phi_\pi^\vee}}\rangle_{\lambda_{\phi^\vee}} \langle M(\eta_{\phi^\alpha}), \mathbb{1}^\natural_{C_{\phi^\alpha}}\rangle_{\lambda_{\phi^\alpha}}.$$

Now, $\langle M(\eta_{\phi^\alpha}), \mathbb{1}^\natural_{C_{\phi^\alpha}}\rangle_{\lambda_{\phi^\alpha}} = \langle \eta_{\phi^\alpha}, \mathcal{IC}(\mathbb{1}_{C_{\phi^\alpha}})\rangle_{\lambda_{\phi^\alpha}} \neq 0$ by Lemma 4.10 and Proposition 3.3. On the other hand, by Lemma 4.11, we have

$$\langle M(\eta_{\phi^\vee}), \mathbb{1}^\natural_{C_{\phi_\pi^\vee}}\rangle_{\lambda_{\phi^\vee}} = \langle \eta_{\phi^\vee}, \mathcal{IC}(\mathbb{1}_{C_{\phi_\pi^\vee}})\rangle_{\lambda_{\phi^\vee}} \neq 0.$$

Therefore, we obtain that

$$\langle M(\eta_{\phi^\times}), \mathbb{1}^\natural_{C_{\phi_\pi^\vee \times \phi^\alpha}}\rangle_{\lambda^\times} \neq 0$$

which proves the theorem. □

We remark on some consequences of Theorem 4.34. First, in [16, §0B], it is claimed that there exists a nonsingleton ABV-packet of $\mathrm{GL}_n(F)$ for any $n \geq 16$. For $\mathrm{GL}_{16}(F)$ this is proved in [16, Corollary 2.7], but for $n \geq 17$, no proof is explicitly given. Their outline is to simply construct a Vogan variety which is isomorphic to the Vogan variety of the nonsingleton ABV-packet of $\mathrm{GL}_{16}(F)$. However, we are able to obtain more complicated examples using Theorem 4.34.

Corollary 4.35. *There exists a nonsingleton ABV-packet of* $\mathrm{GL}_n(F)$ *for* $n = 16, 18, 20$ *or any* $n \geq 21$.

Proof. Let ϕ_{KS} be the L-parameter of $\mathrm{GL}_{16}(F)$ described in [16, §1B]. We have that $\phi_{\mathrm{KS}}^\vee = \phi_{\mathrm{KS}}$ and

$$\exp_{\mathbb{1}_{W_F}}(\phi_{\mathrm{KS}}^\vee) = \{2, 1, 0, -1, -2\}.$$

Note that $\frac{x-1}{2} = 2$ implies that $x = 5$. Thus we have that $\beta = \frac{\alpha-1}{2} \geq m^\beta_{\mathbb{1}_{W_F}}(\phi_{\mathrm{KS}}^\vee)$ if and only if $\alpha = 2, 4$ or $\alpha \geq 5$. By [16, Corollary 2.7], $\Pi^{\mathrm{ABV}}_{\phi_{\mathrm{KS}}}$ consists of two representations, say $\Pi^{\mathrm{ABV}}_{\phi_{\mathrm{KS}}} = \{\pi_1, \pi_2\}$. By Theorem 4.34, for any $i = 1, 2$, we have $\theta_{-\alpha}(\pi_i) \in \Pi^{\mathrm{ABV}}_{(\phi_{\mathrm{KS}})_\alpha}$ for any $\alpha = 2, 4$ or $\alpha \geq 5$. Therefore, there exists a nonsingleton ABV-packet of $\mathrm{GL}_n(F)$ for $n = 16, 18, 20$ or any $n \geq 21$. We note that the Vogan variety for $(\phi_{\mathrm{KS}})_\alpha$ is not isomorphic to that of ϕ_{KS}. □

A second consequence is partial evidence for Conjecture 4.5(2).

Corollary 4.36. *Suppose* $\pi \in \Pi^{\mathrm{ABV}}_\phi$. *Assume that* $\beta = \frac{\alpha-1}{2} \geq m^\beta_{\mathbb{1}_{W_F}}(\phi^\vee)$. *Then the following hold.*

(1) *If* $\theta_{-\alpha}(\pi) \in \Pi^{\mathrm{ABV}}_{\phi_\alpha}$, *then* $\theta_{-\alpha+2}(\pi) \in \Pi^{\mathrm{ABV}}_{\phi_{\alpha+2}}$.

(2) *If* $\beta + \frac{1}{2} = \frac{\alpha}{2} \geq m^{\beta+\frac{1}{2}}_{\mathbb{1}_{W_F}}(\phi^\vee)$ *and* $\theta_{-\alpha}(\pi) \in \Pi^{\mathrm{ABV}}_{\phi_\alpha}$, *then* $\theta_{-\alpha+1}(\pi) \in \Pi^{\mathrm{ABV}}_{\phi_{\alpha+1}}$.

