Bond Prices and Interest Rates

A bond is an IOU. That is, a bond is a promise to pay, in the future, fixed amounts that are stated on the bond. The interest rate that a bond actually pays therefore depends on how these payments compare to the price that is paid for the bond.[1] That price is determined in a market, so as to equate the implicit rate of interest paid on the bond to the rate of interest that buyers could get on other bonds of comparable risk and time to maturity. Figuring out what the interest rate on a bond is can be a quite tricky, since most bonds make payments for several years and of different sizes. Less tricky is to go the other direction, from the interest rate to the price of the bond.

This handout will work through two examples of how bond prices and interest rates would vary for two particularly simple kinds of bonds. Then it will provide the general formula for the price of a bond.

Example 1: A One-Year Bond

Consider a bond – I'll call it $B1$ – with principal equal to \$1000 and interest payment of \$70. That is, the bond is a promise to pay the principal plus interest, or \$1000+\$70=\$1070, one year from now. If the price of the bond were \$1000, then clearly it would be paying an interest rate for that year of 7%. By buying the bond you would be, in effect, lending \$1000 (the price of the bond), and getting repaid one year later both the amount that you lent (the principal) plus interest of \$70, which as a percent of what you lent is 70/1000 or 7%. That is, letting P_{B1} be the price of the bond and i be the implicit interest rate, then

$$\text{If } P_{B1} = \$1000, \text{ then } i = \frac{1070-1000}{1000} = \frac{70}{1000} = 0.07 = 7\%$$

Now suppose instead that the price of the bond were higher – that you had to pay, say, \$1010 for it. Then you would be lending more, but still getting back the same, so that the percentage return would be lower. How much lower? We can figure that out as follows:

$$\text{If } P_{B1} = \$1010, \text{ then } i = \frac{1070-1010}{1010} = \frac{60}{1010} = 5.94\%$$

Notice that the price of the bond goes into the formula for i in two places. It is in the top of the fraction because, the higher is the price, the less will be the extra that you get back over

[1] Bonds routinely also have an interest rate stated on the bond itself, but this is hardly ever the actual interest rate. It *would* be the actual interest rate only if the price of the bond were its face value – i.e., its principal – which it almost never is. The interest rate written on the bond itself is therefore pretty much meaningless, indicative perhaps of what the issuers of the bond hoped or expected the market rate would be.

and above what you paid in the first place. It is also in the bottom of the fraction because we want to calculate what you've earned as a percentage of what you paid.

What if the price of the bond were less – say \$990? Then

$$\text{If } P_{B1} = \$990, \text{ then } i = \frac{1070 - 990}{990} = \frac{80}{990} = 8.08\%$$

One of the things this tells us is that, the higher is the price of the bond, the lower is the interest rate that it pays. The reason is simply that the payment is fixed while the price changes. Since this is true also of more complicated bonds, it is a general property of bond prices and interest rates: **The higher are bond prices, the lower are interest rates, and *vice versa*.**

Suppose now that we do not know the price of the bond, but that we do know that other comparable bonds are paying an interest rate of 5%. Then what must the price of this bond be in order for it also to pay 5%? We can set up the same formula that we used above, but this time we know i and we don't know P_{B1}:

$$i = 5\% = 0.05 = \frac{1070 - P_{B1}}{P_{B1}}$$

We can solve the last equation here for P_{B1} by first multiplying both sides of it by P_{B1}, then adding it to both sides and solving:

$$\begin{aligned} 0.05P_{B1} &= 1070 - P_{B1} \\ P_{B1} + 0.05P_{B1} &= 1070 \\ 1.05P_{B1} &= 1070 \\ P_{B1} &= \frac{1070}{1.05} = \$1019 \end{aligned}$$

Therefore, if other comparable bonds (similar risk and time to maturity) are paying 5% interest, then this bond will have to sell on the market for \$1019 in order to pay the same interest rate of 5%.

Using this reasoning more generally, any one-year bond that promises to make only a single payment of \$$X$ in one year (called principal plus interest, but that does not matter for the calculation) will have a price, call it P_{BX1}, that depends on the market interest rate, i, as follows:

$$P_{BX1} = \frac{X}{1+i}$$

Notice again that the bond price and the interest rate are inversely related: when one rises, the other falls.