Proof. Both parts are immediate consequences of Theorem 4.34. We remark that the requirement $\beta + \frac{1}{2} = \frac{\alpha}{2} \geq m^{\beta+\frac{1}{2}}_{\mathbb{1}_{W_F}}(\phi^\vee)$ is needed in Part (2) as the condition $\beta = \frac{\alpha-1}{2} \geq m^{\beta}_{\mathbb{1}_{W_F}}(\phi^\vee)$ does not necessarily imply that $\beta + \frac{1}{2} = \frac{\alpha}{2} \geq m^{\beta+\frac{1}{2}}_{\mathbb{1}_{W_F}}(\phi^\vee)$. □

Of course, the above corollary is not saying anything substantial as Conjecture 4.5(1) (which is Theorem 4.34) implies Conjecture 4.5(2) for $\alpha \gg 0$. However, the above corollary explicates the condition $\alpha \gg 0$ from Theorem 4.34.

Appendix A. Proof of the fixed point formula

The goal of this appendix is to prove the fixed point formula (Theorem 4.33). For brevity, we let $G_n = \mathrm{GL}_n(F)$ throughout this appendix. The argument is a generalization of a proof of a fixed point formula for local Arthur parameters of G_n given by Cunningham and Ray in [20, Proposition 4.6]. Their results are stated in terms of local Arthur parameters for which the geometry is significantly simpler. For example, both the generic and microlocal fundamental groups (recalled below) are trivial in their situation. In our situation, this is not guaranteed and is known to fail in our situation, e.g., [16].

We recall some notation from §4.2. Fix an infinitesimal parameter λ of G_n and let $\phi \in \Phi_\lambda(G_n)$. We let C_ϕ be the corresponding H_λ-orbit for ϕ in V_λ. We consider the pairing $\langle \cdot, \cdot \rangle : K\Pi_\lambda(G_n) \times K\mathrm{Per}_\lambda(G_n) \to \mathbb{Z}$ defined in Equation 4.1. We further consider

$$\eta_\phi := \eta_{C_\phi} = (-1)^{d(C_\phi)} \sum_{\pi \in \Pi_\lambda(G_n)} (-1)^{d(\pi)} \mathrm{rank}(\mathrm{Evs}_{C_\phi}(\mathcal{P}(\pi)))[\pi] \in K\Pi_\lambda(G_n)$$

We begin by generalizing [20, Proposition 1.6]. We remark that Cunningham and Ray's argument generalizes to any local L-parameter of G_n. For the sake of completeness, we provide the proof. Recall that $D_{H_\lambda}(V_\lambda)$ denotes the H_λ-equivariant derived category of ℓ-adic sheaves on V_λ.

Proposition A.1. *For any* $\mathcal{F} \in D_{H_\lambda}(V_\lambda)$, *we have*

$$\langle \eta_\phi, \mathcal{F} \rangle = (-1)^{d(C_\phi)} \mathrm{rank}(\mathrm{Evs}_{C_\phi} \mathcal{F}).$$

Proof. As in the proof of [20, Proposition 1.6], the Grothendieck groups $K\mathrm{Per}_\lambda(V_\lambda)$ and $KD_\lambda(V_\lambda)$ coincide and so it is enough to prove the proposition for simple objects in $\mathrm{Per}_\lambda(V_\lambda)$.

Let $\mathcal{F} \in \mathrm{Per}_\lambda(V_\lambda)$ be simple. From Vogan's perspective on the local Langlands correspondence ([55]; see §4.2), we have that $\mathcal{F} = \mathcal{P}(\pi')$ for some

$\pi' \in \Pi_\lambda(G_n)$. We obtain that

$$\begin{aligned}\langle \eta_\phi, \mathcal{F}\rangle &= \langle (-1)^{d(C_\phi)} \sum_{\pi\in\Pi_\lambda(G_n)} (-1)^{d(\pi)}\mathrm{rank}(\mathrm{Evs}_{C_\phi}(\mathcal{P}(\pi)))[\pi], [\mathcal{P}(\pi')]\rangle \\ &= (-1)^{d(C_\phi)} \sum_{\pi\in\Pi_\lambda(G_n)} (-1)^{d(\pi)}\mathrm{rank}(\mathrm{Evs}_{C_\phi}(\mathcal{P}(\pi)))\langle[\pi], [\mathcal{P}(\pi')]\rangle \\ &= (-1)^{d(C_\phi)}\mathrm{rank}(\mathrm{Evs}_{C_\phi}(\mathcal{F})).\end{aligned}$$

where the last equality follows from the definition of $\langle\cdot,\cdot\rangle$. □

Next we recall the definition of the (equivariant) microlocal fundamental group from [3, Definition 1.33]. Fix an orbit C_ϕ of V_λ. Fix $y \in \Lambda_{C_\phi}$ and consider $\Lambda_{C_\phi,y} = \{x \in C_\phi \mid [x,y] = 0\}$. Given any $x \in \Lambda_{C_\phi,y}$, we consider the centralizer $Z_{H_\lambda}(x,y)$ and set $A_{y,x} = Z_{H_\lambda}(x,y)/Z_{H_\lambda}(x,y)^0$. By [3, Lemma 24.3], this family is locally constant over most of $\Lambda_{C_\phi,y}$. The (equivariant) microlocal fundamental group is defined to be $A^{mic}_{C_\phi} := Z_{H_\lambda}(x,y)/Z_{H_\lambda}(x,y)^0 = \pi_0(Z_{H_\lambda}(x,y))$ for generic $x \in C_\phi$.

We also need to consider the generic conormal bundle $\Lambda^{gen}_{C_\phi}$ which is defined in [17, §7.9]. Rather than explicating its definition, it suffices to recall some properties of $\Lambda^{gen}_{C_\phi}$. First, we have that $\emptyset \neq \Lambda^{gen}_{C_\phi} \subseteq \Lambda^{reg}_{C_\phi}$. Second, we have

$$\mathrm{Evs}_{C_\phi} : \mathrm{Per}_{H_\lambda}(V_\lambda) \to \mathrm{Loc}_{H_\lambda}(\Lambda^{gen}_{C_\phi}),$$

where $\mathrm{Loc}_{H_\lambda}(\Lambda^{gen}_{C_\phi})$ denotes the category of H_λ-equivariant local systems on $\Lambda^{gen}_{C_\phi}$. The generic fundamental group is $A^{gen}_{C_\phi} := \pi_1(\Lambda^{gen}_{C_\phi}, (x,y))$, where $(x,y) \in \Lambda^{gen}_{C_\phi}$ is a choice of base point. We have

$$\mathrm{Loc}_{H_\lambda}(\Lambda^{gen}_{C_\phi}) \cong \mathrm{Rep}(A^{gen}_{C_\phi}).$$

We warn the reader that the isomorphism for $\mathrm{Loc}_{H_\lambda}(\Lambda^{gen}_{C_\phi})$ is incorrectly stated in [17, §8.4]. Given $(x,y) \in \Lambda^{gen}_{C_\phi}$, let $\mathcal{O}_{H_\lambda}(x,y)$ denote its corresponding H_λ-orbit. We have

$$\mathrm{Loc}_{H_\lambda}(\mathcal{O}_{H_\lambda}(x,y)) \cong \mathrm{Rep}(A_{y,x}).$$

Fix $(x,y) \in \Lambda^{gen}_{C_\phi}$. Given $s \in Z_{H_\lambda}(x,y)$, we let $a_s \in A_{y,x}$ denotes its image. The restriction map $\mathrm{Loc}_{H_\lambda}(\Lambda^{gen}_{C_\phi}) \to \mathrm{Loc}(\mathcal{O}_{H_\lambda}(x,y))$ induces a map $\mathrm{Rep}(A^{gen}_\phi) \to \mathrm{Rep}(A^{mic}_{C_\phi})$.

Given $s \in Z_{H_\lambda}(x,y)$ for $(x,y) \in \Lambda^{gen}_{C_\phi}$, we let $a_s \in A^{mic}_{C_\phi}$ denote its image. We consider the distribution

$$\eta_{\phi,s} := (-1)^{d(C)} \sum_{\pi\in\Pi_\lambda(G_n)} (-1)^{d(\pi)}\mathrm{trace}(a_s, \mathrm{Evs}_C(\mathcal{P}(\pi)))[\pi]. \tag{A.1}$$

Here, we have identified $\mathrm{Evs}_C(\mathcal{P}(\pi))$ with a representation of $A^{mic}_{C_\phi}$ via $\mathrm{Loc}_{H_\lambda}(\Lambda^{gen}_{C_\phi}) \to \mathrm{Loc}(\mathcal{O}_{H_\lambda}(x,y)) \cong \mathrm{Rep}(A^{mic}_{C_\phi})$, where these are the maps discussed above. We note that if a_s is trivial, then $\eta_{\phi,s} = \eta_\phi$. See [18] for further details (we note that the normalization of the Evs functor is trivial, i.e., the functor NEvs defined in [17, §7.10] agrees with Evs).