Example 2: A Perpetuity

A perpetuity is a bond that pays the same amount every year forever, never paying back the principal. Consider a perpetuity, call it *B*2, that pays $70 every year forever. This payment is the same as the interest payment of the one-year bond above, but here you get it every year. On the other hand, it never pays back the principal. So is it worth more or less than the one-year bond? That, it turns out, depends on the market interest rate.

Again we will look at the implicit interest rate that this bond pays for several prices, then turn this around to see what price is implied by any market interest rate.

Suppose the price of the bond were $1000. Then by buying it you would again be lending out $1000, and then you would get back interest payments every year of $70. Since $70 is 7% of $1000, it seems clear that you are earning an interest rate of 7% per year, and that's right. The interest rate you get on a perpetuity is just the payment it makes, divided by the price. In this case

$$\text{If } P_{B2} = \$1000, \text{ then } i = \frac{70}{1000} = 0.07 = 7\%$$

Now suppose that the price of the bond were higher, say $1100.[2] To buy it you will now have to give up more, $1100, but you will get back only the same interest payments of $70 a year. The interest rate is therefore lower:

$$\text{If } P_{B2} = \$1100, \text{ then } i = \frac{70}{1100} = 0.064 = 6.4\%$$

Similarly, if the bond price were lower, the interest rate would be higher:

$$\text{If } P_{B2} = \$500, \text{ then } i = \frac{70}{500} = 0.14 = 14\%$$

In general, if a perpetuity pays $$X$ per year, then its implicit interest rate is just the ratio of X to its price, P_{XPerp}.

$$i = \frac{X}{P_{XPerp}}$$

From this, solving the equation for P_{XPerp}, you can see immediately that the price of a perpetuity is the ratio of the interest payment to the interest rate:

$$P_{XPerp} = \frac{X}{i}$$

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[2] What would be the implicit interest rate on the one-year bond above if its price were $1100? Does this make sense?

Present Value

These two cases are really just special cases of the more general idea of "present value." The **present value** of an amount of money that will be paid in the future – any future payment or receipt – is defined as the amount that could be lent today to get back that amount in the future (or equivalently the amount that could be borrowed today and paid back with that money in the future) at the prevailing interest rate. We've already seen that the formula for the price of a one-year bond is really, then, just the present value of what the bond will pay back one year from now. That is, if you lend an amount Y today for one year at interest rate i, you will get back $X_1=Y+iY=(1+i)Y$ a year from now. Thus $Y=\frac{X_1}{(1+i)}$ is the present value of payment, X_1, one year in the future.

By similar reasoning, if you lend out an amount Y today for *two* years, it is as though you lent it out twice, but getting interest on the interest the second time. That is, you get back X_1 after the first year, then $X_2=X_1+iX_1$ after the second. Now $X_2=(1+i)X_1=(1+i)(1+i)Y=(1+i)^2Y$. Therefore $Y=\frac{X_2}{(1+i)^2}$ is the present value of a payment X_2 two years from now.

In general, for any stream of payments $X=X_1, X_2,\ldots,X_t,\ldots$ for any number of years, the present value of X is given by the formula

$$PV(X)=\frac{X_1}{(1+i)}+\frac{X_2}{(1+i)^2}+\ldots+\frac{X_t}{(1+i)^t}+\ldots$$

More simply, for those who are familiar with the symbol Σ to represent a summation,

$$PV(X)=\sum_t \frac{X_t}{(1+i)^t}$$

We now see what bond prices really are: **The price of a bond is the present value of the payments promised by the bond.** If a bond promises a stream of payments X as above, then the price of the bond is $PV(X)$.

Actual bonds typically promise a fixed interest payment, called the coupon payment, C, each year until maturity, then pay back the entire principal, P_0, in the year the bond matures. If the term to maturity of the bond is denoted T, then the price (present value) of the bond is

$$\begin{aligned} P_B &= \sum_{t=1}^{T}\frac{C}{(1+i)^t}+\frac{P_0}{(1+i)^T} \\ &= \frac{C}{(1+i)}+\frac{C}{(1+i)^2}+\ldots+\frac{C}{(1+i)^T}+\frac{P_0}{(1+i)^T} \end{aligned}$$

Notice again that because the interest rate, i, is always in the bottom of these fractions, if the interest rate goes up, the price of the bond will go down, and *vice versa.*