We recall further notation from §4.2. For $i = 1, \dots, r$, let $\phi_i \in \Phi(G_{n_i})$, and $n = n_1 + \cdots + n_r$. We set $G^\times := G_{n_1} \times \cdots \times G_{n_r}$, $\phi^\times = \phi_1 \times \cdots \times \phi_r$, and $\lambda^\times = \lambda_1 \times \cdots \times \lambda_r$ be the corresponding infinitesimal parameter, where $\lambda_i = \lambda_{\phi_i}$. The Vogan variety is $V_{\lambda^\times} = V_{\lambda_1} \times \cdots \times V_{\lambda_r}$ and we have $H_{\lambda^\times} = H_{\lambda_1} \times \cdots \times H_{\lambda_r}$.

We let $\phi = \phi_1 + \cdots + \phi_r \in \Phi(G_n)$, $\lambda = \lambda_\phi$, and $s \in \widehat{G}_n(\mathbb{C})$ be of finite order (and hence semi-simple) such that $Z_{\widehat{G}_n(\mathbb{C})}(s) \cong \widehat{G}^\times$. The resulting inclusion $\widehat{G}^\times \hookrightarrow \widehat{G}_n(\mathbb{C})$ induces inclusions $H_{\lambda^\times} \hookrightarrow H_\lambda$ and

$$\varepsilon : V_{\lambda^\times} \hookrightarrow V_\lambda$$

which is equivariant for the action by $H_{\lambda^\times}$. We have that

$$V_{\lambda^\times} = V_\lambda^s := \{x \in V_\lambda \mid \mathrm{Ad}(s)x = x\}.$$

Furthermore, we have an inclusion of the dual Vogan varieties

$${}^t\varepsilon : V_{\lambda^\times}^* \hookrightarrow V_\lambda^*$$

and hence an inclusion

$$\varepsilon' = \varepsilon \times {}^t\varepsilon : V_{\lambda^\times} \times V_{\lambda^\times}^* \hookrightarrow V_\lambda \times V_\lambda^*.$$

Let $\varepsilon^* : \mathrm{D}_{H_\lambda}(V_{\lambda_\alpha}) \to \mathrm{D}_{H_{\lambda^\times}}(V_{\lambda^\times})$ denote the equivariant restriction functor for the equivariant derived categories. As a shorthand, we write

$$\mathcal{F}|_{V_{\lambda^\times}} := \varepsilon^*\mathcal{F}.$$

Recall from §4.3 that for an H_λ-orbit C_ϕ of V_λ, we consider its conormal bundle $\Lambda_{C_\phi}^{reg} \subseteq V_\lambda \times V_\lambda^*$. We have the following generalization of [20, Lemma 3.1]. We warn the reader that the phrase "the image of s in A_ϕ^{mic} is trivial" implicitly assumes that $s \in Z_{H_\lambda}(x, y)$ for some $(x, y) \in \Lambda_{C_\phi}^{gen}$.

Proposition A.2. *We continue with the above notation and setting. Suppose further that there exists $(x_{\phi^\times}, y_{\phi^\times}) \in \Lambda_{C_{\phi^\times}}^{reg}$ such that $\varepsilon'(x_{\phi^\times}, y_{\phi^\times}) \in \Lambda_{C_\phi}^{reg}$ and that the image of s in A_ϕ^{mic} is trivial. Then, we have*

$$(-1)^{d(C_\phi)}\mathrm{rank}(\mathrm{Evs}_{C_\phi}\mathcal{F}) = (-1)^{d(C_{\phi^\times})}\mathrm{rank}(\mathrm{Evs}_{C_\phi^\times}\mathcal{F}|_{V_{\lambda^\times}}),$$

for any $\mathcal{F} \in D_{H_\lambda}(V_\lambda)$.

Proof. The proof is a straightforward adaptation of the proof of [20, Lemma 3.1]. However, Cunningham and Ray worked in the setting of local Arthur parameters which implied that the microlocal fundamental group is trivial. In our situation, we assume that the image of s in the microlocal fundamental group is trivial. ∎

We remark that a more general result could be proven than stated above. Namely, one may want to show that

$$(-1)^{d(C_\phi)}\mathrm{trace}(a_s, \mathrm{Evs}_{C_\phi}\mathcal{F}) = (-1)^{d(C_{\phi^\times})}\mathrm{trace}(a'_s, \mathrm{Evs}_{C^\times_\phi}\mathcal{F}|_{V_{\lambda^\times}}),$$

where a_s and a'_s denotes the images of s in $A^{mic}_{C_\phi}$ and $A^{mic}_{C_{\phi^\times}}$, respectively. However, this would requires further assumptions on the compatibility of generic part of the conormal bundle. This issue is avoided in the above proposition as we only considered the case that $a_s = a'_s = 1$. Our next goal is to verify that, in our setting, we do indeed have that $a_s = a'_s = 1$.

We proceed with a technical lemma which describes certain conormal elements.

Lemma A.3. *Suppose that $\beta = \frac{\alpha-1}{2} \geq m^\beta_{\mathbb{1}_{W_F}}(\phi^\vee)$. Let $y_{\lambda^\vee} \in V^*_{\lambda^\vee}$, $y_{\phi^\alpha} \in C_{\widehat{\phi^\alpha}} \subseteq V^*_{\lambda^\alpha}$, and consider $y = {}^t\varepsilon(y_{\lambda^\vee}, y_{\phi^\alpha})$. Then $x \in \Lambda_y$ if and only if $x = \varepsilon(x', 0)$ for some $x' \in V_{\lambda^\vee}$.*

Proof. We recall some setup from §4.2. By [17, Theorem 5.1.1], we may assume that λ_α is unramified, i.e., trivial on I_F, and $\chi(\lambda_\alpha(\mathrm{Fr})) \in \mathbb{R}_{>0}$ for any character $\chi : \widehat{T} \to \mathrm{GL}_1(\mathbb{C})$, where $\widehat{T}$ is any torus in $\mathrm{GL}_n(\mathbb{C})$ containing $\lambda_\alpha(\mathrm{Fr})$. We write

$$\lambda_\alpha = m_1|\cdot|^{e_1} + m_2|\cdot|^{e_2} + \cdots + m_r|\cdot|^{e_r}$$

where $m_i \in \mathbb{Z}_{\geq 1}$ denotes the multiplicity and $e_i \in \mathbb{R}$ with $e_i > e_{i+1}$ for $i = 1, \ldots, r-1$. Furthermore, we may assume that $r = \alpha$ and $e_1 = \frac{\alpha-1}{2}, e_2 = \frac{\alpha-3}{2}, \ldots, e_\alpha = \frac{1-\alpha}{2}$ since $\beta = \frac{\alpha-1}{2} \geq m^\beta_{\mathbb{1}_{W_F}}(\phi^\vee)$. Indeed, in general V_{λ_α} decomposes as a direct product of Vogan varieties based on the exponents modulo $\mathbb{Z}$ and only those exponents lying in the coset $\frac{\alpha-1}{2} + \mathbb{Z}$ will play a nontrivial role in the following arguments.

Now, for $i = 1, \ldots, \alpha$, let E_i denote the $q_F^{e_i}$-eigenspace of $\lambda_\alpha(\mathrm{Fr})$. We have $m_i = \dim(E_i)$ and

$$V_{\lambda_\alpha} \cong \mathrm{Hom}(E_1, E_2) \times \mathrm{Hom}(E_2, E_3) \times \cdots \times \mathrm{Hom}(E_{\alpha-1}, E_\alpha).$$

Given $x \in V_{\lambda_\alpha}$, using the above isomorphism, we write $x = (x_1, \ldots, x_{\alpha-1})$ where

$$(x_1, x_2, \ldots, x_{\alpha-1}) \in \mathrm{Hom}(E_1, E_2) \times \mathrm{Hom}(E_2, E_3) \times \cdots \times \mathrm{Hom}(E_{\alpha-1}, E_\alpha).$$

We identify the dual Vogan variety $V^*_{\lambda_\alpha}$ with ${}^tV_{\lambda_\alpha}$ (recalling that V_{λ_α} lies in the Lie algebra of $\mathrm{GL}_n(\mathbb{C})$, i.e., the spaces of $n\times n$ matrices, the transpose is the usual one). We obtain an isomorphism

$$V^*_{\lambda^\vee_\alpha} \cong \mathrm{Hom}(E_2, E_1) \times \mathrm{Hom}(E_3, E_2) \times \cdots \times \mathrm{Hom}(E_\alpha, E_{\alpha-1}).$$

Given $y \in V^*_{\lambda_\alpha}$, using the above isomorphism, we write $y = (y_1, \ldots, y_{\alpha-1})$ where

$$(y_1, y_2, \ldots, y_{\alpha-1}) \in \mathrm{Hom}(E_2, E_1) \times \mathrm{Hom}(E_3, E_2) \times \cdots \times \mathrm{Hom}(E_\alpha, E_{\alpha-1}).$$

We have similar isomorphisms for $V_{\lambda^\vee}$ and $V_{\lambda_{\phi^\alpha}}$ which we make explicit below. For $i=1,\dots,\alpha$, we let $E_{\lambda^\vee,i}$ be the corresponding $q_F^{e_i}$-eigenspace, possibly zero, of $\lambda^\vee(\mathrm{Fr})$. We have

$$V_{\lambda^\vee} \cong \mathrm{Hom}(E_{\lambda^\vee,1}, E_{\lambda^\vee,2}) \times \cdots \times \mathrm{Hom}(E_{\lambda^\vee,\alpha-1}, E_{\lambda^\vee,\alpha}).$$

Given $x \in V_{\lambda^\vee}$, we identify $x=(x_1,\dots,x_{\alpha-1})$. Note that if $q_F^{e_i}$ or $q_F^{e_{i+1}}$ is not an eigenvalue $\lambda^\vee(\mathrm{Fr})$, then $x_i=0$. The isomorphism for the dual variety is obtained similarly.

For $i=1,\dots,\alpha$, we have that $q_F^{e_i}$ is always an eigenvalue of $\lambda^\alpha(\mathrm{Fr})$ and the corresponding eigenspace $E_{\lambda^\alpha,i}$ is 1-dimensional. We have

$$V_{\lambda^\alpha} \cong \mathrm{Hom}(E_{\lambda^\alpha,1}, E_{\lambda^\alpha,2}) \times \cdots \times \mathrm{Hom}(E_{\lambda^\alpha,\alpha-1}, E_{\lambda^\alpha,\alpha}).$$

Given $x \in V_{\alpha^\vee}$, we identify $x=(x_1,\dots,x_{\alpha-1})$. The isomorphism for the dual variety is obtained similarly.

Conjugating if necessary, we may choose s such that the inclusion $\varepsilon : V_{\lambda^\times} \hookrightarrow V_{\lambda_\alpha}$ is given as follows. Let $x_{\lambda^\vee}=(x_{\lambda^\vee,1},\dots,x_{\lambda^\vee,\alpha-1}) \in V_{\lambda^\vee}$. Also, let $x_{\lambda^\alpha} \in V_{\lambda^\alpha}$ and write $x_{\lambda^\alpha}=(x_{\lambda^\alpha,1},\dots,x_{\lambda^\alpha,\alpha-1})$. For $i=1,\dots,\alpha-1$, we define $\varepsilon_i(x_{\lambda^\vee,i}, x_{\lambda^\alpha,i})$ as follows

$$\varepsilon_i(x_{\lambda^\vee,i}, x_{\lambda^\alpha,i}) := \begin{pmatrix} x_{\lambda^\vee,i} & 0_{\dim(E_{\lambda^\vee,i})\times 1} \\ 0_{1\times\dim(E_{\lambda^\vee,i+1})} & x_{\lambda^\alpha,i} \end{pmatrix}.$$

Note that if $\dim(E_{\lambda^\vee,i})=0$, then we omit the corresponding rows. Similarly, if $\dim(E_{\lambda^\vee,i+1})=0$, then we omit the corresponding columns. The inclusion is then given by

$$\varepsilon(x_{\lambda^\vee}, x_{\lambda^\alpha}) = (\varepsilon_1(x_{\lambda^\vee,1}, x_{\lambda^\alpha,1}),\dots,\varepsilon_{\alpha-1}(x_{\lambda^\vee,\alpha-1}, x_{\lambda^\alpha,\alpha-1})).$$

Note that this inclusion corresponds to taking

$$s=\mathrm{diag}(I_{\dim E_{\lambda^\vee,1}}, -1, I_{\dim E_{\lambda^\vee,2}}, -1, \dots, I_{\dim E_{\lambda^\vee,\alpha}}, -1),$$

where I_k denotes the $k\times k$ identity matrix.

The inclusion of the dual Vogan varieties is given similarly. Let $y_{\lambda^\vee} \in V^*_{\lambda^\vee}$ and write

$$y_{\lambda^\vee}=(y_{\lambda^\vee,1},\dots,y_{\lambda^\vee,\alpha-1}) \in \mathrm{Hom}(E_{\lambda^\vee,2}, E_{\lambda^\vee,1}) \times \cdots \times \mathrm{Hom}(E_{\lambda^\vee,\alpha}, E_{\lambda^\vee,\alpha-1}).$$

Also, let $y_{\lambda^\alpha} \in V^*_{\lambda^\alpha}$ and write

$$y_{\lambda^\alpha}=(y_{\lambda^\alpha,1},\dots,y_{\lambda^\alpha,\alpha-1}) \in \mathrm{Hom}(E_{\lambda^\alpha,2}, E_{\lambda^\alpha,1}) \times \cdots \times \mathrm{Hom}(E_{\lambda^\alpha,\alpha}, E_{\lambda^\alpha,\alpha-1}).$$

We have

$${}^t\varepsilon(y_{\lambda^\vee}, y_{\lambda^\alpha}) = ({}^t\varepsilon_1(y_{\lambda^\vee,1}, y_{\lambda^\alpha,1}),\dots,{}^t\varepsilon_{\alpha-1}(y_{\lambda^\vee,\alpha-1}, y_{\lambda^\alpha,\alpha-1})),$$

where

$${}^t\varepsilon_i(y_{\lambda^\vee,i}, y_{\lambda^\vee,i}) := \begin{pmatrix} y_{\lambda^\vee,i} & 0_{\dim(E_{\lambda^\alpha,i})\times 1} \\ 0_{1\times\dim(E_{\lambda^\vee,i+1})} & y_{\lambda^\alpha,i} \end{pmatrix}.$$

Since C_{ϕ^α} is the 0-orbit in $V_{\lambda_{\phi^\alpha}}$, from Lemma 4.28, we have $y \in C_{\widehat{\phi^\alpha}} \subseteq V^*_{\lambda^\alpha}$ if and only if $y_{\phi^\alpha}=(y_{\phi^\alpha,1},\dots,y_{\phi^\alpha,\alpha-1})$, where $y_{\phi^\alpha,\alpha-i}\neq 0$ for any $i=1,\dots,\alpha-1$. For simplicity, we take $y_{\phi^\alpha,\alpha-i}=1$ for any $i=1,\dots,\alpha-1$

and so $y_{\phi^\alpha} = (1, \ldots, 1)$, although, this is not necessary for the rest of the argument.

Now we fix some $x \in V_{\lambda_\alpha}$ and write $x = (x_1, \ldots, x_{\alpha-1})$ as above. For each $i = 1, \ldots, \alpha - 1$, write

$$x_i = \begin{pmatrix} x_{i,1} & x_{i,2} \\ x_{i,3} & x_{i,4} \end{pmatrix},$$

where $x_{i,4}$ is a 1×1 matrix (which determines the dimensions of the rest of the matrices). Let $y_{\lambda^\vee} \in V_{\lambda^\vee}$. To prove the lemma, we must show that

$$[x, \varepsilon^*(y_{\lambda^\vee}, y_{\phi^\alpha})] = 0$$

if and only if $x = \varepsilon(x_{\lambda^\vee}, 0)$ for some $x_{\lambda^\vee} \in V_{\lambda^\vee}$.

Indeed, the reverse direction follows from direct computation. The forwards direction also follows from direct computation, but with a bit of tedious bookkeeping, largely in cases based on whether $x_i = x_{i,4}$ or not. We will give the details under the assumption that $x_i \neq x_{i,4}$ for any i as the other cases follow from similar arguments. Thus, we assume $x_i \neq x_{i,4}$ for any i and $[x, \varepsilon^*(y_{\lambda^\vee}, y_{\phi^\alpha})] = 0$. We must show that $x_{i,4} = 0$, $x_{i,2} = 0$, and $x_{i,3} = 0$ for each $i = 1, \ldots, \alpha - 1$. Write $y_{\lambda^\vee} = (y_1, \ldots, y_{\alpha-1})$ using the above convention.

From the assumptions, for $i = 2, \ldots, \alpha - 1$, we obtain

$$\begin{pmatrix} x_{1,1}y_1 & x_{1,2} \\ x_{1,3}y_1 & x_{1,4} \end{pmatrix} = 0,$$
$$\begin{pmatrix} x_{i,1}y_i & x_{i,2} \\ x_{i,3}y_i & x_{i,4} \end{pmatrix} = \begin{pmatrix} y_{i-1}x_{i-1,1} & y_{i-1}x_{i-1,2} \\ x_{i-1,3} & x_{i-1,4} \end{pmatrix},$$
$$0 = \begin{pmatrix} y_{\alpha-1}x_{\alpha-1,1} & y_{\alpha-1}x_{\alpha,2} \\ x_{\alpha-1,3} & x_{\alpha-1,4} \end{pmatrix}.$$

From the first equation above, we have that $x_{1,4} = 0$, from which the second equation implies that $x_{i,4} = 0$ for $i = 2, \ldots, \alpha - 1$. Similarly, the first equation implies that $x_{1,2} = 0$, from which the second equation implies that $x_{i,2} = 0$ for $i = 2, \ldots, \alpha-1$. Finally, the last equation implies that $x_{\alpha-1,3} = 0$. The middle equation then implies that $x_{i,3} = 0$ for $i = 1, \ldots, \alpha - 2$. Thus $x_i = \begin{pmatrix} x_{i,1} & 0 \\ 0 & 0 \end{pmatrix} = \varepsilon(x_{\lambda^\vee}, 0)$, where $x_{\lambda^\vee} = (x_{i,1}, \ldots, x_{i,\alpha-1}) \in V_{\lambda^\vee}$. This completes the proof of the lemma. □

We verify that the image of s is trivial in the microlocal fundamental group below.

Lemma A.4. *Suppose that* $\beta = \frac{\alpha-1}{2} \geq m^{\beta}_{\mathbb{1}_{W_F}}(\phi^\vee)$. *Then the image of* s *in* $A^{\mathrm{mic}}_{C_{\phi_\alpha}}$ *is trivial.*

Proof. Since H_{λ_α} acts on $\Lambda_{C_{\phi_\alpha}}$, there exists $(x', y') \in \Lambda_{C_{\phi_\alpha}}$ such that $y' \in C^*_{\phi_\alpha}$. Indeed, we have $\emptyset \neq \Lambda^{reg}_{C_{\phi_\alpha}} \subseteq \Lambda_{C_{\phi_\alpha}}$ and $\Lambda^{reg}_{C_{\phi_\alpha}} \subseteq C_{\phi_\alpha} \times C^*_{\phi_\alpha}$ by [17, Lemma 6.4.2]. Thus we may take $(x', y') \in \Lambda^{reg}_{C_{\phi_\alpha}}$. By Lemma 4.29, we have

$y' \in C^*_{\phi_\alpha} = C_{\widehat{\phi^\vee}+\widehat{\phi^\alpha}} \subseteq V^*_{\lambda_\alpha}$. Let ${}^t\varepsilon : V^*_{\lambda^\vee} \times V^*_{\lambda^\alpha} \hookrightarrow V^*_{\lambda_\alpha}$ denote the inclusion. Then ${}^t\varepsilon(C_{\widehat{\phi^\vee}}, C_{\widehat{\phi^\alpha}}) \subseteq C_{\widehat{\phi^\vee}+\widehat{\phi^\alpha}}$. Again, since H_{λ_α} acts on $\Lambda_{C_{\phi_\alpha}}$, there exists $y = {}^t\varepsilon(y_{\phi^\vee}, y_{\phi^\alpha}) \in \Lambda_{C_{\phi_\alpha}}$, where $y_{\phi^\vee} \in C_{\widehat{\phi^\vee}} \subseteq V^*_{\lambda^\vee}$ and $y_{\phi^\alpha} \in C_{\widehat{\phi^\alpha}} \subseteq V^*_{\lambda^\alpha}$.

By Lemma A.3, we have that $x \in \Lambda_y$ if and only if $x = \varepsilon(x'', 0)$ for some $x'' \in V_{\lambda^\vee}$. It follows that $Z(\mathrm{GL}_n(\mathbb{C})) \times Z(\mathrm{GL}_\alpha(\mathbb{C})) \subseteq H_{y,x}$ for any $x \in \Lambda_y$. Since $Z(\mathrm{GL}_n(\mathbb{C})) \times Z(\mathrm{GL}_\alpha(\mathbb{C}))$ is connected and both the identity and s lie in $Z(\mathrm{GL}_n(\mathbb{C})) \times Z(\mathrm{GL}_\alpha(\mathbb{C}))$, it follows that $s \in H^0_{y,x}$.

Now let $\nu \in \Lambda_y$ such that $A^{\mathrm{mic}}_{C_{\phi_\alpha}} = H_{y,\nu}/H^0_{y,\nu}$. We have that $\nu = \varepsilon(\nu', 0)$ for some $\nu' \in V_{\lambda^\vee}$. From the above observations, we have that the image of s in $A^{\mathrm{mic}}_{C_{\phi_\alpha}}$ is trivial. This completes the proof of the lemma. □

Finally we prove our fixed point formula (Theorem 4.33).

Theorem A.5. *Again, we suppose that* $\beta = \frac{\alpha-1}{2} \geq m^\beta_{\mathbb{1}_{W_F}}(\phi^\vee)$. *Then, we have*

$$\langle \eta_{\phi_\alpha}, [\mathcal{F}] \rangle_{\lambda_\alpha} = \langle \eta_{\phi^\times}, [\mathcal{F}|_{V_{\lambda^\times}}] \rangle_{\lambda^\times}.$$

for any $\mathcal{F} \in D_{H_\lambda}(V_\lambda)$.

Proof. The proof is the same as that of [20, Proposition 4.6] but using the above generalizations. From Proposition 4.32 and Lemma A.4, we have that Proposition A.2 applies in our setting. Combining Propositions A.1 and A.2, we obtain that

$$\begin{aligned} \langle \eta_{\phi_\alpha}, [\mathcal{F}] \rangle_{\lambda_\alpha} &= (-1)^{d(C_\phi)} \mathrm{rank}(\mathrm{Evs}_{C_\phi} \mathcal{F}) \\ &= (-1)^{d(C_{\phi^\times})} \mathrm{rank}(\mathrm{Evs}_{C^\times_\phi} \mathcal{F}|_{V_{\lambda^\times}}) \\ &= \langle \eta_{\phi^\times}, [\mathcal{F}|_{V_{\lambda^\times}}] \rangle_{\lambda^\times}. \end{aligned}$$

□

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